The shape of Riemannian manifolds with large Steklov eigenvalues (with A. Girouard).

"large eigenvalues" $\Rightarrow$ "concentration".

Case of the Laplacian on a R.M. (M^n, g).

I. <u>Theorem</u> (Cheng 75). (M^n, g)

$\mathrm{Ricci}(M,g) \geq -K(n-1)g$, $K \geq 0$

$$\lambda_k(M,g) \leq A_n \cdot K + B(n) \frac{k^2}{\mathrm{diam}^2(M,g)}$$

(M, g_t) $\quad 0 < t < 1$

If $\lambda_k(M, g_t) \to \infty$ $\quad t \to 0$

$\Rightarrow \mathrm{diam}(M, g_t) \to 0$.

II (Gromov Milman 83).

(M^n, g) $\quad 0 < a < 1$

$A \subset M$ $\quad |A| = a|M|$.

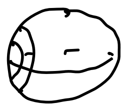
$$N_\varepsilon(A) = \{ x \in M,\ d_M(x, A) \leq \varepsilon \}$$

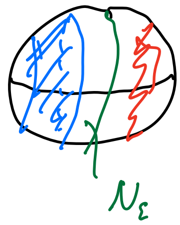
$$|N_\varepsilon(A)| \geq \left(1 - (1-a^2)\exp\left(-\varepsilon\sqrt{\lambda_1}\ \ln(1+a)\right)\right)|M|$$

$a = 10^{-3}$ $\quad \varepsilon = 10^{-3}$ $\quad \lambda_1 = 10^{100}$

$$|N_\varepsilon(A)| \geq (1 - \exp(-10^{45}))|M|.$$

S^n $\qquad n \to \infty \qquad \lambda_1(S^n) = n$





Today: What can be said for Steklov?

(M^{n+1}, g_k) $\quad 0 < k < 1 \quad \partial M = \Sigma_1 \sqcup \dots \cup \Sigma_b$

$$\mathrm{diam}_M(\Sigma_j, g_k) \le \frac{1}{\sigma_k(M, g_k)} \frac{K(n)\,|M|\,k^{n+1}}{(\mathrm{inj}\,\Sigma_j)^n}.$$

If $\sigma_k(M, g_k) \to \infty$ as $k \to \infty$

and $|(M, g_k)| \le V_0$ $\qquad \mathrm{Inj}\,\Sigma_j \ge i_0$

$$\mathrm{diam}_M(\Sigma_j, g_k) \to 0$$

$$\mathrm{diam}^2_M \Sigma \le \frac{1}{\sigma_1(M, g_k)} \frac{K(n)\,|(M, g_k)|}{\mathrm{inj}(M, g_k)^n}$$

extrinsic.

Thm. (M^n, g) $\quad 0 < a, b < 1$

$A, B \subset \Sigma \qquad |A| = a|\Sigma|$

$\qquad |B| = b|\Sigma|$

$$\rho = \inf\{d_M(x,y);\ x \in A,\ y \in B\}$$

$$\rho \leq \frac{|M|}{\sigma_1(M)}\left(\frac{1}{a}+\frac{1}{b}\right)\frac{1}{|\Sigma|}.$$

$A \subset M \quad |A| = a|\Sigma|,$

$A_\varepsilon = \{ x \in \Sigma : d_M(x, A) \leq \varepsilon \}$

$B_\varepsilon = \Sigma \setminus A_\varepsilon \qquad \rho(A, B_\varepsilon) = \varepsilon$

$$|A_\varepsilon| \leq |\Sigma|\left(1 - \frac{|M|}{a\sigma_1(M)\varepsilon^2|\Sigma| - |M|}\right)$$

$|M| \leq V_0 \quad \sigma_1 \nearrow \infty$

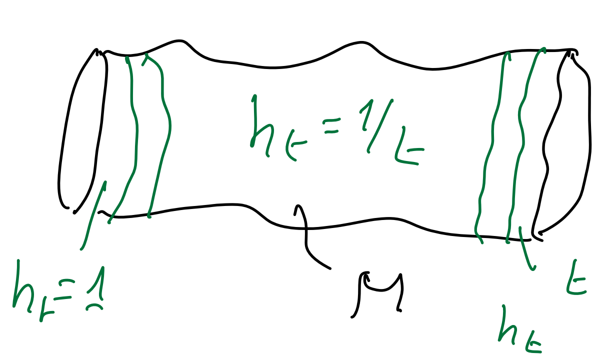
Question Do we need to control $|M|$?

Yes: $(M^{n+1 \geq 3}, g) \quad \Sigma$

$g_t = h_t^2 g \qquad h_t = 1$ on $\partial\Sigma$.

$|(M, g_t)| \nearrow \infty \quad \sigma_1(M, g_t) \nearrow \infty$

$t \to 0$



This leads to the question:

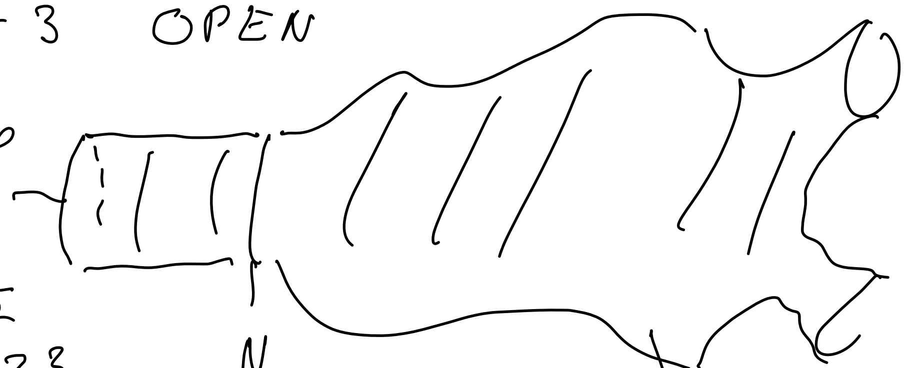
Given $M^{n+1 \geq 3}$ is it possible to construct a family g_t $0 < t < 1$ of metric on M with large eigenvalue σ_1?

$|\Sigma, g_t| = 1 \quad |(M, g_t)| \leq V_0$?

$n+1 \geq 4$. YES.

$1+n = 3$ OPEN

$\lambda_1 g^\Sigma$



Σ $\quad N \quad \Sigma_g^2$

$\underline{\dim \Sigma \geq 3}$