A variational formula for Dirac operators in bounded domains and applications to spectral geometric inequalities

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Rafael Benguria
Instituto de Fisica
Pontificia Universidad Católica de Chile

This is joint work with

Pedro S. Antunes (Lisbon),

Vladimir Lotoreichik (Prague), and

Thomas Ourmières–Bonafos (Marseille),

See: arXiv:2003.04061v1,

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Consider $\Omega \subset \mathbb{R}^2$, a C^∞ simply connected domain, and let $n = (n_1, n_2)^T$ be the outward pointing normal on $\partial\Omega$.

The Dirac operator with *infinite mass boundary conditions* in $L^2(\Omega, \mathbb{C}^2)$ is defined as,

$$D^\Omega \equiv \begin{pmatrix} 0 & -2\, i\, \partial_z \\ -2\, i\, \partial_{\bar{z}} & 0 \end{pmatrix}.$$

Here,

$$\mathrm{dom}(\mathrm{D}^\Omega) = \{\mathrm{u} = (\mathrm{u}_1, \mathrm{u}_2)^\mathrm{T} \in \mathrm{H}^1(\Omega, \mathbb{C}^2) | \mathrm{u}_2 = \mathrm{i}\mathbf{n}\, \mathrm{u}_1 \text{ on } \partial\Omega\}.$$

We have set $\mathbf{n} = n_1 + i\, n_2$ and

$$\partial_z = \frac{1}{2}\left(\partial_1 - i\partial_2\right), \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_1 + i\partial_2\right).$$

The Dirac operator with *infinite mass boundary conditions* is self-adjoint (E.Stockmeyer & S.Vugalter, 2019), and (R.B., S.Fournais, E. Stockmeyer, H. Van den Bosch, 2017). Its spectrum is symmetric with respect to the origin, consisting of eigenvalues of finite multiplicity, with,

$$... \leq E_k(\Omega) \leq \cdots \leq -E_1(\Omega) < 0 < E_1(\Omega) < \ldots E_k(\Omega) \leq \ldots$$

In 2017, R.B., S.Fournais, E. Stockmeyer, H. Van den Bosch proved the following geometrical bound,

$$E_1(\Omega) \geq \sqrt{\frac{2\pi}{|\Omega|}},$$

Where $|\Omega|$ is the area of the domain Ω. By analogy with the Rayleigh-Faber–Krahn inequality it is natural to conjecture:

Conjecture 1.

$$E_1(\Omega) \geq \sqrt{\frac{\pi}{|\Omega|}}\, E_1(\mathbb{D}),$$

Where $\mathbb{D}$ is the unit disk. There is equality if and only if Ω is a disk.

Remark. The eigenstructure of the unit disk is explicit. In fact, $E_1(\mathbb{D}) = k \equiv 1.435\ldots$, where k is the first positive root of the equation $J_0(k) = J_1(k)$. Here $J_n(t)$ denotes the n–th Bessel function of the first kind. Moreover, using polar coordinates (ρ, θ), the first eigenfunction is given by

$$\begin{pmatrix} J_0(k\,\rho) \\ i\,e^{i\theta} J_1(k\,\rho) \end{pmatrix}.$$

Our main results

Theorem 1 [P. Antunes, RB, T. Ourmières-Bonafos, V. Lotoreichik, 2020]. Let $\Omega \subset \mathbb{R}^2$ be a C^∞ simply connected domain. Then we have

$$E_1(\Omega) \leq \frac{|\partial\Omega|}{\pi r_i^2 + |\Omega|} E_1(\mathbb{D}),$$

with equality if and only if Ω is a disk.

Here $|\Omega|$ is the area, $|\partial\Omega|$ the perimeter and r_i the inradius of the domain Ω.

What we actually prove is
Theorem 2. Let $\Omega \subset \mathbb{R}^2$ be a C^∞ simply connected domain. Then we have

$$E_1(\Omega) \leq \frac{|\partial\Omega| + \sqrt{|\partial\Omega|^2 + 8\pi E_1(\mathbb{D})(E_1(\mathbb{D}) - 1)(\pi r_i^2 + |\Omega|)}}{2(\pi r_i^2 + |\Omega|)}$$

with equality if and only if Ω is a disk.

Theorem 1 follows from Theorem 2 using $\pi r_i^2 \leq |\Omega| \leq |\partial\Omega|^2/(4\pi)$.

The proof is obtained by combining a new variational characterization of $E_1(\Omega)$, inspired by min-max techniques for operators with gaps introduced by J. Dolbeault, M. Esteban, and E. Seré, JFA **174** (2000), 208–226, and the classical proof of Szegő about the first nontrivial Neumann eigenvalue of the Laplacian in $\mathbb{R}^2$ (1954).

Consider the quadratic form

$$q_{E,0}^{\Omega}(u) \equiv 4\int_{\Omega} |\partial_{\bar{z}} u|^2 \, dx - E^2 \int_{\Omega} |u|^2 \, dx + E \int_{\partial\Omega} |u|^2 \, ds,$$

with $\operatorname{dom}\left(q_{E,0}^{\Omega}\right) = C^{\infty}(\bar{\Omega}, \mathbb{C})$.

For $E > 0$, $q_{E,0}^{\Omega}$ is bounded below with dense domain and we consider q_E^{Ω} the closure in $L^2(\Omega)$ of $q_{E,0}^{\Omega}$. Then, we define the first min-max level,

$$\mu^{\Omega}(E) = \inf_u \frac{q_E^{\Omega}(u)}{\int_{\Omega} |u|^2 \, dx}.$$

where the infimum is taken over $\operatorname{dom}(q_E^{\Omega}) \setminus \{0\}$.

Theorem 3. $E > 0$ is the first non–negative eigenvalue of D^Ω if and only if $\mu^\Omega(E) = 0$.

Heuristics: Let $(u,v)^T \in \mathrm{dom}(D^\Omega)$ be an eigenfunction with eigenvalue E. In Ω the eigenvalue equation reads,

$$-2i\partial_z v = E\,u, \qquad -2i\partial_{\bar{z}} u = E\,v.$$

Assuming the equations are valid up to the boundary, using the infinite mass boundary conditions, we get the following boundary condition on u,

$$\bar{n}\partial_{\bar{z}} u + \frac{E}{2} u = 0, \qquad \text{on } \partial\Omega.$$

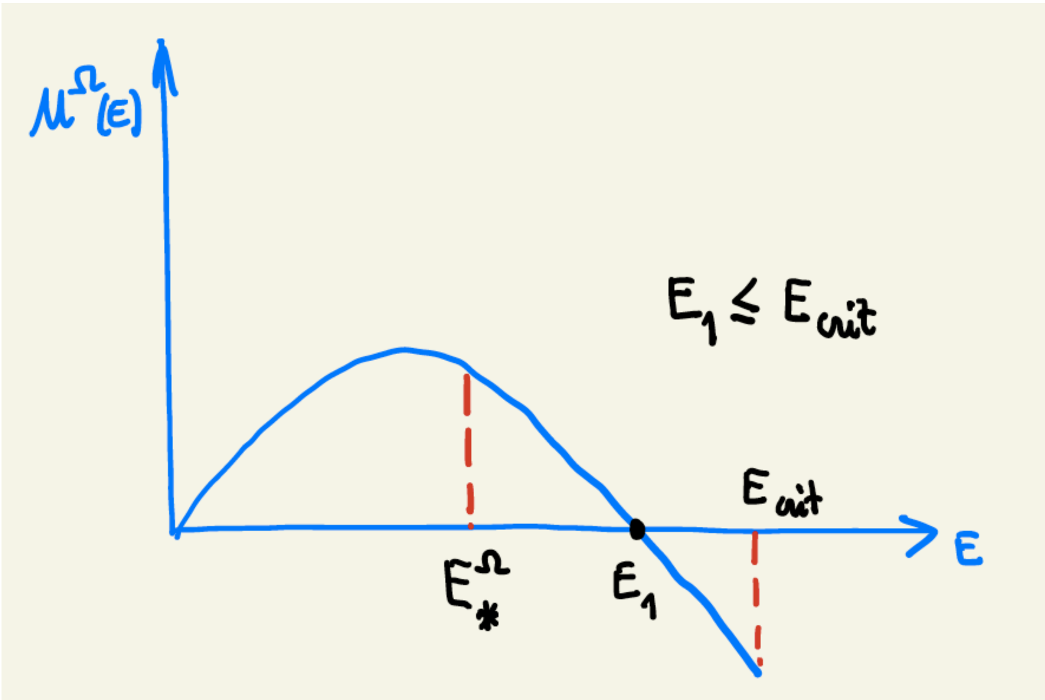
Now from the equations for u and v we get,

$$-4\partial_z \partial_{\bar{z}} u = E^2 u, \qquad \text{in } \Omega.$$

Taking the scalar product with u, integrating by parts, and using the boundary condition formally gives $q_E^\Omega(u)$. This is the reason for introducing q_E^Ω.

In order to use the function $\mu^\Omega(E)$ to estimate $E_1(\Omega)$ we need the following:

Lemma 1. The map $\mu^\Omega : E \geq 0 \to \mu^\Omega(E)$ satisfies:
i) $\mu^\Omega(E)$ is a continuous and concave function on $\mathbb{R}_+$.
ii) We have $\mu^\Omega(0) = 0$, and there exists $E_*^\Omega > 0$ such that for all $(0, E_*^\Omega)$, $\mu^\Omega(E) > 0$.



A simple application: a first geometric (non sharp) upper bound.

Lemma. Let $\Omega \subset \mathbb{R}^2$ be C^∞ and simply connected. Then,

$$E_1(\Omega) \le \frac{|\partial\Omega|}{|\Omega|}.$$

Proof. Let $E > 0$ and $u \equiv 1$. By the min-max principle

$$\mu^\Omega(E) \le \frac{q_E^\Omega(u)}{\|u\|^2_{L^2(\Omega)}} = E\left(\frac{|\partial\Omega|}{|\Omega|} - E\right).$$

We see that for $E_{\text{crit}} = |\partial\Omega|/|\Omega|$, $\mu^\Omega(E_{\text{crit}}) \le 0$. Then,

$$E_1(\Omega) \le E_{\text{crit}} = \frac{|\partial\Omega|}{|\Omega|}.$$

Proof of Theorem 2

Geometric Preliminaries

Koebe's estimate.

Let $f(z)$ be a conformal mapping from the unit disc $\mathbb{D}$ onto a simply connected domain Ω. Then for all $z \in \mathbb{D}$,

$$\frac{1}{4}|f'(z)|\left(1-|z|^2\right) \leq \text{dist}(f(z), \partial\Omega) \leq |f'(z)|\left(1-|z|^2\right).$$

(See, e.g., J.B. Garnett & D. E. Marshall, *Harmonic Measure*, CUP, 2005, Theorem 4.3, p. 19).

In particular if $f(0)=0$ and $f(z)=\sum_{n=1}^{\infty} c_n\, z^n$, We have

$$r_i \leq |c_1|.$$

The area formula.
In terms of f one can write the area of the domain Ω as follows,

$$|\Omega| = \frac{1}{2}\oint_{\partial\Omega}(xdy - ydx) \quad = \tfrac{1}{2i}\int_{\partial\Omega}\bar{\omega}\,d\omega = \tfrac{1}{2i}\int_{|z|=1}\bar{f}\,df = \\ = \pi\sum_{n=1}^{\infty} n|c_n|^2. \qquad (1)$$

The perimeter.
In terms of the conformal map one also has the perimeter of the domain Ω,

$$|\partial\Omega| = \int_0^{2\pi} |f'(e^{i\theta})|\,d\theta.$$

Proof of Theorem 2.
We construct a proper test function for q_E^Ω transplanting the first eigenfunction of the Dirac operator on the unit disc $\mathbb{D}$ to the domain Ω using the conformal map f. We obtain an upper bound on $\mu^\Omega(E)$, which is a quadratic in $E > 0$ with coefficients depending on the geometry of Ω. Finally we optimize on E. We do this procedure in 5 steps.

Step 1: "Choice of trial function".
Take

$$u_0(x) = J_0(E_1|x|) \in H_1(\mathbb{D}) \subset \mathrm{dom}\left(q^D_{E_1(\mathbb{D})}\right).$$

Recall that this u_0 is the upper spinor of the first eigenfunction of the Dirac operator on the unit disc. Theorem 3 implies that $q^D_{E_1(\mathbb{D})}(u_0) = 0$, which can be written explicitly as,

$$E_1(\mathbb{D}) \int_0^1 J_1(E_1(\mathbb{D})\, r)^2\, r\, dr - E_1(\mathbb{D}) \int_0^1 J_0(E_1(\mathbb{D})\, r)^2\, r\, dr + J_0(E_1(\mathbb{D}))^2 = 0.$$

Step 2: "Choice of trial function"
For $x = (x_1, x_2) \in \Omega$, consider

$$v_0(x_1, x_2) = u_0(f^{-1}(x_1 + ix_2)) \in H_1(\Omega) \subset \mathrm{dom}(q_E^\Omega).$$

By the min-max principle, we have

$$\mu^\Omega(E) \leq \frac{q_E^\Omega(v_0)}{\|v_0\|_{L^2(\Omega)}^2} = \frac{\|\nabla v_0\|_{L^2(\Omega)}^2 + E\|v_0\|_{L^2(\partial\Omega)}^2}{\|v_0\|_{L^2(\Omega)}^2} - E^2.$$

Here we used that v_0 is real valued to insure that $\|\nabla v_0\|_{L^2(\Omega)}^2 = 4\|\partial_{\bar{z}} v_0\|_{L^2(\Omega)}^2$.

Step 3. "Computation of norms of the trial function".

Since f is a conformal map, we have

$$\|v_0\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\mathbb{D})} = 2\pi\, E_1(\mathbb{D})^2 \int_0^1 J_1(E_1(\mathbb{D})\, r)^2\, r\, dr.$$

Using the perimeter formula, we also have,

$$\|v_0\|^2_{L^2(\partial\Omega)} = \int_0^{2\pi} |v_0(f(e^{i\theta}))|^2\, |f'(e^{i\theta})|\, d\theta = J_0(E_1(\mathbb{D}))^2\, |\partial\Omega|.$$

Finally, we also have

$$\|v_0\|^2_{L^2(\Omega)} = \int_0^1 \int_0^{2\pi} u_0(r)^2 |f'(re^{i\theta})|^2\, r\, dr\, d\theta$$

$$= \int_0^1 u_0(r)^2 \left(\int_0^{2\pi} |\sum_{n\geq 1} n\, c_n r^{n-1} e^{i(n-1)\theta}|^2\, d\theta \right) r\, dr = 2\pi \sum_{n\geq 1} n|c_n|^2 M_n,$$

by Parseval's inequality. Here,

$$M_n \equiv n \int_0^1 J_0(E_1(\mathbb{D} r)^2\, r^{2n-1}\, dr,$$

for all $n \geq 1$.

Step 4. "Properties of Bessel functions and their consequences".

Inserting the computations of the three norms back in the right side of (1) we have

$$\mu^{\Omega}(E) \leq 2\pi E_1(\mathbb{D})^2 \frac{\int_0^1 J_1(E_1(\mathbb{D})r)^2 r\, dr}{2\pi \sum_{n\geq 1} n|c_n|^2 M_n} + E\frac{J_0(E_1(\mathbb{D}))^2|\partial\Omega|}{2\pi \sum_{n\geq 1} n|c_n|^2 M_n} - E^2.$$

Next, we prove the following lower bound on M_n:

$$M_n \geq \frac{n}{2n-1} M_1,$$

For all $n \geq 1$. From the definition of M_n and properties of Bessel functions we have

$$M_1 = J_0(E_1(\mathbb{D}))^2,$$

and

$$M_n = \frac{1}{2} J_0(E_1(\mathbb{D}))^2 + \frac{E_1(\mathbb{D})}{2} \int_0^1 J_0(E_1(\mathbb{D})r) J_1(E_1(\mathbb{D})r)\, r^{2n}\, dr,$$

Which follows from the definition of M_n, an integration by parts, and the fact that $J_0' = -J_1$.

Step 4 (continued)

$$M_n = \frac{1}{2} M_1 + \frac{E_1(\mathbb{D})}{2} \int_0^1 J_0(E_1(\mathbb{D})r) J_1(E_1(\mathbb{D})r)\, r^{2n}\, dr.$$

Now, for $n \geq 1$, we use that the functions

$$g(r) \equiv r^2 (J_0 J_1)(E_1(\mathbb{D})r),$$

and

$$h(r) = r^{2n-2},$$

are non-decreasing on $[0, 1]$ and by Chebyschev's inequality (*) for non-decreasing functions we get

$$M_n \geq \frac{M_1}{2} + \frac{M_1}{2} \int_0^1 r^{2n-2}\, dr = \frac{n}{2n-1} M_1.$$

(*) In this case,

$$\int_0^1 g(r) h(r)\, dr \geq \int_0^1 g(r)\, dr \int_0^1 h(r)\, dr.$$

Step 4 (continued)

Using the lower bound on M_n we get,

$$L \equiv 2\pi \sum_{n\geq 1} n|c_n|^2 M_n \geq M_1 \left(2\pi |c_1|^2 + 2\pi \sum_{n \geq 2} \frac{n^2}{2n-1} |c_n|^2 \right) \equiv R$$

and, successively

$$R \geq M_1 \left(2\pi |c_1|^2 + \pi \sum_{n\geq 2} n|c_n|^2 \right) = M_1 \left(\pi |c_1|^2 + |\Omega| \right) \geq M_1 \left(\pi r_i^2 + |\Omega| \right).$$

Recall that $M_1 = J_0(E_1(\mathbb{D}))^2$. There is equality in the above inequalities if and only if $c_n = 0$, for all $n \geq 2$ and if $|c_1| = r_i$, i.e., if and only if $f(z) = c_1 z$ and Ω is a disk centered at 0 with radius r_i.

Step 4 (continued)

Using the above in (1), and recalling that

$$\int_0^1 J_0(E_1(\mathbb{D})\, r)^2\, r\, dr = J_0(E_1(\mathbb{D}))^2,$$

we get that,

$$\mu^{\Omega}(E) \leq \frac{P(E)}{\pi r_i^2 + |\Omega|},$$

where $P(E)$ is a quadratic polynomial given by

$$P(E) = -E^2(\pi r_i^2 + |\Omega|) + E|\partial\Omega| + 2\pi\, E_1(\mathbb{D})\, (E_1(\mathbb{D}) - 1)\,.$$

Step 5. "Conclusion".

From the exact solution when $\Omega = \mathbb{D}$ we have

$$E_1(\mathbb{D}) - 1 \geq \sqrt{2} - 1 > 0.$$

Thus, the discriminant of P,

$$\delta P = |\partial\Omega|^2 + 8\pi E_1(\mathbb{D})\,(E_1(\mathbb{D}) - 1)\,(\pi\, r_i^2 + |\Omega|) \geq 0.$$

Therefore, P has two real roots and as $P(0) > 0$, the only positive root is

$$E_{\text{crit}} = \frac{|\partial\Omega| + \sqrt{\delta P}}{2(\pi\, r_i^2 + |\Omega|)}.$$

One gets,

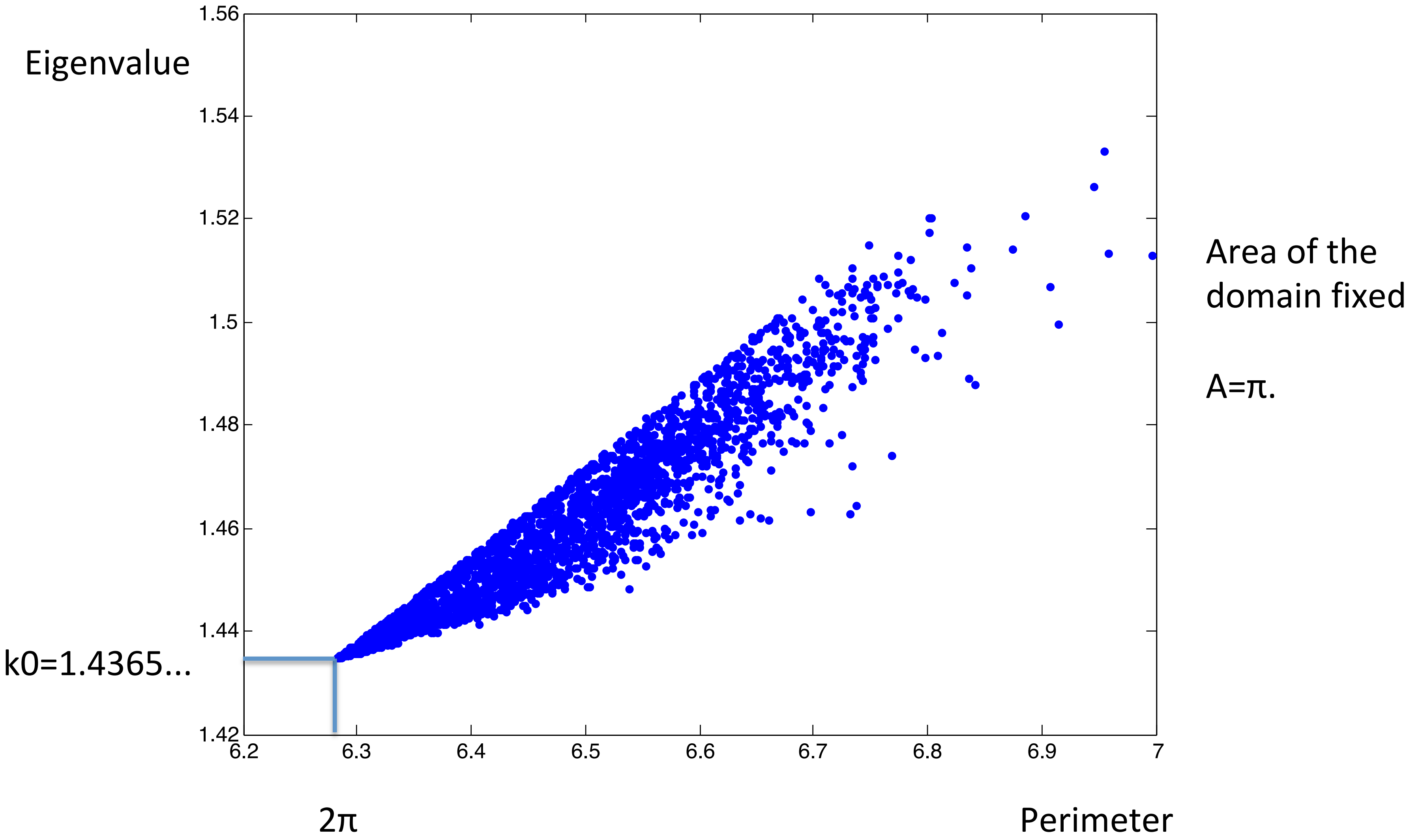
$$\mu^{\Omega}(E_{\text{crit}}) \leq \frac{P(E_{\text{crit}})}{\pi\, r_i^2 + |\Omega|} = 0,$$

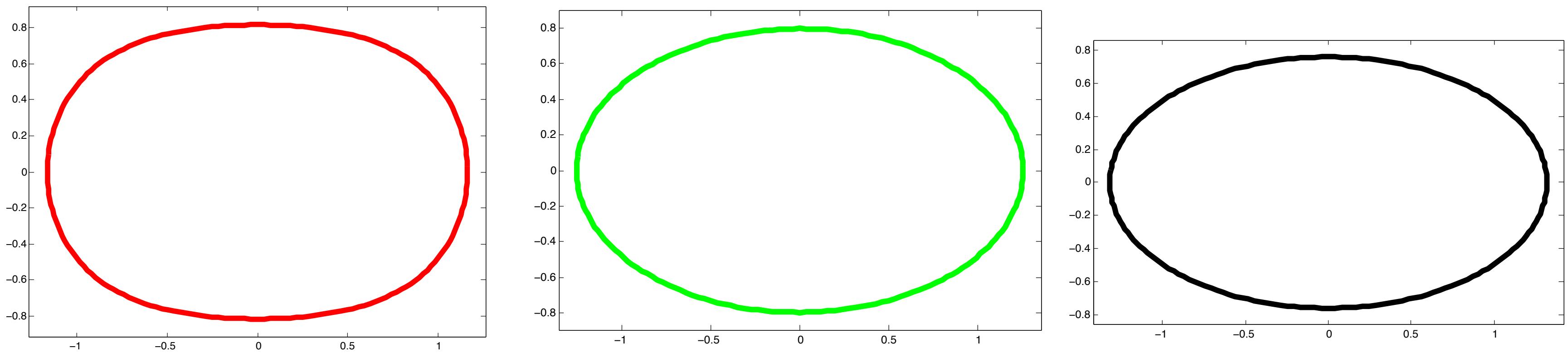
and by Lemma 1 and Theorem 3 we finally get,

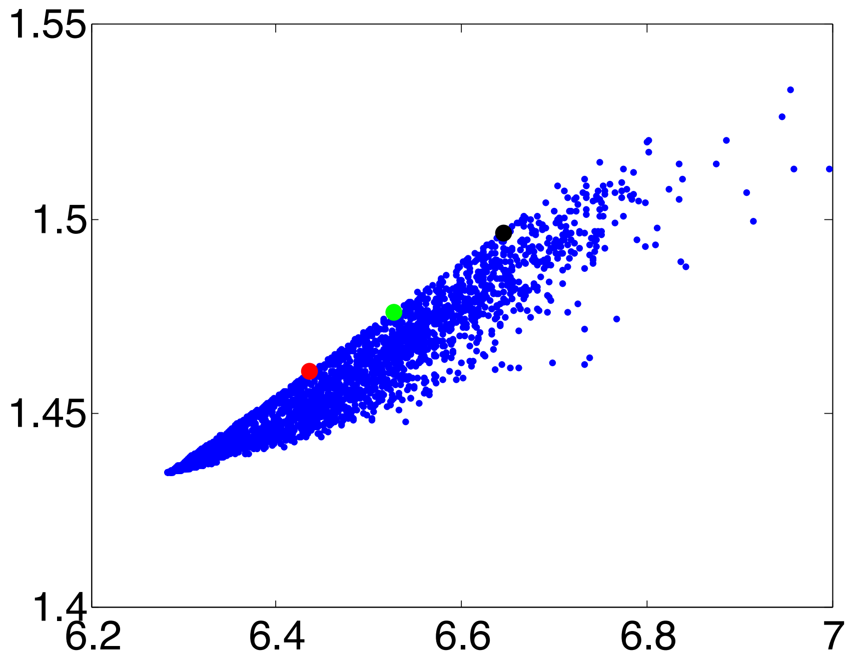
$$E_1(\Omega) \leq E_{\text{crit}},$$

and we are done.

Numerical evidence supporting the RFK conjecture


Eigenvalue
1.56
1.54
1.52
1.5
1.48
1.46
1.44
1.42
k0=1.4365...
6.2
6.3
6.4
6.5
6.6
6.7
6.8
6.9
7
2π
Perimeter
Area of the domain fixed
A=π.




1.55
1.5
1.45
1.4
6.2
6.4
6.6
6.8
7

THANKS