Fourier convergence and homeomorphisms of the circle

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1 Introduction

Can one improve convergence properties of Fourier series by change of variable?

Theorem 1 (J. Pál, 1914; H. Bohr, 1935). *For every real function $f \in C(\mathbb{T})$ there exists a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that the Fourier series of the superposition $f \circ h$ converges uniformly.*

Proof. Let $f > 0$. Consider $\Omega = \{z = re^{i\theta},\ r < f(\theta)\}$. Let $F(z)$ be the conformal map: $D \to \Omega$. Then $|F(e^{it})| = f \circ h\,(t)$. F belongs to the Sobolev space $W(2, \frac{1}{2})$, and $|F|$ also does.

The homeomorphism h in this proof is, in general, singular.

Problems (N.N. Luzin):

1. Is it possible for any f to find an *absolutely continuous* homeomorphism h such that $f \circ h$ has uniformly convergent Fourier series?

2. Is it possible to find h so that the superposition has absolutely convergent Fourier series?

Nina Bari "Trigonometric series" (1959)

The second problem was resolved negatively in 1981, using different approaches, by J.P. Kahane & Y. Katznelson and by Olevskii.

"Real" proofs of Pál - Bohr thm (K - K 1978, A. Saakyan 1979) extended the theorem in different aspects. However in all these proofs h is singular. So the first Luzin problem has been open so far.

One cannot require h to be $C^{1+\epsilon}$ - smooth (Kahane - Katznelson 1983).

THE RESULT

Theorem 2 (G. Kozma, A.O., 2021). *For every continuous real function f there exists an absolutely continuous (AC) homeomorphism h such that Fourier series of the superposition $f \circ h$ converges uniformly.*

I will discuss some ideas involved in the proof.

2 Random Homeomorphisms

Recall our earlier result in the subject (G. Kozma, A.O. 1998):

For any $f \in C(\mathbb{T})$ there is a Holder homeomorphism h such that $\|S_n(f \circ h)\|_\infty = o(\log \log n)$.

We used the Dubbins - Freedman random homeomorphism:

Take $\phi(0) = 0$, $\phi(1) = 1$. Take $\phi(1/2)$ to be uniform between 0 and 1. Then take $\phi(1/4)$ to be uniform between 0 and $\phi(1/2)$, and $\phi(3/4)$ to be uniform between $\phi(1/2)$ and 1, and otherwise independent. Continue similarly, taking $\phi(k/2^n)$ to be uniform between $\phi((k-1)/2^n)$ and $\phi((k+1)/2^n)$ for all odd $k \in \{1, 3, ..., 2^n - 1\}$. Almost surely this can be extended to a Holder homeomorphism.

Theorem (G. Kozma, A.O. 1998). *For any continuous function f, if ϕ is a Dubbins -Freedman random homeomorphism then the Fourier partial sums of the superposition $f \circ \phi$ have norms $o(\log \log n)$ almost surely.*

The estimate is sharp.

We did not expect a purely random construction to solve Luzin's problem.

3 Random signs

Given f, $\|f\| \le 1$, consider partial sums:

$$S_n(f \circ h; x) = \int f(h(t)) D_n(x - t) dt \,.$$

To make the problem discrete assume $x = j/n,\ j \in \{0, .., n-1\}$.

Most serious simplification: replace homeomorphism by multiplication with signs. Instead of $f \circ h$ we replace f in the interval $(k/n, (k+1)/n)$ by $\epsilon_k f$ for some $\epsilon_k = \pm 1$.

Question: are there ϵ_k such that

$$|\sum_{[0,n-1]} \epsilon_k \int_{[k/n,(k+1)/n]} f(t)\, D_n(\frac{j}{n} - t) dt\, | < C \quad \forall j\,?$$

The integrals above can be bounded by $C/(|k-j|+1)$. So the following lemma gives the answer:

Lemma 1. *Let $v_{k,j}$ be numbers satisfying $|v_{k,j}| \le 1/(|k-j|+1)$. Then there are signs ϵ_k such that*

$$|\sum \epsilon_k v_{k,j}| < C \quad \forall j\,.$$

In what follows, we will use an advanced (more technical) version of the lemma.

Taking ϵ_k be i.i.d. does not work: the maximum becomes $\approx \log\log n$.

For the proof of lemma 1 we use a "hierarchical random construction". A similar one was used by B. Kashin (1979) in his discrete version of Menshov correction theorem.

It is interesting to compare Lemma 1 with J. Komlos conjecture (1980-s):

If $\|v_k\|_2 \leq 1\ \forall k$ *then there are signs* ϵ_k *such that*

$$\| \sum \epsilon_k v_k \|_\infty < C\,.$$

J. Spenser proved that it is true if we allow $\epsilon_k \in [-1, 1]$ and half of them in $\{-1, 1\}$.

4 Removing randomness

Come back to random homeomorphisms. Here I describe the first of two main ideas involved in the proof of Theorem 2.

Starting with a random homeomorphism of Dubbins - Freedman type, we are going to "remove randomness" step by step, keeping the behavior of the average $\mathbb{E}(f \circ \phi)$ and its partial Fourier sums under control.

Fix a number q, $0 < q < 1$. We define a random homeomorphism ψ_q. The only difference from D-F is: given a dyadic interval $I = [(k-1)/2^n, (k+1)/2^n]$, the image of the point $d = k/2^n$ is defined uniformly distributed on the interval concentric to $\psi(I)$ and of length $q|\psi(I)|$.

One can see that ψ_q is Holder *for sure*, and for small q the smoothness is close to 1.

Let a function f be given, $\|f\| = 1$. For dyadic points $d = k/2^n$, k is odd, we call n the rank of d. Let n be fixed.

Assume that we have a modification of ψ_q (denote it by ϕ), satisfying the following conditions:

(i) for all dyadic points d of rank $< n$ $\phi(d)$ is not random;

(ii) for all d of rank n $\phi(d)$ is uniformly distributed over some interval $J(d)$ of length $|\phi(I)|\,q/2^j$;
(iii) for all d of higher rank the conditional distributions of $\phi(d)$ remains as it was at the beginning.

Now we make the next modification (denote it by ϕ'), changing only the condition (ii). It is replaced by

(ii') for all d of rank n the image $\phi'(d)$ is uniformly distributed over a half of $J(d)$, upper or lower.

The choice of the half, depending on f, is based on Lemma 1 (advanced version). The corresponding matrix $V = (v_{i,k})$ has 2^{n-1} rows (the number of points d of rank n) and infinite number of columns.

The lemma allows one to choose for each d the corresponding half so that all Fourier sums of the function

$$F = \mathbb{E}(f \circ \phi') - \mathbb{E}(f \circ \phi)$$

will be small.

More precisely:

$$\|S_r(F)(x)\| < e^{-c(j+|n-u|)}$$

for $r \in [2^{u-1}, 2^u)$, $u = 1, 2, ...$

Now by induction over j (for a fixed n) we get a random homeomorphism ϕ_n, for which all points of rank n become non-random. Then another induction, over n, provides a deterministic homeomorphism h.

Summing all the deviations, we get

Theorem 3. *There is a Holder homeomorphism h (of any order < 1) such that $f \circ h$ has bounded Fourier sums.*

5 Absolutely continuous homeomorphisms

The scheme above, started with ϕ_q, where q is a constant, allows us to get Holder but not AC homeomorphism.

To get AC we need to reduce "randomness" somehow. For this purpose, we make q not a constant number but a function of dyadic rationals d, such that the local "size of randomness" would correspond to local oscillations of f.

Roughly, if f is "flat" on a dyadic interval centered at d, then we do not need much randomness there and we can take the value $q(d)$ small. This makes the homeomorphism smoother which allows us to hope eventually for the AC property.

However the problem is whether any f has enough "flatness". On the other hand, if $q(d)$ is too small then it provides not enough randomness to kill resonances with the Dirichlet kernel.

The Haar decomposition of f plays an important role in our construction.

We set:

$$q_f(d) = |I|^{-3/2} \sum_{w \in I} <f, \chi_w>^2 |w|^{1/2},$$

where I is the dyadic interval centered at d, and w is the support of the Haar function χ_w.

Having $\{q_f\}$ for all dyadic d in $[0, 1]$, we define the "starting" homeomorphism ψ_f (in fact, ψ_f^{-1}) as above.

We prove:

1) the random homeomorphism ψ_f is absolutely continuous for sure;

2) it admits the process of "removing of randomness", similar to one described in section 4, which finally gives a deterministic homeomorphism h required in Theorem 2.

In the proof of the first claim, the John - Nirenberg inequality for dyadic BMO space is used.