Noisy linear inverse problems under convex constraints: Exact risk asymptotics in high dimensions

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Abstract:

In the standard Gaussian linear measurement model $Y=X\mu_0+\xi \in \R^m$ with a fixed noise level $\sigma>0$, we consider the problem of estimating the unknown signal μ_0 under a convex constraint $\mu_0 \in K$, where K is a closed convex set in $\R^n$. We show that the risk of the natural convex constrained least squares estimator (LSE) $\hat{\mu}(\sigma)$ can be characterized exactly in high dimensional limits, by that of the convex constrained LSE $\hat{\mu}_K^{\seq}$ in the corresponding Gaussian sequence model at a different noise level.

The characterization holds (uniformly) for risks in the maximal regime that ranges from constant order all the way down to essentially the parametric rate, as long as certain necessary non-degeneracy condition is satisfied for $\hat{\mu}(\sigma)$.

The precise risk characterization reveals a fundamental difference between noiseless (or low noise limit) and noisy linear inverse problems in terms of the sample complexity for signal recovery. A concrete example is given by the isotonic regression problem: While exact recovery of a general monotone signal requires $m\gg n^{1/3}$ samples in the noiseless setting, consistent signal recovery in the noisy setting requires as few as $m\gg \log n$ samples. Such a discrepancy occurs when the low and high noise risk behavior of $\hat{\mu}_K^{\seq}$ differ significantly. In statistical languages, this occurs when $\hat{\mu}_K^{\seq}$ estimates 0 at a faster `adaptation rate' than the slower `worst-case rate' for general signals. Several other examples, including non-negative least squares and generalized Lasso (in constrained forms), are also worked out to demonstrate the concrete applicability of the theory in problems of different types.