Nodal count via topological data analysis

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joint work with

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The corresponding eigenfunctions f_j,

$$\Delta f_j = \lambda_j f_j,$$

form an orthonormal basis in $L^2(M)$.

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Nodal pattern of an eigenfunction on $\mathbb{S}^2$ corresponding to an eigenvalue $\lambda = 17 \cdot 18$.
(*Picture credit: M. Levitin.)*

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Theorem (R. Courant, 1923)

A Laplace eigenfunction f_j has at most j nodal domains.

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4. higher topological invariants: Betti numbers m_r instead of m_0 (Arnold, 2005)

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Theorem (Buhovsky–Logunov–Sodin, 2020)

There exists a Riemannian metric g *on a* 2*-torus and a sequence* f_j *of eigenfunctions of the Laplacian* Δ_g*, such that the functions* $f_j + 1$ *have infinitely many nodal domains.*

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Idea: What if we ignore *small* oscillations?

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Our first main result shows that $m_0(f, \delta)$ is controlled by the appropriate Sobolev norms of f.

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Theorem (BP^{3}S^2, 2022)

Let $f \in W^{k,p}(M)$ *for* $k > \frac{n}{p}$, *where* $n = \dim M$. *Then for any* $\delta > 0$,

$$m_0(f, \delta) \leq C\delta^{-\frac{n}{k}} \|f\|_{W^{k,p}}^{\frac{n}{k}},$$

where C *depends on* M, k, p *but not on* δ.

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Note that the estimate blows up as $\delta \to 0$, and one can check that the constant C blows up as $k \to \infty$.

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Theorem

Let $k > \frac{n}{2}$ be an integer. Then for any $\delta > 0$ and any $f \in \mathcal{F}_\lambda$ with $\|f\|_{L^2} = 1$,

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Remark

All other extensions mentioned earlier can be also obtained in the *coarse* setting.

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In order to use our approach, we need to introduce the notion of coarse zero count.

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Let $f = (f_1, \ldots, f_n)$, where $f_j \in \mathcal{F}_\lambda$, $j = 1, \ldots, n$, and let $k > n/2$ be an integer. Then for any $\delta > 0$,

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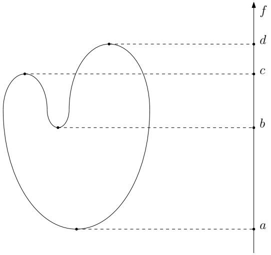
Examples:

$$d_{bottle}(\{(0,2],[0,1]\},\{(0,2.1)\}) = \frac{1}{2}$$

$$d_{bottle}(\{(0,2],[0,1]\},\{(0,+\infty)\}) = +\infty$$

Example: barcode of a height function

$f : \mathbb{S}^1 \to \mathbb{R}$ is a height function on deformed circle given by:



(*Picture credit: V. Stojisavljević.*)

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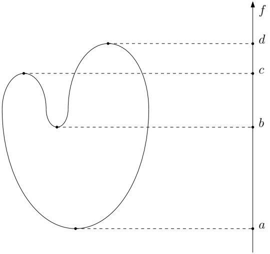

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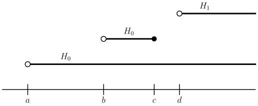

Barcode $\mathcal{B}(f)$. *(Picture credit: M. Levitin.)*

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- Infinite bars are of the form $(a, +\infty)$, where a is a critical value. They represent classes that are born but never die, i.e. genuine homology classes. This means that

$$\text{number of infinite bars} = \beta_M := \sum_{r=0}^{\dim M} b_r(M),$$

where β_M is the total Betti number of M.

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Let f, g be two Morse functions on M. Then

$$d_{bottle}(\mathcal{B}(f), \mathcal{B}(g)) \leq d_{C^0}(f, g).$$

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Stability theorem is a key feature of the theory.

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Indeed, 0 is the minimal value of $|f|$, and its maximal value in a δ-deep nodal domain is $\geq \delta$. Hence

$$m_0(f, \delta) \leq N_\delta(|f|).$$

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Let $f \in W^{k,p}(M)$ *for* $k > \frac{n}{p}$, *where* $n = \dim M$. *Then for any* $\delta > 0$,

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Theorem (Morrey–Sobolev)

Let $Q \subset \mathbb{R}^n$ be a cube and let $\mathcal{P}_k(Q) \subset C^0(Q)$ denote the subspace of polynomials of degree $\leq k$. Then

$$d_{C^0}\left(f, \mathcal{P}_{k-1}(Q)\right) \leq C_{n,k,p}\left(\mathrm{Vol}(Q)\right)^{\frac{k}{n}-\frac{1}{p}} \|f\|_{W^{k,p}(Q)}.$$

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- Multiscale dyadic partition into small cubes until functions are well aproximated by polynomials.
- Nice behavior of N_δ under unions and stability theorem.

Thank you for your attention!