Maximization of Neumann eigenvalues

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joint works with
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Shape Optimisation and Geometric Spectral Theory

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$\Omega \subseteq \mathbb{R}^N$, open, bounded, Lipschitz

$$\begin{cases} -\Delta u = & \mu u \text{ in } \Omega \\ \frac{\partial u}{\partial n} = & 0 \ \partial\Omega \end{cases}$$

$$0 = \mu_0 \leq \mu_1 \leq \mu_2 \leq \dots \leq \mu_k \leq \to +\infty$$

are the first k-eigenvalues of the Neumann Laplacian,

$$\mu_k(\Omega) = \min_{S \in \mathscr{S}_{k+1}} \max_{u \in S \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx},$$

where $\mathscr{S}_k$ is the family of all subspaces of dimension k in $H^1(\Omega)$.

Problem

$$\max\{\mu_k(\Omega) : \Omega \subseteq \mathbb{R}^N, |\Omega| = m\}.$$

Equivalently

$$\max_{\Omega \subseteq \mathbb{R}^N} |\Omega|^{\frac{2}{N}} \mu_k(\Omega).$$

- Existence of a solution: optimal domain
- Qualitative properties of the optimal domain: regularity of the boundary, symmetry, topological properties
- Identify the optimal shape: analytically (is it the ball ?), otherwise numerical approximations

Related to the Pólya conjecture

$$\forall k \in \mathbb{N}, \quad \mu_k(\Omega) \leq \frac{4\pi^2 k^{\frac{2}{N}}}{(\omega_N |\Omega|)^{\frac{2}{N}}}.$$

Numerical computations for Neumann eigenvalues

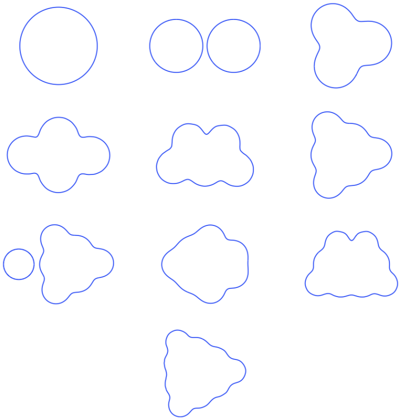


Figure: P. Antunes, E. Oudet, *Numerical results for extremal problem for eigenvalues of the Laplacian. Shape optimization and spectral theory*, 398?411, De Gruyter Open, Warsaw, 2017 (previous computations by Antunes-Freitas 2012, Berger 2015)

Collection of results on Neumann eigenvalues

- Szegö 1954: the disc maximizes $|\Omega|\mu_1(\Omega)$, in $\mathbb{R}^2$ among smooth simply connected sets
- Weinberger 1956: the ball maximizes $|\Omega|^{\frac{2}{N}}\mu_1(\Omega)$, in $\mathbb{R}^N$
- Girouard, Nadirashvili, Polterovich 2008: the union of two equal disjoint discs maximizes $|\Omega|\mu_2(\Omega)$, in $\mathbb{R}^2$ among smooth simply connected sets.
- Polyquin and Roy-Fortin 2010, μ_{22} is not maximized by a ball or union of balls.
- B., Henrot 2019: the union of two equal disjoint balls maximizes $|\Omega|^{\frac{2}{N}}\mu_2(\Omega)$, in $\mathbb{R}^N$ among Lipschitz sets and more.
- Freitas, Laugesen 2020: a different topological argument for μ_2.

What means more?

Let $\rho : \mathbb{R}^N \to [0,1]$, $\rho \in L^1(\mathbb{R}^N)$.

For every $k \geq 1$, we define

$$\tilde{\mu}_k(\rho) := \inf_{S \in \mathscr{L}_k} \max_{u \in S} \frac{\int_{\mathbb{R}^N} \rho |\nabla u|^2 dx}{\int_{\mathbb{R}^N} \rho u^2 dx},$$

where $\mathscr{L}_k$ is the family of all subspaces of dimension k in

$$\{u \cdot 1_{\{\rho(x)>0\}} : u \in C_c^\infty(\mathbb{R}^N), \int_{\mathbb{R}^N} \rho u dx = 0\}. \tag{1}$$

If Ω is bounded and Lipschitz then for

$$\rho = 1_\Omega$$

we have

$$\forall k \in \mathbb{N}, \; \tilde{\mu}_k(\rho) = \mu_k(\Omega).$$

What means more?

Theorem (B., Henrot 2019)

Let $\rho : \mathbb{R}^N \to [0,1]$, $\rho \in L^1(\mathbb{R}^N)$. *Then*

- $\mu_1(\rho)(\int_{\mathbb{R}^N} \rho\, dx)^{\frac{2}{N}} \le |B_1|^{\frac{2}{N}} \mu_1(B_1)$
- $\mu_2(\rho)(\int_{\mathbb{R}^N} \rho\, dx)^{\frac{2}{N}} \le 2^{\frac{2}{N}} |B_1|^{\frac{2}{N}} \mu_1(B_1)$

Main (intuitive) conclusion: if an inequality is proved by mass transplantation for open, Lipschitz sets, then it occurs for densities.

Natural questions

- Does problem

$$\sup\{\mu_k(\rho) : \rho : \mathbb{R}^N \to [0,1], \int_{\mathbb{R}^N} \rho\, dx = m\},$$

  or its scale invariant version

$$\mu_k^* := \sup\{\Big(\int_{\mathbb{R}^N} \rho\, dx\Big)^{\frac{2}{N}} \mu_k(\rho) : \rho : \mathbb{R}^N \to [0,1]\},$$

  have a solution for every k?

- What is the geometry of the optimal densities?

- Does the Pólya conjecture hold for densities?
  I mean: is there any chance for this assertion to hold?

Existence of an optimal density

Take a maximizing sequence $(\rho_n)_n$. For a subqsequence we have

$$\rho_n \rightharpoonup \rho \quad \text{weakly} - *\text{in } L^\infty(\mathbb{R}^N).$$

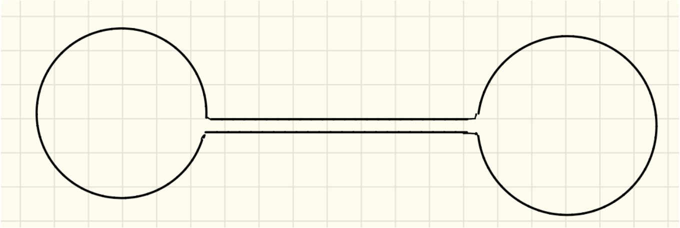
Moreover

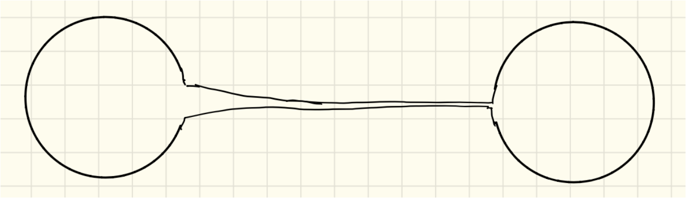
$$\mu_k(\rho) \geq \limsup_{n\to\infty} \mu_k(\rho_n)!$$

But the constraint is not preserved !!!

We do not have, in general, $\int_{\mathbb{R}^N} \rho\, dx = m$, since

$$1 \notin L^1(\mathbb{R}^N).$$





Surgery not possible...

- *Compactness.* There exists a subseqence $(\rho_{n_j})_j$ and a sequence of vectors $y_{n_j} \in \mathbb{R}^N$ such that

$$\rho_{n_j}(y_{n_j} + \cdot) \rightharpoonup \rho, \text{ weakly-* in } L^\infty(\mathbb{R}^N)$$

and $\int_{\mathbb{R}^N} \rho \, dx = m$. This is good !

- *Vanishing.* For every $R > 0$, we have that

$$\sup_{y \in \mathbb{R}^N} \int_{B(y,R)} \rho_n dx \to 0, \text{ when } n \to \infty.$$

- *Dichotomy.* There exists a subseqence $(\rho_{n_j})_j$ and a sequence of vectors $y_{n_j} \in \mathbb{R}^N$ such that

$$\rho_{n_j}(y_{n_j} + \cdot) \rightharpoonup \rho, \text{ weakly-* in } L^\infty(\mathbb{R}^N)$$

and $0 < \int_{\mathbb{R}^N} \rho \, dx < m$ (and is maximal).

Lemma

Let $\rho \in L^1(\mathbb{R}^N, [0,1])$, $\int_{\mathbb{R}^N} \rho\, dx = m$, *such that* $\mu_k(\rho) > 0$. *There exists a ball* B_{x,R^*} *with*

$$R^* = \sqrt{\frac{4(k+1)}{c_N \mu_k(\rho)}}$$

such that

$$\int_{B_{x,R^*}} \rho\, dx \geq \frac{c_N m}{k+1}.$$

Consequence: no vanishing!

Lemma

Let $\rho \in L^1(\mathbb{R}^N, [0,1])$, $\int_{\mathbb{R}^N} \rho dx = m$, *such that* $\rho = \rho_0 + \rho_1 + \cdots + \rho_j$ *and let* $R > 0$. *Assume that*

$$\forall 1 \leq l \neq i \leq j, \ dist(\{\rho_l > 0\}, \{\rho_i > 0\}) \geq 3R.$$

Moreover, assume $\forall 1 \leq l \leq j$, $m_l = \int \rho_l dx > 0$ *and denote* $m_0 = \int \rho_0 dx$. *Then, for every* $l \in 1, \ldots, j$, *there exists* $R_l^* > 0$ *and* $x_l \in \mathbb{R}^N$ *satisfying*

$$\frac{1}{R_l^*} \geq \Big[\frac{1}{2}\Big(\frac{\mu_k(\rho) c_N m_l}{(k+1)(m_l + m_0)}\Big)^{\frac{1}{2}} - \frac{1}{2R}\Big]^+ \tag{2}$$

and

$$\int_{B_{x_l, R_l^*}} \rho_l \geq \frac{c_N m_l}{k+1}.$$

Consequence: there exists at mots k concentration of masses.

Theorem (B.-Martinet-Oudet 2022)

The following problem

$$\max\{\mu_k(\rho) : 0 \le \rho \le 1, \int_{\mathbb{R}^N} \rho\, dx = m\}$$

has a solution (as a collection of at most k densities).

- Key question : is it true that the optimal $\rho = 1_\Omega$ for some characteristic function ?

In general, regularity not possible here...

Assume $\Omega =$ union of intervals.

Then $\max \mu_k(\Omega)|\Omega|^2$ is maximal on a union of k equal intervals, possibly joining at their extremities.

What about

$$\max_{\rho} \mu_k(\rho)\Big(\int_{\mathbb{R}} \rho\, dx\Big)^2 ?$$

Theorem (B.M.O. 2022)

In $\mathbb{R}$, $\forall k \in \mathbb{N}$

$$\mu_k(\rho)\Big(\int_{\mathbb{R}} \rho\, dx\Big)^2 \leq \pi^2 k^2.$$

Equality is attained for ρ being the characteristic function associated to the union of at most k open, pairwise disjoint segments of total length equal to $m := \int_{\mathbb{R}} \rho\, dx$, each one with length an entire multiple of $\frac{m}{k}$.

Corollary (Sturm-Liouville eigenvalues)

Let $(\alpha,\beta) \subseteq \mathbb{R}$ be an interval and $\rho_1, \rho_2 : [\alpha,\beta] \to \mathbb{R}$ be positive C^1 functions. We consider the eigenvalue problem

$$\begin{cases} -(\rho_1 u')' = \mu_k \rho_2 u \text{ on } (\alpha,\beta) \\ \qquad u'(\alpha) = u'(\beta) = 0 \end{cases}$$

Then

$$\forall k \geq 0,\ \mu_k \leq \frac{\|\rho_2\|_\infty}{\|\rho_1\|_\infty} \frac{\pi^2 k^2}{\min\Big(\frac{\int_\alpha^\beta \rho_1}{\|\rho_1\|_\infty}, \frac{\int_\alpha^\beta \rho_2}{\|\rho_2\|_\infty}\Big)^2}$$

- It holds in $\mathbb{R}^1$

$$\forall k \in \mathbb{N},\ \mu_k(\rho) \leq \frac{\pi^2 k^2}{\left(\int_{\mathbb{R}} \rho\, dx\right)^2}.$$

- Kröger inequalities 1992 in $\mathbb{R}^N$

$$\mu_k(\Omega) \leq 4\pi^2 \left(\frac{(N+2)k}{2\omega_N} \frac{1}{|\Omega|} \right)^{2/N}$$

Theorem (B.M.O. 2022)

Let $N \geq 2$, $\rho \in L^1(\mathbb{R}^N, [0,1])$, $\rho \not\equiv 0$. Then

$$\forall k \in \mathbb{N}, \quad \mu_k(\rho) \leq 4\pi^2 \left(\frac{(N+2)k}{2\omega_N} \frac{1}{||\rho||_1} \right)^{2/N},$$

where ω_N is the volume of the unit ball of $\mathbb{R}^N$.

Previous results obtained by Colbois, El Soufi, Savo 2015 with generic constant.

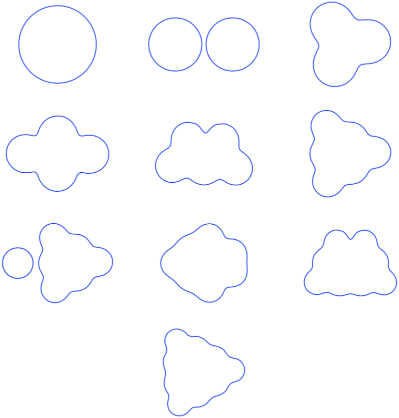


Figure: P. Antunes, E. Oudet, *Numerical results for extremal problem for eigenvalues of the Laplacian. Shape optimization and spectral theory*, 398?411, De Gruyter Open, Warsaw, 2017 (previous computations by Antunes-Freitas 2012, Berger 2015)

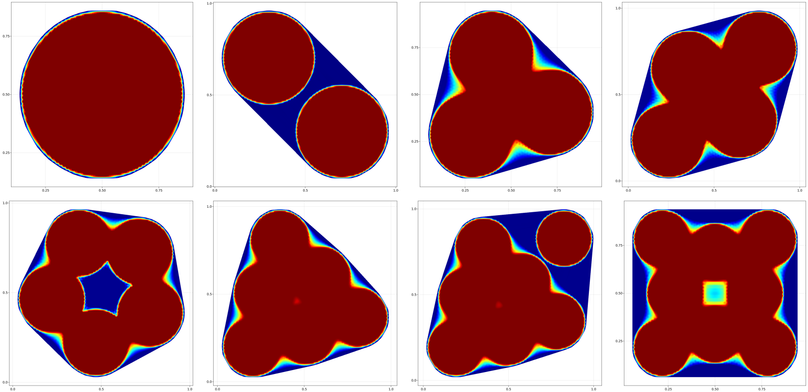


Figure: Approximation of the first eight optimal densities, D. Bucur, E. Martinet, E. Oudet, *Maximization of Neumann eigenvalues*, Arxiv arXiv:arXiv:2204.11472v2, 2022.

	μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8
Multiplicity	2	2	3	3	3	4	4	4
Optimal densities	10.65	21.28	32.92	43.90	54.47	67.25	77.96	89.47
Optimal shapes, Antunes-Freitas [2]			32.79	43.43	54.08	67.04	77.68	89.22
Union of discs	10.65	21.30	31.95	42.60	53.25	63.90	74.55	88.85

For $\Omega \subseteq \mathbb{S}^n$,

$$\begin{cases} -\Delta_{\mathbb{S}^n} u = \mu_k(\Omega) u \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{cases}$$

Theorem (Ashbaugh-Benguria 1995)

Let $m \leq \frac{|\mathbb{S}^n|}{2}$. If $\Omega \subseteq \mathbb{S}^n \cap \{x_1 \geq 0\}$ (i.e. in a hemisphere) and $|\Omega| = m$, then

$$\mu_1(\Omega) \leq \mu_1(B^m).$$

Can one remove the hemisphere inclusion condition ?

Can one consider $|\mathbb{S}^n| > m > \frac{|\mathbb{S}^n|}{2}$?

Numerical computations by Eloi Martinet...

... and intriguing answers.

Theorem (B., Martinet, Nahon, 2022)

Let $0 < m < |\mathbb{S}^n|$. *If* $\Omega \subseteq \mathbb{S}^n$ *and* $|\Omega| = m$, *then*

$$\mu_2(\Omega) \le \mu_1(B^{\frac{m}{2}}).$$

Or, more general

$$\sum_{k=2}^{n} \frac{1}{\mu_k(\Omega)} \ge \frac{n-1}{\mu_1(B^{m/2})}.$$

Main observation : no mass or hemisphere constraint...

Consequence on μ_1.

Theorem (B., Martinet, Nahon, 2022)

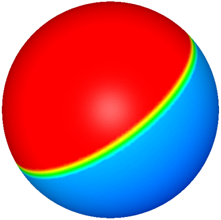
Let $m \leq \frac{|\mathbb{S}^n|}{2}$. If $\Omega \subseteq \mathbb{S}^n \setminus B^m$ (i.e. in a complement of a geodesic ball of mass m) and $|\Omega| = m$, then

$$\mu_1(\Omega) \leq \mu_1(B^m).$$

And also true for densities.

Can one remove the inclusion condition ?

Simplified (better) density



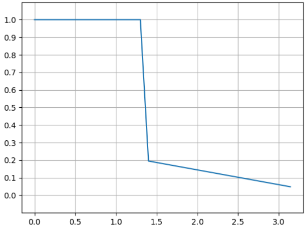


Figure: Approximation of the first eight optimal densities, D. Bucur, E. Martinet, M. Nahon, *Sharp inequalities for Neumann eigenvalues on the sphere*, Arxiv arXiv:2208.11413v1, 2022.

What about $m > \frac{|\mathbb{S}^n|}{2}$?

The maximality of the spherical shell is false:

- Numerical computations by Martinet.
- Analytical proof by B., Laugesen, Martinet, Nahon (ongoing work).

Thank you for your attention!