Concavity properties of solutions to Robin problems

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Outline

- *Motivation: the fundamental gap of the Laplacian*
- *The importance of being log-concave*
- *Is the Robin ground state log-concave?*
- *Concavity properties of the Robin torsion function*
- *Open problems*

Laplacian: eigenvalues and eigenfunctions

$\Omega \subset \mathbb{R}^N$ open bounded connected domain, with smooth boundary. For the Laplacian (with Dirichlet or Robin Boundary Conditions) we know that the spectrum is purely discrete, with increasing sequence of eigenvalues

$$0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

and corresponding eigenfunctions (ϕ_j) solutions of

$$\begin{cases} -\Delta \phi_j = \lambda_j \phi_j & \text{in } \Omega, \\ \text{Boundary Conditions} & \text{on } \partial\Omega. \end{cases}$$

Boundary Conditions:

- Dirichlet: $\phi_j = 0$ on $\partial\Omega$;
- Robin: $\dfrac{\partial \phi_j}{\partial \nu} + \beta\, \phi_j = 0$ on $\partial\Omega$, with $\beta \geq 0$ a given constant.

The fundamental gap

- The difference $\lambda_2 - \lambda_1$ is the fundamental gap.

- Can we provide an estimate from below of the fundamental gap? For example, in the study of the asymptotics of the solution $u(t)$ of the associated parabolic problem $\partial_t u - \Delta u = 0$ as $t \to +\infty$:

$$u(t) = \sum_{j=1}^{\infty} c_j \, e^{-\lambda_j t} \phi_j \quad \Longrightarrow \quad e^{\lambda_1 t} u(t) = c_1 \, \phi_1 + \sum_{j=2}^{\infty} c_j \, e^{-(\lambda_j - \lambda_1) t} \phi_j$$

- if λ_1 is simple (which is the case for the Laplacian with the given boundary conditions), then $e^{\lambda_1 t} u(t) \to c_1 \, \phi_1$ as $t \to +\infty$;
- the fundamental gap allows to estimate the rate of this convergence, since the remaining part goes to zero at the rate $e^{-(\lambda_2 - \lambda_1) t}$.

But also: statistical mechanics (condensation in free-Boson gas), QFT, control of error in numerical methods, refined Poincaré inequality, *a-priori* estimates, ...

Estimate of the Fundamental Gap

Let $\Omega \subset \mathbb{R}^N$ be a convex domain.

- Neumann Boundary Conditions: $\lambda_1 = 0$, the fundamental gap is the first nontrivial eigenvalue

[Payne–Weinberger 1960, Zhong–Yang 1984] Optimal estimate

$$\lambda_2 \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2}$$

- Dirichlet Boundary Conditions

[Andrews–Clutterbuck 2011] Optimal estimate

$$\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{\mathrm{diam}(\Omega)^2}$$

(also for Schrödinger operators $-\Delta + V(x)$, V convex potential).

They improved the already known non-optimal estimate $\lambda_2 - \lambda_1 \geq \pi^2 / \mathrm{diam}(\Omega)^2$ by Yu–Zhong 1986 and a previous estimate by Singer–Wong–Yau–Yau 1985.

Robin boundary conditions

Ω convex, $\beta > 0$:

$$\begin{cases} -\Delta\phi = \lambda\,\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\nu} + \beta\,\phi = 0 & \text{on } \partial\Omega. \end{cases}$$

The first eigenfunction does not change sign, hence we assume $\phi_1 > 0$.
If $\log\phi_1$ is concave, then the fundamental gap estimate

$$\lambda_2 - \lambda_1 \geq \frac{\pi^2}{\mathrm{diam}(\Omega)^2}$$

holds.

- The estimate holds also for Schrödinger operators $-\Delta + V(x)$.
- Improved log-concavity would give a better estimate.

Sketch of the proof

Define $v(x,t) := \dfrac{e^{-\lambda_2 t}\phi_2(x)}{e^{-\lambda_1 t}\phi_1(x)}$. Then

$$\begin{cases} \partial_t v = \Delta v + \underbrace{2\nabla(\log\phi_1)}_{X(x)} \cdot \nabla v & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,\infty). \end{cases}$$

$\log\phi_1$ concave $\Longrightarrow (X(x)-X(y))\cdot(y-x)\leq 0.$

Hence, by [Andrews-Clutterbuck 2011], a modulus of continuity of $v(\cdot,t)$ is given by $\varphi(\cdot,t)$, where $\varphi(s,t)$ satisfies

- $\partial_t\varphi \geq \varphi''$ in $[0,d/2]\times(0,\infty)$, $d := \mathrm{diam}(\Omega)$;
- $\varphi' > 0$ in $(0,d/2)\times(0,\infty)$;
- $\varphi(0,t)\geq 0$ for every $t\geq 0$;
- $\varphi(s,0)$ is a modulus of continuity for $v(\cdot,0)$.

A direct computation shows that, for every $C > 0$, the function

$$\varphi(s,t) := C\, e^{-(\pi^2/d^2)t} \sin\left(\frac{\pi s}{d}\right), \quad s \in \left[0, \frac{d}{2}\right],\ t \geq 0,$$

satisfies the first three requirements, and, if C is large enough, then $\varphi(s,0) = C \sin(\pi s/d)$ is a modulus of continuity for

$$v(x,0) = \phi_2(x)/\phi_1(x), \qquad x \in \Omega.$$

Hence,

$$e^{-(\lambda_2-\lambda_1)t} \operatorname{osc}_{\overline{\Omega}} \left(\frac{\phi_2}{\phi_1}\right) \leq C\, e^{-(\pi^2/d^2)t}, \qquad \forall t \geq 0,$$

which implies $\lambda_2 - \lambda_1 \geq \pi^2/d^2$.
(This estimate is optimal in the case $\beta = 0$, i.e., for Neumann B.C.)

The importance of being log-concave

In conclusion, to apply the scheme above it is enough to prove the log-concavity of ϕ_1.

Dirichlet Boundary Conditions

For the Laplacian with Dirichlet B.C. it is known that $\log \phi_1$ is concave [Brascamp–Lieb 1976].

Andrews–Clutterbuck 2011 used an improved log-concavity estimate

$$(X(x,t) - X(y,t)) \cdot \frac{y-x}{|y-x|} \leq -2\omega\left(\frac{|y-x|}{2}, t\right)$$

to obtain the optimal estimate.

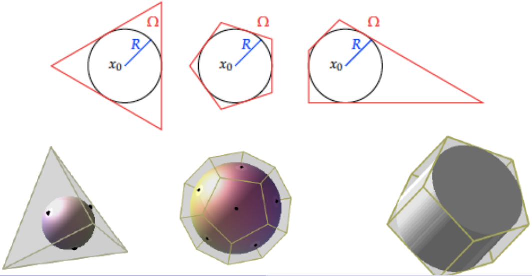
...but unfortunately...

Andrews–Clutterbuck–Hauer 2020: for small values of β, the first Robin eigenfunction need not be log-concave!

The result involves the geometry of the domain!

Let Ω be a convex polyhedron in $\mathbb{R}^N$.

- Ω is a circumsolid if there is a ball which is tangent to all its faces.
- Ω is a product of circumsolids if it is the cartesian product of circumsolids contained into orthogonal subspaces of $\mathbb{R}^N$.
  Call $\mathscr{P}$ this class of polyhedra.



[Andrews-Clutterbuck-Hauer '20] Let Ω be a convex polyhedral domain in $\mathbb{R}^N$, $N \geq 2$.

If $\Omega \notin \mathscr{P}$, i.e. it is not a product of circumsolids, then for sufficiently small β

the Robin ground state is not log-concave.

Proof: perturbation of the Neumann problem ($\beta = 0$);
$\log \phi_1 = \beta\, v + o(\beta)$, with v concave if and only if $\Omega \in \mathscr{P}$.
v is the solution of

$$\begin{cases} -\Delta v = \frac{|\partial\Omega|}{|\Omega|}\,, & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = -1\,, & \text{on } \partial\Omega. \end{cases}$$

- Adding more boundary regularity there is no hope to avoid non-concavity
- Strengthened hypotheses yield nonconvex level sets for sufficiently small β

Conjecture for large β

For a given bounded convex domain $\Omega \subset \mathbb{R}^N$:

- is the Robin ground state log-concave for $\beta \geq \beta^*$?
- If affirmative, how β^* depends on N and on the geometry of the domain Ω?

A positive result

[C.G.–Fragalà 2021] Let $\Omega \subset \mathbb{R}^N$ be a uniformly convex open set of class C^m, with $m \geq 4 + \frac{N}{2}$.

There exists a positive threshold β^* such that, for $\beta \geq \beta^*$, the Robin ground state of Ω is strictly log-concave.

Moreover, β^* is uniform for convex domains of class C^m satisfying

$$\operatorname{diam}(\Omega) \leq \overline{d}\,, \qquad \kappa_{min}(\Omega) \geq \overline{\kappa} > 0\,, \qquad \delta_m(\Omega) \leq \overline{\delta}\,.$$

- $d(\Omega) :=$ the diameter of Ω;
- $\kappa_{min}(\Omega) := \min_{x \in \partial\Omega} \min_{i=1,\dots,N-1} \{\kappa_i(x)\}$ (κ_i= principal curvatures);
- $\delta_m(\Omega) := \sum_{|\alpha| \leq m} \max_{x_0 \in \partial\Omega} |\partial^\alpha \varphi_{x_0}(0)|$ (max of "higher order curvatures").

A glance at the proof

Let $v_1 := -\log\phi_1$. Then $D^2 v_1$ is positive definite in Ω (for large β).

The approach via continuity method [Caffarelli-Friedman 1985]:

Let B be a ball, and for $t \in [0,1]$ set

$$\Omega_t = (1-t)B + t\Omega,$$

$$\phi_t = \text{first Robin eigenfunction in } \Omega_t, \qquad v_t = -\log\phi_t$$

Claim: $D^2 v_t$ is positive definite in Ω_t for every $t \in [0,1]$.

• Step 1: For every $t \in [0,1]$, $D^2 v_t$ is positive definite in an interior tubular neighborhood of $\partial\Omega_t$.
We address this point in the next slides!

• Step 2: Assume by contradiction that the Claim is false.
Since $D^2 v_0$ is positive definite (explicit solution on the ball),
$\exists\, s \in (0,1)$: $D^2 v_s$ is positive semidefinite, but *not* positive definite in Ω_s.
Since

$$\nabla \log \phi_s = \frac{\nabla \phi_s}{\phi_s}\,, \qquad D^2 \log \phi_s = \frac{1}{\phi_s} D^2 \phi_s - \frac{1}{\phi_s^2} \nabla \phi_s \otimes \nabla \phi_s\,,$$

$$\Longrightarrow \quad \Delta v_s = -\operatorname{trace}(D^2 \log \phi_s) = \lambda_1(\Omega_s) + |\nabla v_s|^2\,.$$

By the *Constant Rank Theorem* of Korevaar-Lewis 1987,
$D^2 v_s$ has *constant rank* in Ω_s $\Rightarrow$ contradiction by Step 1.

Ingredients

Let us comment on Step 1: For every $t \in [0,1]$, $D^2 v_t$ is positive definite in an interior tubular neighborhood of $\partial\Omega_t$.

- Global regularity and asymptotic estimates as $\beta \to +\infty$

[C.G.–Fragalà 2021] For $\Omega \in C^m$, the Robin ground state ϕ_1 is in $H^m(\Omega)$.
If in addition $m \geq 4 + \frac{N}{2}$, then denoting by ϕ_1^D, (λ_k^D) respectively the Dirichlet ground states and the Dirichlet eigenvalues, it holds that

$$\|\phi_1^\beta - \phi_1^D\|_{C^{2,\theta}(\overline{\Omega})} \leq \frac{M}{\sqrt{\beta}}, \qquad |\lambda_k^\beta(\Omega) - \lambda_k^D(\Omega)| \leq \frac{\Lambda_k}{\sqrt{\beta}}.$$

We need M and Λ_k depend only on the geometric bounds on diameter, minimal curvature and higher curvatures.

[C.G.–Fragalà 2021]
$\Omega \subset \mathbb{R}^N$ convex satisfying a uniform interior sphere condition.
Then ϕ_1^D satisfies the following boundary gradient estimate:

$$q(\Omega) := \min_{\partial\Omega} |\nabla \phi_1^D| \geq \frac{C_2}{\rho} \left(\max_{\overline{\Omega}} \phi_1^D\right) C_1^{-(\sqrt{\lambda_1^D}/2 + 2\sqrt{N}/\sigma)\, \mathrm{diam}(\Omega)},$$

where C_1, C_2 are dimensional constants, while

- $r = r(\Omega)$ is the inradius of Ω
- $\sigma := \min\left\{\frac{\rho}{2}, C_0\, r\right\}$, with $\rho =$ radius of interior sphere, and $C_0 = \sqrt{\frac{0.833}{\lambda_1^D(B_1)}}$

Proof [thanks to David Jerison!] Use a Harnack chain of balls, and an estimate for the location of the hot spot $\overline{x}$:
$\mathrm{dist}(\overline{x}, \partial\Omega) \geq C_0\, r(\Omega)$ [Biswas-Lörinczi '19] □

- Tracking the concavity threshold

By exploiting the previous convergence result, we obtain

$$\langle D^2 v_t \eta, \eta \rangle > 0 \qquad \forall x \in \mathscr{U}(\partial\Omega_t), \ \eta \in S^{N-1}, \ \beta \geq \beta^*$$

with

$$\beta^* = \beta^*(\mathrm{diam}(\Omega_t)_\uparrow, \delta_m(\Omega_t)_\uparrow, \kappa_{min}(\Omega_t)_\downarrow, \lambda_1^D(\Omega_t)_\uparrow, q(\Omega_t)_\downarrow)$$

where $q(\Omega_t) = \min_{\partial\Omega_t} |\nabla \phi_t^D|$.

Crucial question: is the threshold β^* **uniform** in the family of sets

$$\Omega_t = [(1-t)B + t\Omega] \qquad \text{for } t \in [0,1] \quad ?$$

Yes! The sets Ω_t are of class C^m [Ghomi '12], and

- **red quantities are bounded (with $\uparrow$ from above, with $\downarrow$ from below)**
- **blue quantities are controlled by the red ones.**

The torsion problem

[C.G.–Fragalà 2021]

Let $\Omega \subset \mathbb{R}^N$ be a uniformly convex open set of class C^m, with $m \geq 4 + \frac{N}{2}$.

There exists a positive threshold β^{**} such that, for $\beta \geq \beta^{**}$, the Robin torsion function of Ω, i.e. the unique solution to

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega\,, \\ \frac{\partial u}{\partial \nu} + \beta\, u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

is strictly $(1/2)$-concave.

Moreover, β^{**} is uniform for convex domains of class C^m satisfying

$$d(\Omega) \leq \overline{d}\,, \qquad \delta_m(\Omega) \leq \overline{\delta}\,, \qquad \kappa_{min}(\Omega) \geq \overline{\kappa}\,.$$

Open problems

- Can the regularity assumptions of our results be removed or weakened?
- What happens in the plane for triangles or circular sectors?
- Is it possible to characterize convex domains on which Robin solutions enjoy concavity properties for all values of the parameter?
- What about Brunn–Minkowski type inequalities for Robin energies?

THANK YOU FOR YOUR ATTENTION!

References

Andrews–Clutterbuch, *Proof of the fundamental gap conjecture*, J. Amer. Math. Soc. 8 (2011)

Andrews–Clutterbuch–Hauer, *Non-concavity of the Robin ground state*, Camb. J. Math. 8 (2020)

Crasta–Fragalà, *Concavity properties of solutions to Robin problems*, Camb. J. Math. 9 (2021)

Other contributions to concavity

Korevaar '83: Concavity Maximum Principle
Caffarelli-Friedman '85: continuity method
Korevaar-Lewis '87: constant rank theorem
Caffarelli-Spruck '82, Kennington '85, Kawohl '86, ...

For fully nonlinear PDEs:
Alvarez-Lasry-Lions '97
Caffarelli-Guan-Ma '07, Bian-Guan '09: constant rank theorem for viscosity solutions