On the minimization of the drag force in Stokes fluids: a free discontinuity approach.

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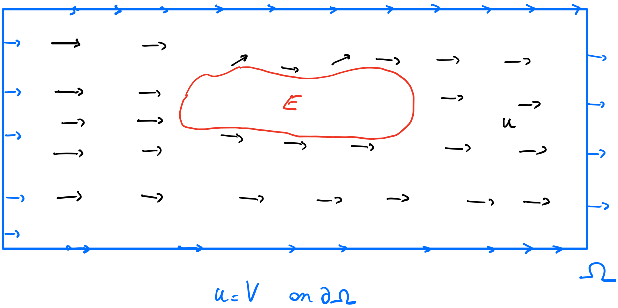
September 22, 2022

Joint work with D. Bucur, A. Chambolle and M. Nahon

The stationary flow under Navier conditions

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary, $E \subset \Omega$, and let $V \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ be a divergence free vector field.



If $u : \Omega \setminus E \to \mathbb{R}^d$ is the velocity field, we require that the following items hold true.

(a) *Incompressibility*: $\operatorname{div} u = 0$ in $\Omega \setminus E$.

(b) *Boundary conditions*: we have

$$u = V \text{ on } \partial\Omega \qquad \text{and} \qquad u \cdot \nu = 0 \text{ on } \partial E,$$

(c) *Equilibrium*: considering the stress

$$\sigma := -pI_d + 2\mu e(u),$$

we require

$$\operatorname{div} \sigma = 0 \qquad \text{in } \Omega \setminus E.$$

(d) *Navier conditions on the obstacle*: we have

$$(\sigma\nu)_\tau = \beta u \qquad \text{on } \partial E,$$

where $\beta > 0$.

The stationary flow has the following variational characterization: u is the minimizer of the energy

$$\mathcal{E}(u) := 2\mu \int_{\Omega \setminus E} |e(u)|^2 \, dx + \beta \int_{\partial E} |u|^2 \, d\mathcal{H}^{d-1}$$

among the class of (sufficiently regular) admissible fields

$$\mathcal{V}^{\mathrm{reg}}_{E,V}(\Omega) := \Big\{ v \in H^1(\Omega \setminus E, \mathbb{R}^d) \, : \, v \text{ satisfies incompressibility and boundary conditions} \Big\} .$$

The drag force and the optimization problem

Assume now that the external vector field V is equal to a constant $V_\infty \in \mathbb{R}^d \setminus \{0\}$, i.e. the obstacle E is immersed in a uniform flow. The flow is perturbed near E, and the obstacle experiences a force whose component in the direction V_∞ is given by

$$Drag(E) := \int_{\partial E} \sigma\nu \cdot \frac{V_\infty}{|V_\infty|} \, d\mathcal{H}^{d-1}.$$

It turns out that

$$\begin{aligned} Drag(E) &= \frac{1}{|V_\infty|}\mathcal{E}(u) \\ &= \frac{1}{|V_\infty|}\left[2\mu \int_{\Omega\setminus E} |e(u)|^2 \, dx + \beta \int_{\partial E} |u|^2 \, d\mathcal{H}^{d-1}\right]. \end{aligned}$$

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Let $c > 0$ and let $f : (0, |\Omega|) \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous functions that is not identically equal to $+\infty$. We are interested in the following optimization problem:

$$\min_{E} \left\{ Drag(E) + c\mathcal{H}^{d-1}(\partial E) + f(|E|) \right\}.$$

Letting V be arbitrary, the drag minimization problem above is a particular case of the following shape optimization problem

$$\min_{E, u \in \mathcal{V}^{\text{reg}}_{E,V}(\Omega)} \left\{ \int_{\Omega \setminus E} |e(u)|^2 \, dx + \beta \int_{\partial E} |u|^2 \, d\mathcal{H}^{d-1} \right.$$
$$\left. + c\mathcal{H}^{d-1}(\partial E) + f(|E|) \right\}.$$

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Goal: prove existence of a minimizer through the Direct Method of the Calculus of Variations in a suitable relaxed context.

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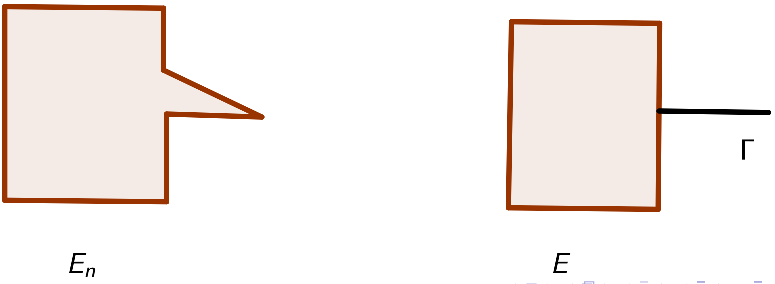

We are led to consider a velocity field discontinuous across Γ. We also expect an extra term in the surface integral related to the Navier conditions, which amounts at least to

$$\beta \int_{\Gamma \setminus \partial E} [|u^+|^2 + |u^-|^2] \, d\mathcal{H}^{d-1},$$

where $u^\pm$ are the two traces from both sides of Γ. An other extra term comes from the perimeter penalization and reads

$$2c\mathcal{H}^{d-1}(\Gamma).$$

A natural functional space to be considered is the space of functions of bounded deformation SBD: $u \in SBD(\Omega)$ if $u \in L^1(\Omega; \mathbb{R}^d)$ and

$$Eu = e(u)\,dx + (u^+ - u^-) \odot \nu_u \mathcal{H}^{d-1} \lfloor J_u$$

as matrix valued measures on Ω.

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Compactness in *SBD* (Bellettini, Coscia, Dal Maso)

Let $\Omega \subseteq \mathbb{R}^d$ be open, bounded and with a Lipschitz boundary, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBD(\Omega)$ such that

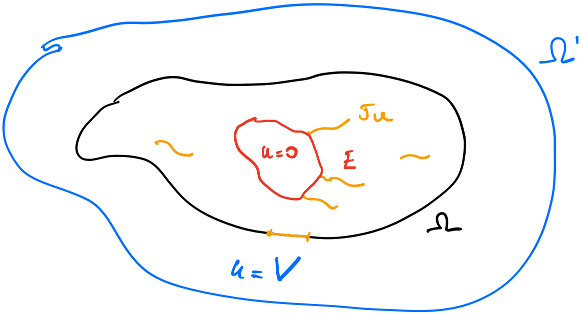
$$\sup_n \Big[|Eu_n|(\Omega) + \|u_n\|_{L^1(\Omega;\mathbb{R}^d)} + \|e(u_n)\|_{L^p(\Omega;\mathbb{M}^d_{\mathrm{sym}})} + \mathcal{H}^{d-1}(J_{u_n}) \Big] < +\infty$$

for some $p > 1$. Then there exists $u \in SBD(\Omega)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that

$$u_{n_k} \to u \qquad \text{strongly in } L^1(\Omega;\mathbb{R}^d),$$

$$e(u_{n_k}) \rightharpoonup e(u) \qquad \text{weakly in } L^p(\Omega;\mathbb{M}^d_{\mathrm{sym}}),$$

$$\mathcal{H}^{d-1}(J_u) \leq \liminf_{k\to+\infty} \mathcal{H}^{d-1}(J_{u_{n_k}}).$$


Ω'
Ω
J_u
E
u=0
u = V

The functional setting

Let $\Omega' \subset\subset \mathbb{R}^d$ be open and bounded with $\Omega \subset\subset \Omega'$.

Admissible pairs

We say that (E, u) is an admissible pair and write $(E, u) \in \mathcal{A}(V)$, if

(a) $E \subseteq \Omega$ has finite perimeter;

(b) $u \in SBD(\Omega')$ with $u = 0$ a.e. in E, $u = V$ on $\Omega' \setminus \Omega$ and

- Incompressibility: $div\, u = 0$ in $\mathcal{D}'(\Omega')$;
- Tangency constraint: $u^\pm \perp \nu_{E \cup J_u}$ on $\partial^* E \cup J_u$.

The relaxed optimization problem

The relaxed formulation for the optimization problem involves the energy

$$\mathcal{J}(E,u) := \int_{\Omega\setminus E} |e(u)|^2 \, dx + \beta \int_{\partial^* E} |u^+|^2 \, d\mathcal{H}^{d-1} + \beta \int_{J_u \setminus \partial^* E} [|u^+|^2 + |u^-|^2] \, d\mathcal{H}^{d-1} + c\mathcal{H}^{d-1}(\partial^* E) + 2c\mathcal{H}^{d-1}(J_u \setminus \partial^* E) + f(|E|).$$

The main result

Theorem (Bucur, Chambolle, G., Nahon, 2022)

The minimum problem

$$\min_{(E,u)} \mathcal{J}(E,u)$$

has a solution.

Compactness

Theorem (Bucur, Chambolle, G., Nahon, 2022)

For every $(E, u) \in \mathcal{A}(V)$ we have

$$\mathcal{H}^{d-1}(\partial^* E) + \|e(u)\|_{L^2} + \mathcal{H}^{d-1}(J_u) \\ + |Eu|(\Omega) + \|u\|_{L^2} + \leq C\mathcal{J}(E, u).$$

As a consequence compactness in L^1 for the obstacles and in SBD for the velocities are available.

The main difficulties to get the full result are the following.

- The tangency constraint:

$$u_n^{\pm} \perp \nu_{E_n \cup J_{u_n}} \Longrightarrow u^{\pm} \perp \nu_{E \cup J_u}.$$

- Lower semicontinuity of Navier energies:

$$\int_{\partial^* E} |u^+|^2 \, d\mathcal{H}^{d-1} + \int_{J_u \setminus \partial^* E} |u^+|^2 + |u^-|^2 \, d\mathcal{H}^{d-1}$$
$$\leq \liminf_n \left[\int_{\partial^* E_n} |u_n^+|^2 \, d\mathcal{H}^{d-1} + \int_{J_{u_n} \setminus \partial^* E_n} |u_n^+|^2 + |u_n^-|^2 \, d\mathcal{H}^{d-1} \right]$$

Theorem (Bucur, Chambolle, G., Nahon, 2022)

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded open set, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $SBD(\Omega)$ such that

$$\sup_n \left[\int_\Omega |e(u_n)|^2 \, dx + \mathcal{H}^{d-1}(J_{u_n}) \right] < +\infty$$

with $u_n \to u$ in $L^1(\Omega)$ for some $u \in SBD(\Omega)$. Then the following facts hold true.

- We have

$$\int_{J_u} \left[|u^+ \cdot \nu_u| + |u^- \cdot \nu_u| \right] \, d\mathcal{H}^{d-1} \le \liminf_{n\to+\infty} \int_{J_{u_n}} \left[|u_n^+ \cdot \nu_{u_n}| + |u^- \cdot \nu_{u_n}| \right] \, d\mathcal{H}^{d-1}.$$

- If $\phi : \mathbb{R}^d \to [0, +\infty]$ is a lower semicontinuous function, we have

$$\int_{J_u} [\phi(u^+) + \phi(u^-)] \, d\mathcal{H}^{d-1} \leq \liminf_{n \to +\infty} \int_{J_{u_n}} [\phi(u_n^+) + \phi(u_n^-)] \, d\mathcal{H}^{d-1}.$$

Thank you for your attention!

Regularity

Theorem (Bucur, Chambolle, G., Nahon, 2022)

Assume $d = 2$, and let (E, u) be a minimizer of the problem. Then

$$\mathcal{H}^1(\overline{\partial^* E \cup J_u} \setminus (\partial^* E \cup J_u)) = 0$$

and

$$u \in H^1(\Omega \setminus (\overline{\partial^* E \cup J_u}); \mathbb{R}^2) \cap C^\infty(\Omega \setminus (\overline{\partial^* E \cup J_u}); \mathbb{R}^2).$$

In other words, the optimal obstacle is given by the closed set

$$\overline{\partial^* E \cup J_u}.$$