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Metric upper bounds for Steklov and Laplace eigenvalues

joint work with

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Steklov eigenvalue problem

Let M be a compact Riemannian manifold with smooth $\Sigma := \partial M$.

$$\begin{cases} \Delta u = 0 & \text{in } M, \\ \partial_\nu u = \sigma u & \text{on } \Sigma. \end{cases}$$

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$$0 = \sigma_0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \nearrow +\infty$$

$$\sigma_k = \min_{V \in E(k+1)} \max_{u \in V \setminus \{0\}} \frac{\int_M |\nabla u|^2 \, dV}{\int_\Sigma u^2 \, dA}$$

where $E(k+1) = \{V \subset H^1(M) \, : \, \dim(V) = k+1\}$

Geometric upper bounds for Steklov eigenvalues

Fraser–Schoen 2011

$$\sigma_1(M)|\Sigma||M|^{\frac{1-n}{1+n}} \leq (n+1)V_{rc}(M)^{2/n+1}$$

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The isoperimetric inequality leads to

$$\sigma_k(M)|\Sigma|^{1/n} \leq K(n)k^{2/(n+1)}$$

Here $\Sigma = \partial M$ is the boundary of a domain: an hypersurface

Colbois–Girouard–Gittins 2019

Let $\Sigma \subset \mathbb{R}^d$ be a n-dimensional closed submanifold.

There is $A_\Sigma > 0$ such that each submanifold $M \subset \mathbb{R}^d$ with $\Sigma = \partial M$ satisfies

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What is the nature of the constant A_Σ?

Theorem A (Colbois–Girouard 2021)
Let M be a smooth connected compact Riemannian manifold of dimension $n+1$ with boundary Σ.

The following holds for each $k \geq 1$,

$$\boxed{\sigma_k(M) \leq \left(\frac{b^2 N^3 \Gamma \Lambda^2}{|\Sigma|^{1+\frac{2}{n}}}\right)|M| k^{2/n}}$$

where

$b =$ number of connected components of the boundary

$N =$ packing constant of Σ for d_M

$\Gamma =$ growth constant of Σ

$\Lambda =$ distortion of Σ in M

a) **The packing constant $N \in \mathbb{N}$ for (Σ, d_M):**

For each $r > 0$ and $x \in \Sigma$, the extrinsic ball $B^M(x,r) \cap \Sigma$ can be covered by N extrinsic balls of radius $r/2$ centred at points $x_1, \cdots, x_N \in \Sigma$:

$$B^M(x,r) \cap \Sigma \subset \bigcup_{i=1}^{N} B^M(x_i, r/2);$$

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b) **The growth constant Γ:**

For each $x \in \Sigma$ and $r > 0$, $|B^\Sigma(x,r)|_\Sigma \leq \Gamma r^n$.

c) **The distortion of the boundary** Λ:

For $x, y \in \Sigma$, we have $d_M(x,y) \leq d_\Sigma(x,y)$.

Let $\Sigma_1, \cdots, \Sigma_b$ be the connected components of the boundary.

The distortion of Σ_j in M is the number $\Lambda_j \in [1, \infty)$ defined by

$$\Lambda_j := \inf\{c \geq 1 \ : \ d_\Sigma(x,y) \leq c d_M(x,y) \quad \forall x, y \in \Sigma_j\}. \qquad (1)$$

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The distortion of Σ in M is

$$\Lambda := \max\{\Lambda_1, \cdots, \Lambda_b\}.$$

The distortion is a measure of how much the geodesic distance d_Σ differs from the induced distance $d_M\big|_\Sigma$.

Theorem A

Let M be a smooth connected compact Riemannian manifold of dimension $n+1$ with boundary Σ.

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Theorem B (Colbois–Girouard 2021)

Let M be a smooth connected compact Riemannian manifold of dimension $n+1$ with boundary $\Sigma = \cup_{j=1}^{b} \Sigma_j$.

Then, for each $j = 1, \cdots, b$ and each $k \geq 1$,

$$\boxed{\sigma_k(M) \leq K(n) \frac{|M|}{\mathrm{Diam}_M(\Sigma_j)^2 \mathrm{inj}(\Sigma_j)^n} k^{n+1}}$$

where
- $\mathrm{Diam}_M(\Sigma_j)$ is the extrinsic diameter of Σ_j
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Moreover
$$\sigma_1(M) \leq K(n) \frac{|M|}{\mathrm{Diam}_M(\Sigma)^2 \mathrm{inj}(\Sigma)^n}$$

Applications to eigenvalues of the Laplace operator

Let Σ **be a closed Riemannian manifold**.

Let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots$ be the nonzero **eigenvalues of the Laplacian** on Σ.

Let $M = [-L, L] \times \Sigma$.

The **Steklov eigenvalues of** M are

$$0, \; 1/L, \; \sqrt{\lambda_k}\tanh(\sqrt{\lambda_k}L), \; \sqrt{\lambda_k}\coth(\sqrt{\lambda_k}L)$$

For $L > 0$ small enough, $\sigma_k = \sqrt{\lambda_k}\tanh(\sqrt{\lambda_k}L)$.

Notice that $b = 2, \quad \Lambda = 1, \quad N = N_\Sigma \quad$ and $\quad |M| = L|\Sigma|$.

Theorem A $\Longrightarrow$

$$\sqrt{\lambda_k}\,\tanh(\sqrt{\lambda_k}L) \le {N_\Sigma}^3\Gamma\frac{L|\Sigma|}{|\Sigma|^{\frac{n+2}{n}}}k^{2/n}.$$

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Divide by L:

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Take $L \to 0$: $\boxed{\lambda_k(\Sigma)|\Sigma|^{\frac{2}{n}} \le {N_\Sigma}^3\Gamma k^{2/n}}$

This is in the spirit of Grigor'yan–Netrusov–Yau and Korevaar.

Theorem B $\Longrightarrow$

$$\sqrt{\lambda_k}\,\mathrm{tanh}(\sqrt{\lambda_k}L) \leq K(n)\frac{L|\Sigma|}{\mathrm{Diam}(\Sigma)^2}\left(\frac{1}{\mathrm{inj}(\Sigma)^n}\right)k^{n+1},$$

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Divide by L and take $L \to 0$ and obtain. . .

Theorem C (Colbois–Girouard 2021)

$$\boxed{\lambda_k(\Sigma)\mathrm{Diam}(\Sigma)^2 \leq K(n)\frac{|\Sigma|}{\mathrm{inj}(\Sigma)^n}k^{n+1}}$$

This is an improvement of results from Berger, Croke and Kokarev.

Berger, Croke and Kokarev

Let Σ be a closed Riemannian manifold

Berger 1979

If Σ admits an isometric involution without fixed points, then

$$\lambda_1(\Sigma) \leq K(n) \frac{|\Sigma|}{\mathrm{inj}(\Sigma)^{n+2}}.$$

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Improvements:

The exponent on k is better.

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$$\lambda_1 \sim 1/L^2 \qquad \mathrm{Diam}(\Sigma_L) \sim L \qquad |\Sigma_L| = L|\mathbb{S}^{n-1}|.$$

Injectivity and convexity radii = π.

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Berger and Croke:

$$\lambda_1(\Sigma) \leq K(n)L|\mathbb{S}^{n-1}|/\pi^{n+2} \qquad \text{and} \qquad \lambda_1 \leq K(n)L^2|\mathbb{S}^{n-1}|^2/\pi^{2n+2}.$$

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Our bound:

$$\lambda_1 \leq K(n)\frac{|\mathbb{S}^{n-1}|}{L\pi^n} \xrightarrow{L\to\infty} 0.$$

Optimal exponent on k for negative curvature

Croke 1980 and Kokarev 2019

$$\lambda_k(\Sigma) \leq K(n) \frac{|\Sigma|^2}{\text{conv}(\Sigma)^{2n+2}} k^{2n}.$$

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Colbois-Girouard 2021

$$\lambda_k(\Sigma) \leq K(n) \frac{|\Sigma|}{\mathrm{Diam}(\Sigma)^2 \mathrm{inj}(\Sigma)^n} k^{n+1}$$

Kokarev 2019

Let Σ be a closed Riemannian manifold with non-positive sectional curvature. Then,

$$\lambda_k(\Sigma) \leq K(n) \frac{|\Sigma|^2}{\mathrm{inj}(\Sigma)^{2n+2}} k^{2/n}.$$

Some references

Bruno Colbois, Alexandre Girouard. *Metric upper bounds for Steklov and Laplace eigenvalues*.
arXiv:2108.03101

Gerasim Kokarev. *Berger inequality for Riemannian manifolds with an upper sectional curvature bound*.
arXiv:1910.06647

Bounds on σ_k in terms of the diameter have also been studied elsewhere:

Abdelkader Al Sayed; Beniamin Bogosel; Antoine Henrot; Florent Nacry. *Maximization of the Steklov eigenvalues with a diameter constraint*.
SIAM J. Math. Anal. 53 (2021)

Beniamin Bogosel; Dorin Bucur; Alessandro Giacomini. *Optimal shapes maximizing the Steklov eigenvalues*.
SIAM J. Math. Anal. 49 (2017), no. 2, 1645–1680.