# Eigenvalue estimates for the Aharonov-Bohm Laplacian in 2D

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# Introduction

- In this talk, we give some eigenvalue estimates for the magnetic Laplacian on a Riemannian surface, possibly with boundary (in that case, we use magnetic Neumann boundary conditions).
- The magnetic Laplacian, and its spectrum, depend on the pair $(\Omega, A)$, where $\Omega$ is the surface and $A$ is the *potential* 1-*form*, giving rise to the *magnetic field* $B \doteq dA$, which in dimension 2 is identified with a real valued function by the Hodge-star operator.
- When $A = 0$, or more generally when $A$ is exact, the spectrum coincide with that of the usual, non magnetic case, in particular

$$\lambda_1(\Omega, A) = 0.$$

- However there are situations in which the magnetic field is zero (i.e. $A$ is closed) and yet the ground state energy is positive:

$$\lambda_1(\Omega, A) > 0.$$

- Physically, this corresponds to the so-called *Aharonov-Bohm* effect: consider an impenetrable region (typically, an ideal solenoid) where a magnetic field is confined, while a charged quantum particle is placed outside the impenetrable region.
- It turns out that the corresponding Hamiltonian of the particle feels in some sense a shift which is related to the flux of magnetic potential A along closed paths, even if the magnetic field vanishes outside the solenoid (the flux should not be an integer).
- For a spectral geometer, this relation between the spectrum and the topology is quite interesting, which prompted us to study the case of closed potentials ($dA = 0$).
- In fact, a general easy argument shows that the spectrum does not change if we replace $A$ by its co-closed part in the Hodge decomposition.
- This means that we can assume $A$ to be a *harmonic* one form, i.e. a de Rham cohomology class.

- Eigenvalue estimates should depend on the flux of the magnetic potential $A$ and the geometry of the surface.

- In particular, this gives a one-parameter (or multi-parameter) deformation of spectrum of the standard Laplacian, the parameter being precisely the set of fluxes.

- We first give a lower bound for annuli, that is, domains of type $[0, 1] \times \mathbf{S}^1$ with a Riemannian metric. Then, we apply the lower bound to Euclidean annuli having convex boundary components.

- A lower bound for closed genus one surfaces should follow (work in progress). The case of genus $g \geq 2$ is more complicated.

- Finally, we introduce the so-called Aharonov-Bohm potentials, in which the magnetic field is concentrated at one point of the domain (Dirac mass). In this limit case, we can prove a reverse Faber-Krahn inequality for domains in the plane and, more generally, in the plane endowed with a large class of radially invariant metrics, including those with non-positive curvature.

- Similar classic isoperimetric inequalities for the Steklov problem, as Brock's theorem and Weinstock inequality, are extended to the magnetic case.

# The magnetic Laplacian

- Let $\Omega$ be a Riemannian manifold and $A$ a smooth real 1-form, called *the potential* 1-form. The 2-form

$$B = dA$$

is called the *magnetic field*.

- One defines a modified gradient $\nabla^A$ on the space of complex-valued functions $C^\infty(M, \mathbf{C})$ by

$$\nabla^A_X u = \nabla_X u - iA(X)u.$$

- The *magnetic Laplacian* is the operator acting on $C^\infty(\Omega, \mathbf{C})$ defined by $\Delta_A = (\nabla^A)^\star \nabla^A$.

Explicitly one has:

$$\Delta_A u = \Delta u + |A|^2 u + 2i\langle du, A\rangle + iu\,\mathrm{div} A.$$

- We stress that the potential 1-form is assumed to be *real*.

- If $M$ is closed (compact, without boundary) the magnetic Laplacian has a discrete spectrum, which we denote by

$$\lambda_1(\Omega, A) \leq \lambda_2(\Omega, A) \leq \cdots \leq \lambda_k(\Omega, A) \leq \dots$$

and $\lambda_1(\Omega, A) \geq 0$, because

$$\lambda_1(\Omega, A) = \inf_{0 \neq u \in C^\infty(\Omega, \mathbb{C})} \frac{\int_\Omega |\nabla^A u|^2}{\int_\Omega |u|^2}$$

# Neumann conditions

- If $\Omega$ has a (smooth) boundary we will adopt (magnetic) Neumann conditions: these are

$$\langle \nabla^A u, N \rangle = 0 \quad \text{on} \quad \partial\Omega.$$

The spectrum will be denoted in the same way.

Clearly the spectrum reduces to the spectrum of the usual (non-magnetic) Laplacian when $A = 0$.

- Note that, according to our numbering:

$$\lambda_1(\Omega, 0) = 0$$

so that the first *positive* eigenvalue (in the non-magnetic case) is $\lambda_2(\Omega, 0)$.

# Gauge invariance

It is the identity:

$$\Delta_A e^{-i\phi} = e^{-i\phi} \Delta_{A+d\phi}$$

for all smooth real valued functions $\phi$ on $M$. Therefore, $\Delta_A$ and $\Delta_{A+d\phi}$ are unitarily equivalent, so that they have the same spectrum:

$$\lambda_k(\Omega, A + d\phi) = \lambda_k(\Omega, A)$$

for all $k$. In particular, $\lambda_k(\Omega, d\phi) = \lambda_k(\Omega, 0)$ (the usual non-magnetic Laplacian).

- What happens when the potential is a closed form, that is, the magnetic field is zero? Is it true that then $\lambda_k(\Omega, A) = \lambda_k(\Omega, 0)$ (the non-magnetic case) ?
- In other words: does the magnetic field determine the spectrum?
- Answer: no.
- In particular, there are many situations in which $B = dA = 0$ but $\lambda_1(\Omega, A) > 0$.

The situation was clarified by Shigekawa (closed manifolds) and Helffer et al. (for magnetic Neumann boundary conditions). Given a closed curve $c$ on $M$, consider the flux of $A$ around $c$:

$$\Phi_c^A = \frac{1}{2\pi}\oint_c A$$

### Theorem

*One has $\lambda_1(\Omega, A) = 0$ if and only if $dA = 0$ and the flux of $A$ around any closed curve is an integer.*

- In other words, we can "gauge away" all potential 1-forms in the lattice

$$\mathcal{P} = \{A : dA = 0,\ \Phi_c^A \in \mathbb{Z} \quad \text{for all closed curves } c\}$$

Note that $\mathcal{P}$ is between the subspace of exact forms and that of closed forms.

# Spectrum of the unit circle

Let $M = \mathbf{S}^1$, the circle with length $2\pi$. Let $t$ be the angular coordinate. The one-form

$$A_\nu = \nu dt$$

is closed, not exact, and has flux $\nu$ around the circle. One easily computes the spectrum and gets the family of eigenvalues

$$(k-\nu)^2, \quad k \in \mathbb{Z}$$

with associated eigenfunctions

$$u_k(t) = e^{ikt}.$$

Note that indeed the spectrum reduces to the non-magnetic case when $\nu$ is an integer, and that:

$$\lambda_1(M, A_\nu) = \inf_{k \in \mathbb{Z}} (k - \nu)^2$$

which is positive precisely when $\nu \notin \mathbb{Z}$.

- Actually, by gauge invariance, one could always assume that $\nu \in [0, \frac{1}{2}]$. In that case,

$$\lambda_1(M, A_\nu) = \nu^2$$

- One could see $\lambda_k(\mathbf{S}^1, A_\nu)$ as a continuous deformation of the usual spectrum $\lambda_k(\mathbf{S}^1, 0)$ of $\mathbf{S}^1$.

# Flat tori

Take for simplicity of exposition the square torus $T = \mathbf{S}^1 \times \mathbf{S}^1$ with potential one-form

$$A = \nu_1 dx_1 + \nu_2 dx_2,$$

where $\nu_1, \nu_2 \in \mathbf{R}$. Then $A$ is closed (actually harmonic), and has fluxes $\nu_1, \nu_2$ around the two homology classes which generate the cohomology of $T$.

The spectrum is the union of

$$(k - \nu_1)^2 + (h - \nu_2)^2$$

over $k, h \in \mathbf{Z}$ and the lowest eigenvalue is

$$\lambda_1 = \inf \Big\{ (k - \nu_1)^2 + (h - \nu_2)^2 : (k, h) \in \mathbf{Z} \times \mathbf{Z} \Big\}.$$

One sees that $\lambda_1 > 0$ iff $(\nu_1, \nu_2)$ does not belong to the integer lattice $\mathbf{Z} \times \mathbf{Z}$;

- if $(\nu_1, \nu_2) \in \mathbf{Z} \times \mathbf{Z}$ then $\lambda_1 = 0$ and actually the spectrum reduces to the spectrum of the usual (non-magnetic) Laplacian.

- Compute the lowest eigenvalue of the magnetic Laplacian on any flat torus.

# Aharonov-Bohm potentials

We now focus on dimension 2, and consider the lowest eigenvalue for domains in a space form of constant curvature (that is, $\mathbb{R}^2, \mathbb{H}^2, \mathbb{S}^2$) and for a class of particular potentials, the Aharonov-Bohm potentials.

- We explain the results for planar bounded domains $\Omega \subseteq \mathbb{R}^2$.

Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^2$ with a distinguished point $x_0 = (a, b)$. Consider the one-form

$$\omega = -\frac{y-b}{(x-a)^2+(y-b)^2}dx + \frac{x-a}{(x-a)^2+(y-b)^2}dy. \tag{1}$$

Then, $\omega$ is smooth on $\mathbb{R}^2 \setminus \{x_0\}$ and singular at the pole $x_0$; it is closed (actually harmonic) and has flux 1 around the point $x_0$.

- The one-form $A_{x_0,\nu} = \nu\omega$ will be called *Aharonov-Bohm potential with pole $x_0$ and flux $\nu$*.

- In the punctured plane $\mathbf{R}^2 \setminus \{x_0\}$, the potential $A_{x_0,\nu}$ gives rise to a vanishing magnetic field ($B = dA_{x_0,\nu} = 0$); viewed as a distribution on $\mathbf{R}^2$ the magnetic field is a Dirac mass at $x_0$:

$$\star dA_{x_0,\nu} = \nu \delta_{x_0}.$$

- If the flux $\nu$ is not an integer, the lowest eigenvalue (with magnetic Neumann conditions) is positive.

- By that we mean the eigenvalue problem (the Aharonov-Bohm potential $A_{x_0,\nu}$ is simply denoted by $A$):

$$\begin{cases} \Delta_A u = \lambda u\,, & \text{in } \Omega, \\ \langle \nabla^A u, N \rangle = 0\,, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $N$ is the inner unit normal.
It admits a non-negative, discrete spectrum:

$$\lambda_1(\Omega, A) \leq \lambda_2(\Omega, A) \leq \ldots$$

and the min-max principle reads:

$$\lambda_1(\Omega, A_{x_0,\nu}) = \inf_{0\neq u\in H^1_A(\Omega,\mathbb{C})} \frac{\int_\Omega |\nabla^A u|^2}{\int_\Omega |u|^2}$$

where $H^1_A(\Omega, \mathbb{C})$ is the form domain we work with, the *magnetic Sobolev space*, the closure of $C^\infty_{x_0}(\Omega, \mathbb{C})$ (the space of smooth functions on $\Omega$ vanishing in a neighborhood of $x_0$) with respect to the norm

$$\|u\|^2_A := \int_\Omega |\nabla^A u|^2 + |u|^2, \quad \forall u \in C^\infty_{x_0}(\Omega, \mathbb{C}) : \nabla^A u, u \in L^2(\Omega, \mathbb{C}).$$

- If $\nu \in \mathbb{Z}$ then the Aharonov-Bohm spectrum coincides with the usual non-magnetic Neumann spectrum:

$$\lambda_k(\Omega, A_{x_0,\nu}) = \lambda_k(\Omega, 0)$$

for all $k$.

# Magnetic Szëgo-Weinberger inequality (Euclidean plane)

### Theorem

(Colbois, Provenzano, S)

*Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$ and let $A_{x_0,\nu}$ be the Aharonov-Bohm potential with pole at $x_0$ and flux $\nu$. Let $B = B(x_0, R)$ be the disk centered at the pole $x_0$ such that $|B| = |\Omega|$. Then*

$$\lambda_1(\Omega, A_{x_0,\nu}) \leq \lambda_1(B, A_{x_0,\nu}); \tag{3}$$

*if $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.*

- We remark that the classical, non-magnetic Szëgo-Weinberger inequality regards the *second* Neumann eigenvalue, and not the first (which is zero for every domain), and in our notation can be stated as follows:

$$\lambda_2(\Omega, 0) \leq \lambda_2(B, 0)$$

where $B$ is a ball with the same volume of $\Omega$.

# Idea of proof

- We can assume that $\nu \in (0, \frac{1}{2}]$.

Here is the scheme of the proof:

**Step 1**. We compute the spectrum of a disk centered at the pole $x_0$, and observe that the first eigenfunction is real and radial around $x_0$.

**Step 2.** We reduce the proof of the above inequality to an isoperimetric inequality involving Schrödinger operators of type $\Delta + V$ where $V = V(r)$ is radial around $x_0$.

**Step 3.** We apply the inequality in Step 2 to the case $V = |A_{x_0,\nu}|^2$ and get the final result.

# Step 2: an upper bound by an associated Schrödinger operator

When giving upper bounds we often use test-functions which are *real*. Since the potential one-form $A$ is real, when $u$ is real the Rayleigh quotient writes:

$$\int_\Omega |\nabla^A u|^2 = \int_\Omega \left(|\nabla u|^2 + |A|^2 u^2\right)$$

that is, the first eigenvalue for the magnetic Laplacian is bounded above by that of the Scrhödinger $\Delta + V$ where $V = |A|^2$, for Neumann conditions.

## Theorem

*One has always:*

$$\lambda_1(\Omega, \Delta_A) \leq \lambda_1(\Omega, \Delta + V),$$

*where $V = |A|^2$. Equality holds if and only if there is a first eigenfunction of $\Delta_A$ which is real.*

# An isoperimetric inequality for Schröedinger operators

The following considerations apply in any dimension $n$.
We consider potentials $V$ which are non-negative and radial around a point $x_0$:

$$V = V(r),$$

where $r$ is the distance to $x_0$.

- Note that $V$ could be singular at $x_0$: if $A = A_{x_0,\nu}$ then $V(r) = \dfrac{\nu^2}{r^2}$.

Introduce the natural form domain

$$H^1_V(\Omega) = \{u \in H^1(\Omega) : V^{\frac{1}{2}} u \in L^2(\Omega)\}$$

and define

$$\lambda_1(\Omega, \Delta + V) = \inf_{0 \neq u \in H^1_V(\Omega)} \frac{\int_\Omega \left(|\nabla u|^2 + V u^2\right)}{\int_\Omega u^2},$$

which is non-negative.

Let then $\Omega$ be a smooth bounded domain in $\mathbf{R}^n$ and let $B \doteq B(x_0, R)$ be the ball centered at $x_0$ with the same volume:

$$|\Omega| = |B|.$$

We set $D(\Omega) = \sup\{d(x, x_0) : x \in \Omega\}$ and make the following assumptions:

**Assumption 1**. There exists a first eigenfunction $u$ on $B$ which is non-negative, radial and non-decreasing in the radial direction: $u \geq 0, u' \geq 0$.

**Assumption 2**. $V$ is radial around $x_0$, non-negative and non increasing: $V'(r) \leq 0$ on $(0, D_\Omega)$.

**Assumption 3**. $V' + 2V^2 r \leq 0$ on $(0, R)$.

## Theorem

*Under assumptions* **1**, **2**, **3** *above, we have:*

$$\lambda_1(\Omega, \Delta + V) \leq \lambda_1(B, \Delta + V),$$

*with equality if and only if $\Omega$ is the ball $B$.*

# Step 3: end of proof

We now go back to dimension 2. One easily checks that the assumption hold when $V = |A_{x_0,\nu}|^2$, that is, when

$$V(r) = \frac{\nu^2}{r^2}$$

Therefore,

$$\lambda_1(\Omega, \Delta + |A_{x_0,\nu}|^2) \leq \lambda_1(B, \Delta + |A_{x_0,\nu}|^2).$$

We now conclude easily:

$$\begin{aligned} \lambda_1(\Omega, A_{x_0,\nu}) &\leq \lambda_1(\Omega, \Delta + |A_{x_0,\nu}|^2) \\ &\leq \lambda_1(B, \Delta + |A_{x_0,\nu}|^2) \\ &= \lambda_1(B, A_{x_0,\nu}) \end{aligned}$$

where the second inequality follows from our estimate on Schrödinger operators and the last one follows from the fact that the groundstate on the ball is real.

# Extension to space forms and surfaces of revolution

The above inequality for Schrödinger operators can be extended, under suitable hypothesis, to domains in any manifold of revolution (basically, $\mathbf{R}^n$ with a radial metric around $x_0$).

In dimension 2, consider polar coordinates $(r, t)$ around $x_0$. The 1 form:

$$A_{x_0,\nu} = \nu\, dt$$

is closed and has flux $\nu$ around $x_0$; this two facts characterize the spectrum of the magnetic Laplacian, and $A_{x_0,\nu}$ could be called *Aharonov-Bohm potential* as well.

One can try to extend the magnetic Szëgo-Weinberger inequality to this situation. Note for example that in $\mathbb{H}^2$:

$$|A_{x_0,\nu}|^2 = \frac{\nu^2}{\sinh^2 r}$$

while in $\mathbb{S}^2$:

$$|A_{x_0,\nu}|^2 = \frac{\nu^2}{\sin^2 r}$$

The above scheme yields the following fact.

### Theorem

*Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$ or $\mathbb{H}^2$ and let $A_{x_0,\nu}$ be the Aharonov-Bohm potential with pole at $x_0$ and flux $\nu$. If $B = B(x_0, R)$ is the ball centered at $x_0$ with the same volume of $\Omega$, then*

$$\lambda_1(\Omega, A_{x_0\nu}) \leq \lambda_1(B, A_{x_0,\nu}). \tag{4}$$

*If $\nu \notin \mathbb{Z}$, equality holds if and only if $\Omega = B(x_0, R)$.*
*The same conclusions hold when $\Omega \subseteq \mathbb{S}^2$ is a spherical domain contained in the hemisphere centered at the pole $x_0$.*

- More generally, the Szëgo-Weinberger inequality holds for any metric on $\mathbf{R}^2$, which is radial around $x_0$ and has non-positive Gauss curvature.

# Optimal placement of the pole

Let $M$ be a space form ($\mathbb{R}^2, \mathbb{H}^2$ or $\mathbb{S}^2$) and $A_{x_0,\nu}$ be, as usual, the Aharonov-Bohm potential with pole $x_0$ and flux $\nu$. Fix a disk $B(p, R)$ with center $p$ and radius $R$. It is a fact that:

- If the pole $x_0 \in B(p, R)$ tends to the boundary then

$$\lambda_1(B(p, R), A_{x_0,\nu}) \to \lambda_1(B(p, r), 0) = 0.$$

- What is the optimal position of the pole? In Euclidean or hyperbolic space, the first eigenvalue is maximized when the pole $x_0$ is at the center: $x_0 = p$. In the sphere, we assume that $B(p, R)$ is contained in a hemisphere. This follows immediately from the isoperimetric inequality proved above. In other words:

## Theorem

*Among all geodesic balls of fixed radius in $\mathbb{R}^2, \mathbb{H}^2$ and $\mathbb{S}^2_+$ (the hemisphere) the maximum value of the first Aharonov-Bohm eigenvalue is attained when the pole is at the center (for any value of the flux).*

# Steklov problem

For Aharonov-Bohm potential $A = A_{x_0,\nu}$ we now consider the *magnetic Steklov eigenvalue problem*:

$$\begin{cases} \Delta_A u = 0\,, & \text{in } \Omega, \\ \langle \nabla^A u, N \rangle = \sigma u\,, & \text{on } \partial\Omega \end{cases} \tag{5}$$

which has a discrete, non-negative spectrum:

$$\sigma_1(\Omega, A) \leq \sigma_2(\Omega, A) \leq \cdots \leq \sigma_k(\Omega, A) \leq \ldots$$

its lowest eigenvalue (by standard arguments as above) is positive provided $\nu \notin \mathbb{Z}$.

# Brock theorem

**Theorem**

*Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^2$, $x_0 \in \Omega$ a fixed pole, and let $B = B(x_0, R)$ be the disk with the same measure of $\Omega$. Then:*

$$\sigma_1(\Omega, A_{x_0,\nu}) \leq \sigma_1(B, A_{x_0,\nu}) = \frac{\sqrt{\pi}}{|\Omega|^{\frac{1}{2}}} \inf_{k \in \mathbf{Z}} |\nu - k|$$

*Equality holds if and only if $\Omega = B(x_0, R)$.*

# Magnetic Weinstock inequality

## Theorem

*Let $\Omega$ be bounded simply connected domain in $\mathbb{R}^2$, $x_0 \in \mathbb{R}^2$ be a fixed pole, and let $B \doteq B(x_0, R)$ the disk with the same perimeter of $\Omega$. Then:*

$$\sigma_1(\Omega, A_{x_0,\nu}) \leq \sigma_1(B, A_{x_0,\nu}) = \frac{2\pi}{|\partial\Omega|} \inf_{k \in \mathbf{Z}} |\nu - k|.$$

*Equality holds if and only if $\Omega = B(x_0, R)$.*

- Remark that the non-magnetic Weinstock inequality reads

$$\sigma_2(\Omega, 0) \leq \sigma_2(B, 0).$$

The proof is a simple consequence of: the conformal invariance of the magnetic energy, gauge invariance and the Riemann mapping theorem. The conformal invariance of the magnetic energy states that, if $\Phi : \Omega_1 \to \Omega_2$ is a conformal map between surfaces and $A$ is any potential one-form on $\Omega_2$ then, for all functions $u$:

$$\int_{\Omega_1} |\nabla^{\Phi^\star A}(\Phi^\star u)|^2 \, d\mu = \int_{\Omega_2} |\nabla^A u|^2 \, d\mu$$

where $\Phi^\star A$ is the potential one-form on $\Omega_1$ obtained by pulling back $A$ by $\Phi$.

# A general lower bound for annuli

An *annulus* is a Riemannian manifold diffeomorphic to $[0,1]\times \mathbf{S}^1$. It has two boundary components

$$\Gamma_0=\{0\}\times\mathbf{S}^1,\quad \Gamma_1=\{1\}\times\mathbf{S}^1$$

- If $A$ is a closed potential one-form, it has the same flux around $\Gamma_0$ and $\Gamma_1$: we denote this common flux by $\Phi^A$.
- If $\Phi^A$ is close to an integer, by continuity the first eigenvalue tends to zero. Hence, a lower bound will involve the minimum distance of $\Phi^A$ to the integers:

$$d(\Phi^A,\mathbf{Z})\doteq\inf_{k\in\mathbf{Z}}|\Phi^A-k|.$$

- the plan is to give a geometric lower bound for $\lambda_1(\Omega,A)$.

- On any annulus there exists functions $\psi : \Omega \to \mathbf{R}$ such that $\psi$ is constant on each component of $\partial\Omega$ and has no critical points inside $\Omega$.
- Then, for any such $\psi$ we can define the invariant:

$$K = K_{\Omega,\psi} = \frac{\sup_{\Omega}|\nabla\psi|}{\inf_{\Omega}|\nabla\psi|}.$$

- Of course $K \geq 1$.
- Notice that $K_{\Omega,\psi} = 1$ when $\Omega$ is a flat cylinder, that is, it is isometric to $[0, a] \times \mathbf{S}^1(R)$ with the product metric, and $\psi$ is the "height" function (distance function to one of the boundary components).
- Notice that the level curves of $\psi$ are all regular. We then say that $\Omega$ *is $K$-foliated by the level curves of $\psi$*.

## Theorem

*(Colbois-S 2018)* a) *Let $A$ be a closed 1-form on an annulus $\Omega$ which is foliated by the level curves of $\psi$. Then:*

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{K_{\Omega,\psi} L^2} \inf_{k \in \mathbf{Z}} |\Phi^A - k|^2.$$

*where $L$ is the maximum length of a level curve of $\psi$ and $\Phi^A$ is the flux of $A$ across any boundary component of $\Omega$.*

b) *Equality holds if and only if $\Omega$ is a flat cylinder, i.e. the Riemannian product $[0, a] \times \mathbf{S}^1(R)$ for some $a$ and $R$, in which case $K_{\Omega,\psi} = 1$.*

- Sketch of proof: $\Omega$ is foliated by the level curves of $\psi$; on each of these, say $\{\psi = t\}$, restrict the first eigenfunction and use it as test-function for the magnetic Laplacian of a circle of the same length. Integrate the inequalities by using the co-area formula. The equality case is more involved technically.

# Doubly connected plane domains

We now consider a doubly connected plane domain, bounded by two convex curves $\Gamma$ and $\Gamma'$. We let $\beta$ and $B$ be, respectively, the minimum amd maximum distance of a point $x \in \Gamma'$ (the outer component) to $\Gamma$. By applying the theorem above, we obtain:

## Theorem

*Let $\Omega$ be as above, and let $A$ be a closed potential 1-form with flux $\Phi^A$ around any of the two boundary components. Then:*

$$\lambda_1(\Omega, A) \geq \frac{4\pi^2}{L^2}\frac{\beta^2}{B^2} d(\Phi^A, \mathbf{Z})^2,$$

*where $L$ is the length of the outer component.*

- In order have $\lambda_1$ small, for a fixed outer length, it is necessary that the ratio $\frac{\beta}{B}$, and not just $\beta$, has to be small.
- The dependance on $\frac{\beta}{B}$ is somewhat necessary: if the two components get close somewhere (and $B$ is uniformly bounded below) then the lowest eigenvalue might be very small.
- The dependance on the outer length and $d(\Phi^A, \mathbf{Z})^2$ is also necessary, as explicit examples show.
- The term $\dfrac{\beta^2}{B^2}$ has been improved to $\dfrac{\beta}{B}$ (at some extra cost in the constants) in Colbois-S. 2021. This linear dependance is shown to be the sharp one, when the flux and the outer length have been fixed.

# Closed surfaces

Let $\Sigma$ be a compact surface without boundary with genus $g$ and $A$ a closed potential one-form. We are interested in the lowest eigenvalue

$$\lambda_1(\Omega, A).$$

- By the Hodge decomposition theorem, we can reduce to the case

where $A$ is a harmonic form (i.e. closed and co-closed), hence, a 1-cohomology class. Hence, if $g = 0$, $\Sigma$ is a topological sphere, hence $A$ is also exact and by gauge invariance

$$\lambda_k(\Sigma, A) = \lambda_k(\Sigma, 0),$$

the spectrum of the usual Laplacian. In particular:

$$\lambda_1(\Sigma, A) = \lambda_1(\Sigma, 0) = 0.$$

- Hence we have something new only when $g \geq 1$; in that case a harmonic 1-form is determined by its fluxes across the generators of the homology in dimension 1, which are, in number, $2g$ cycles.
- If at least one of these fluxes is not an integer, we have $\lambda_1(\Omega, A) > 0$.

# Genus one

A topological torus (genus one closed surface) can be split into two annuli. Since we need to estimate the *first* eigenvalue, no orthogonality condition is needed and then $\lambda_1$ is bounded below by the smallest eigenvalue of the two pieces.

- A lower bound for all topological tori follows (work in progress).
- For example, take a revolution torus $\Omega$ embedded in $\mathbf{R}^3$ and having radii $a > b > 0$ (that is, $\Sigma$ is the set of points at distance $b$ to a circle of radius $a$ in 3-space).
- Let $A$ be a closed potential one-form having flux $\Phi_1^A$ around any of the parallels and flux $\Phi_2^A$ around any of the meridians. Using the main theorem, and foliating the torus in "two different orthogonal ways" we get:

$$\lambda_1(\Omega, A) \geq \frac{1}{(a+b)^2} d(\Phi_1^A, \mathbf{Z})^2 + \frac{a-b}{b^2(a+b)} d(\Phi_2^A, \mathbf{Z})^2$$

- The homology of a genus $g$ closed surface $\Sigma$ has $2g$ generators; hence a harmonic potential 1-form $A$ is determined by its fluxes across these generators. Give a lower bound of $\lambda_1(\Sigma, A)$ in terms of these fluxes and the geometry of $\Sigma$.