Level Sets of Eigenfunctions and Harmonic Functions

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Neumann Eigenfunctions

Hot Spots Conjecture (Rauch 1974)
If $\Omega \subset \mathbb{R}^n$ is convex, and ψ_1 is the first non-trivial Neumann eigenfunction, then $\nabla \psi_1(x) \neq 0$ for all $x \in \Omega$.

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How do level sets of (stable) solutions to $\Delta u = f(u)$ fit together?

Dirichlet Eigenfunctions

Rectangle: $\Psi(x) = \prod_{j=1}^{n} \sin(k_j x_j), \quad \Omega = [0, \pi/k_1] \times [0, \pi/k_2] \times \cdots$

$$\frac{\nabla^2 \Psi(x)}{\Psi(x)} \equiv \begin{pmatrix} -k_1^2 & 0 & \cdots & 0 \\ 0 & -k_2^2 & \cdots & 0 \\ \vdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & -k_n^2 \end{pmatrix}$$

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Hessian Conjecture (1994)
If $\Omega \subset \mathbb{R}^n$ is convex, then the first Dirichlet eigenfunction ψ_0 satisfies

$$\frac{\nabla^2 \psi_0(x)}{\psi_0(x)} \approx \text{constant}$$

for $\psi_0(x) > \frac{1}{2} \max \psi_0$.

Polya-Szegö Capacitary Conjecture (1950)
For $n \geq 3$, the extremal for convex $\Omega \subset \mathbb{R}^n$ of

$$\inf_{\Omega} \frac{\mathrm{cap}(\Omega)^{1/(n-2)}}{\sigma(\partial\Omega)^{1/(n-1)}} \quad \text{is a flat disk.}$$

Recall, the equilibrium potential u for Ω in $\mathbb{R}^n$ is the harmonic function in $\mathbb{R}^n \setminus \Omega$ satisfying

$$u = 1 \quad \text{on } \partial\Omega; \quad u(x) \to 0 \quad \text{as} \quad x \to \infty;$$

$$u(x) = \mathrm{cap}(\Omega)\, c_n |x|^{2-n} + O(|x|^{1-n}) \quad \text{as} \quad x \to \infty.$$

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The common feature of these three conjectures is that they involve convex level sets of equations of the form $\Delta u = f(u)$. So far, a rather superficial connection.

A variational principle for level sets

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$$dx/d\tau = (1/|\nabla\psi|)\mathbf{N}, \quad M_\tau := \{\psi(x) = \tau\}$$

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Proof: $$\frac{d}{dt}\int_{M_\tau + tv\mathbf{N}} |\nabla\psi|, d\sigma\Big|_{t=0} = -\lambda\tau\int_{M_\tau} v\, d\sigma$$

Moreover, max $\int v\, d\sigma$ constrained by $\int v^2 |\nabla\psi|\, d\sigma = 1$ is realized by

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Mean curvature flow $dx/dt = H\mathbf{N}$ decreases area fastest, so

$$1/|\nabla\psi| \quad \longleftrightarrow \quad H$$

Analogy with mean curvature and Ricci flow

High hopes for the analogy with MCF and Ricci flow:

- Level sets resemble shrinkers.
- Hamilton's Harnack inequalites, $H(x_1, t_1) \approx H(x_2, t_2)$, might prevent $|\nabla \psi_1|$ from vanishing.
- Hamilton's curvature pinching estimates could stabilize the Hessian of ψ_0.
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But what does this have to do with Polya-Szegö?

Hamilton's approach is to study $\partial_t g_{ij}$, $\partial_t h_{ij}$, Δh_{ij} and other tensors. Key ingredient: his differential inequalities are sharp, and their form is dictated solitons, for which they are equalities.

Our solitons will be confocal ellipsoids, the family of examples motivating the Polya-Szegö Conjecture.

Confocal Ellipsoids

Confocal ellipsoids are defined for $-a_1^2 \le -a_2^2 \le \cdots \le -a_n^2 < s$ by

$$\partial\Omega_s := \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} \frac{x_j^2}{a_j^2 + s} = 1 \right\}.$$

The equilibrium potential U with these levels sets is

$$U(x) = \Phi(s), \qquad \Phi'(s) = \frac{1}{\sqrt{(s + a_1^2)(s + a_2^2)\cdots(s + a_n^2)}}.$$

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FIRST STEP: For $\Delta u = 0$,

Find geometric invariants of $M_\tau = \{x : u(x) = \tau\}$ that are zero for the solitons U.

Classical geometry of hypersurfaces

In $\mathbb{R}^n$, $\partial_j = (\partial/\partial x_j$ is covariant constant:

$$\nabla_{\partial_i}(a\partial_j) = (\partial_i a)\partial_j$$

The Levi-Civita connection D^{LC} and 2nd fundamental form A on M are defined by writing $\nabla_X Y$ in tangential and normal components:

$$\nabla_X Y = D^{LC}_X Y + A(X,Y)\mathbf{N} \quad (X,\ Y \in TM)$$

Classical Geometry and Affine Geometry

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Affine geometry of hypersurfaces

The affine normal μ is a transverse SL_n invariant vector.

$$\nabla_X Y = D_X^{\mu} Y + A^{\mu}(X,Y)\mu \quad (X,\ Y \in TM)$$

We introduce the harmonic normal defined by

$$\nu = |\nabla u|\mathbf{N} - \nabla^A |\nabla u|$$

with $\nabla^A f$ defined by $A(\nabla^A f, Y) = Yf$.
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We define a "harmonic" connection D and quadratic form L on the level surface $M_\tau = \{u = \tau\}$ by

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Harmonic Level Set Geometry

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(The flux $|\nabla u|\, d\sigma$ is covariant constant with respect to the connection D on the surface M_τ.)

Higher Harmonic Invariants

Following the analogy with affine geometry, we differentiate the basic invariants:

$$\text{Cubic Form:} \quad F(X,Y,Z) = (D_X L)(Y,Z).$$

$$\text{Shape Operator:} \quad S(X) = -\nabla_X \nu \quad (S : TM_\tau \to TM_\tau).$$

The shape operator is symmetric with respect to L, so we can define another natural quadratic form

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The properties that indicate that this is the natural choice are

1. $F \equiv 0$ if an only if M is a quadric.
2. In the case $u = U$, $\tilde{L}$ is the metric on the unit sphere for all τ.

What's Next?

Since $\partial_\tau \tilde{L} = 0$ for confocal ellipsoids, we also have, for example,

$$\partial_\tau^2 \widetilde{\mathrm{Riem}} = 0,$$

with $\widetilde{\mathrm{Riem}}$ the Riemann curvature tensor of $\tilde{L}$.

Hamilton's mean curvature and Ricci flow differential inequalities are based on the heat operator, $\partial_t - \Delta_t$, for the intrinsic Laplacian Δ_t on the evolving manifold M_t. We expect ours to be based on the ordinary Laplace operator Δ. For any harmonic function u with $\nabla u \neq 0$, the Laplace operator can be written using the coordinate $u = \tau$ and the intrinsic Laplacian Δ_τ on $\{u = \tau\}$ as

$$\Delta = |\nabla u|^2 \partial_\tau^2 + \Delta_\tau.$$