Comparison between Neumann and Steklov eigenvalues

Antoine Henrot joint work with Marco Michetti (Nancy-University of Lorraine) and partly Enea Parini (Marseille)

Shape Optimisation and Geometric Spectral Theory, Edinburgh

22 September 2022

Notation

Let $\Omega \subset \mathbb{R}^2$ be a plane bounded domain with Lipschitz boundary, we consider Neumann eigenvalues

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

that we denote by

$$0 = \mu_0(\Omega) \leq \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \to +\infty.$$

and the Steklov eigenvalues

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \partial_\nu u = \sigma u & \text{on } \partial\Omega, \end{cases}$$

that we denote by

$$0 = \sigma_0(\Omega) \leq \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots \to +\infty.$$

The initial question

In Montréal (2018), with A. Girouard and J. Lagacé, we were led to the following question:

Question

Is it true that $|\Omega|\mu_1(\Omega) \geq P(\Omega)\sigma_1(\Omega)$?

$|\Omega|\mu_1(\Omega)$ (with the area) and $P(\Omega)\sigma_1(\Omega)$ (with the perimeter) are the natural normalization (scale invariant) for these two eigenvalues.

The initial question

In Montréal (2018), with A. Girouard and J. Lagacé, we were led to the following question:

Question

Is it true that $|\Omega|\mu_1(\Omega) \geq P(\Omega)\sigma_1(\Omega)$?

$|\Omega|\mu_1(\Omega)$ (with the area) and $P(\Omega)\sigma_1(\Omega)$ (with the perimeter) are the natural normalization (scale invariant) for these two eigenvalues.
More generally, we can ask

Question

Are there inequalities that relate $|\Omega|\mu_1(\Omega)$ and $P(\Omega)\sigma_1(\Omega)$??

A first answer

Despite many favorable examples (discs, rectangles,...), the inequality

$$|\Omega|\mu_1(\Omega) \geq P(\Omega)\sigma_1(\Omega)$$

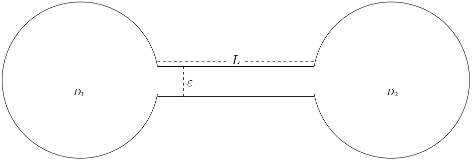
is false.

A first answer

Despite many favorable examples (discs, rectangles,...), the inequality

$$|\Omega|\mu_1(\Omega) \geq P(\Omega)\sigma_1(\Omega)$$

is false. A counter-example is given by



for ϵ small enough and L large enough.

D. Bucur, A. Henrot, M. Michetti: *Asymptotic behaviour of the Steklov problem on dumbbell domains*, Communications in PDE (2021), vol. 46, Issue 2

A shape functional

Let us define the following functional:

$$F(\Omega) = \frac{\mu_1(\Omega)|\Omega|}{\sigma_1(\Omega)P(\Omega)}.$$

We are interested in minimizing or maximizing the functional $F(\Omega)$, where $\Omega \in \mathcal{S}$ and $\mathcal{S}$ is a class of admissible subsets of $\mathbb{R}^2$.

The general case

If we don't put any restriction on the class $\mathcal{S}$ the problem is ill posed:

$$\inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = 0,$$

$$\sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = +\infty.$$

The general case

If we don't put any restriction on the class $\mathcal{S}$ the problem is ill posed:

$$\inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = 0,$$

$$\sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = +\infty.$$

These results follow from the following theorem:

Theorem (D. Bucur, M. Nahon 2020)

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^2$ be two smooth, conformal open sets. Then there exists a sequence of smooth open sets $(\Omega_{1,\epsilon})_{\epsilon>0}$ with uniformly bounded perimeter satisfying a uniform ϵ-cone condition such that

$$\lim_{\epsilon\to 0} d_H(\partial\Omega_{1,\epsilon}, \partial\Omega_1) = 0 \text{ and}$$

$$\lim_{\epsilon\to 0} P(\Omega_{1,\epsilon})\sigma_k(\Omega_{1,\epsilon}) = P(\Omega_2)\sigma_k(\Omega_2)$$

The convex case

We are now interested in the study of the extremum problems in the class of convex plane domains:

$$\inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex}\},$$

$$\sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex }\}.$$

In the convex setting, since every quantity is continuous for the Hausdorff convergence, the alternative is the following:

The convex case

We are now interested in the study of the extremum problems in the class of convex plane domains:

$$\inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex}\},$$

$$\sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ convex }\}.$$

In the convex setting, since every quantity is continuous for the Hausdorff convergence, the alternative is the following:

- There exists an open convex set that minimize (maximize) the functional $F(\Omega)$
- The minimizing (maximizing) sequence converges to a segment (collapsing sequences).

Behavior for collapsing sequences

Let Ω_n be a sequence converging to a segment, we can prove

Theorem

$$\frac{\pi^2}{12} \leq \liminf F(\Omega_n) \leq \limsup F(\Omega_n) \leq 4.$$

The idea is to study the asymptotic behavior of $\mu_1(\Omega_n)$ and $\sigma_1(\Omega_n)$ and we are led to eigenvalues of two different Sturm-Liouville problems.

Upper and lower bounds for $F(\Omega)$

Theorem

There exists an explicit constant C_1 such that, for every convex open set $\Omega \subset \mathbb{R}^2$, the following inequalities hold

$$0.62 \leq \frac{\pi^2}{6\sqrt[3]{18}} \leq F(\Omega) := \frac{\mu_1(\Omega)|\Omega|}{\sigma_1(\Omega)P(\Omega)} \leq C_1 \leq 9.2.$$

Upper and lower bounds for $F(\Omega)$

Theorem

There exists an explicit constant C_1 such that, for every convex open set $\Omega \subset \mathbb{R}^2$, the following inequalities hold

$$0.62 \leq \frac{\pi^2}{6\sqrt[3]{18}} \leq F(\Omega) := \frac{\mu_1(\Omega)|\Omega|}{\sigma_1(\Omega)P(\Omega)} \leq C_1 \leq 9.2.$$

Proof of the lower bound: We define the following class of domains ($2 \leq \delta \leq \pi$):

$$\mathcal{C}_\delta := \{\Omega \subset \mathbb{R}^2 : \Omega \text{ is convex and } P(\Omega) \leq \delta D(\Omega)\}.$$

We separate the set of convex plane domains in two sets $\mathcal{C}_\delta$ and $\mathcal{C}_\delta^c$. We worked separately in the two classes and then we found the optimal δ.

Proof of the upper bound

Let us introduce, the inradius $r(\Omega)$, the minimal width $w(\Omega)$ and the diameter $D(\Omega)$ of the set Ω. For $\sigma_1(\Omega)$ we use a lower bound found in a classical paper of Kuttler-Sigilitto:

$$\sigma_1(\Omega) \geq \frac{\mu_1(\Omega) r(\Omega)}{2(1 + \sqrt{\mu_1(\Omega)} D(\Omega))}$$

We combine it with the following inequality (recent joint work with A. Lemenant and I. Lucardesi) $\mu_1(\Omega) \leq \dfrac{\pi^2 w^2(\Omega)}{|\Omega|^2}$ and the geometric inequality $|\Omega| \leq r(\Omega) P(\Omega)$ to end up with

Proof of the upper bound

Let us introduce, the inradius $r(\Omega)$, the minimal width $w(\Omega)$ and the diameter $D(\Omega)$ of the set Ω. For $\sigma_1(\Omega)$ we use a lower bound found in a classical paper of Kuttler-Sigilitto:

$$\sigma_1(\Omega) \geq \frac{\mu_1(\Omega) r(\Omega)}{2(1 + \sqrt{\mu_1(\Omega)} D(\Omega))}$$

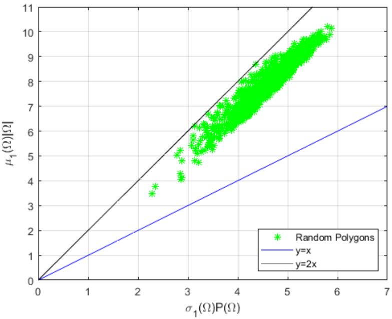
We combine it with the following inequality (recent joint work with A. Lemenant and I. Lucardesi) $\mu_1(\Omega) \leq \dfrac{\pi^2 w^2(\Omega)}{|\Omega|^2}$ and the geometric inequality $|\Omega| \leq r(\Omega) P(\Omega)$ to end up with

$$F(\Omega) \leq 2 \left(1 + \frac{\pi w(\Omega) D(\Omega)}{r(\Omega) P(\Omega)} \right).$$

To conclude, we use the Blaschke–Santaló diagram found by M. A. Hernández Cifre involving the three quantities w, r, D.

Conjecture

We plot the diagram for **random convex** polygons



Conjecture

For every **convex** and open set $\Omega \subset \mathbb{R}^2$ the following bounds hold

$$1 \leq F(\Omega) \leq 2.$$

The inequalities of the conjecture would be sharp.

- If Ω_ϵ is a sequence of isoscele triangles collapsing to a segment, then $F(\Omega_\epsilon) \to 2$.

The inequalities of the conjecture would be sharp.

- If Ω_ϵ is a sequence of isoscele triangles collapsing to a segment, then $F(\Omega_\epsilon) \to 2$.
- If Ω_ϵ is a sequence of rectangles collapsing to a segment, then $F(\Omega_\epsilon) \to 1$.

Another approach: a L^2-hot spot property

Let u be an eigenfunction associated to $\mu_1(\Omega)$, we introduce

Definition

Ω satisfies the L^2-hot spot property if

$$\frac{1}{P(\Omega)} \int_{\partial\Omega)} u^2(\sigma) d\sigma \geq \frac{1}{|\Omega|} \int_{\Omega)} u^2(X) dX$$

Another approach: a L^2-hot spot property

Let u be an eigenfunction associated to $\mu_1(\Omega)$, we introduce

Definition

Ω satisfies the L^2-hot spot property if

$$\frac{1}{P(\Omega)}\int_{\partial\Omega)} u^2(\sigma)d\sigma \geq \frac{1}{|\Omega|}\int_{\Omega)} u^2(X)dX$$

Assume that Ω satisfies the L^2-hot spot property and that u also satisfies $\int_{\partial\Omega} u(\sigma)d\sigma = 0$ (this is the case for example if Ω has two orthogonal axis of symmetry) then

$$\sigma_1(\Omega) \leq \frac{\int_\Omega |\nabla u|^2}{\int_{\partial\Omega} u^2} \leq \frac{|\Omega|}{P(\Omega)}\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} = \frac{|\Omega|}{P(\Omega)}\mu_1(\Omega).$$

The L^2-hot spot property

Conjecture

Any convex domain Ω satisfies the L^2-hot spot property.

The L^2-hot spot property

Conjecture

Any convex domain Ω satisfies the L^2-hot spot property.

We have been able to prove:

Theorem (A.H., M. Michetti, E. Parini)

Let Ω be a convex domain circumscribed to a disk, then Ω satisfies the L^2-hot spot property.

For the proof, we use the torsion function with Neumann boundary condition of such domains which turns out to be proportional to $x^2 + y^2$.

Thank you for your attention!