SURVEY ON ALGEBRAIC COMPLEXITY

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Talk Outline

1. Introduction to Algebraic Complexity
2. Lower bound Techniques
3. Polynomial Identity Testing
4. Proof Complexity
5. Possible Barriers to Lower Bounds

Talk Outline

① Introduction to Algebraic Complexity

Algebraic Complexity Theory

$(P_N(x_1, \ldots, x_N))_N$ - sequence of polynomials

$\deg(P_N) \leq N^{O(1)}$

Input: $a \in \mathbb{C}^N$ [or $\mathbb{F}^N$]

Output: $P_N(a)$

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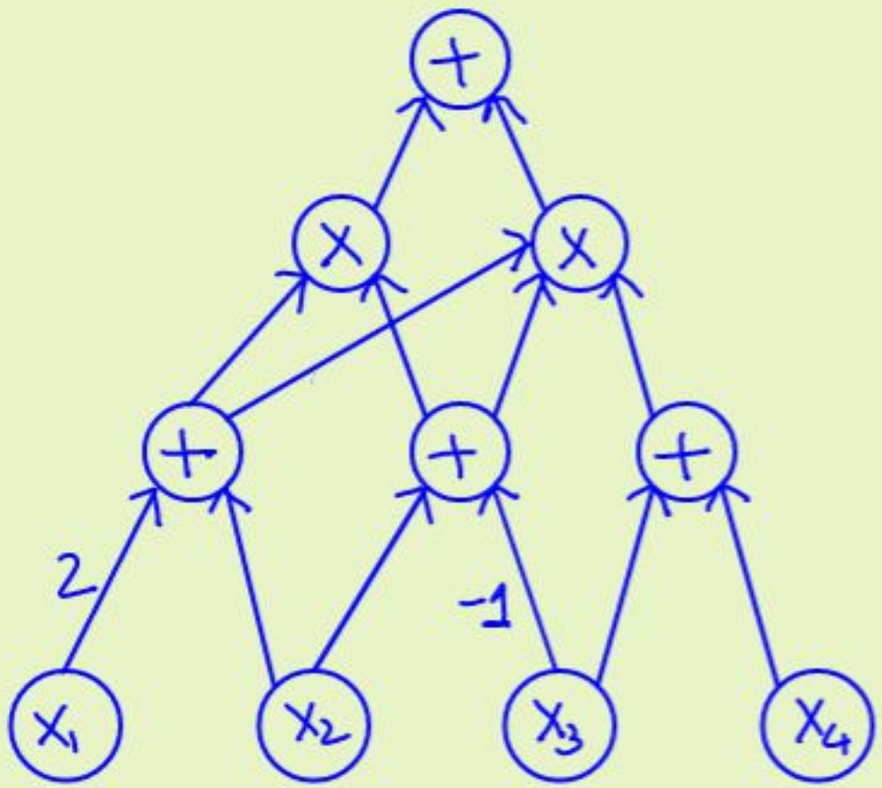
Examples: ① $\det_n\left(x_{i,j}\right)_{i,j} = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, x_{1,\sigma(1)} \cdots x_{n,\sigma(n)}$

② $\operatorname{per}_n\left(x_{i,j}\right)_{i,j}$ $\quad N = n^2,\ \deg = n$

③ $IMM_{n;d} =$ 

$N \approx d \cdot n^2$, $\deg = d$

Algebraic Circuits



Size = # of gates = 6

Depth = length of longest path

= 3

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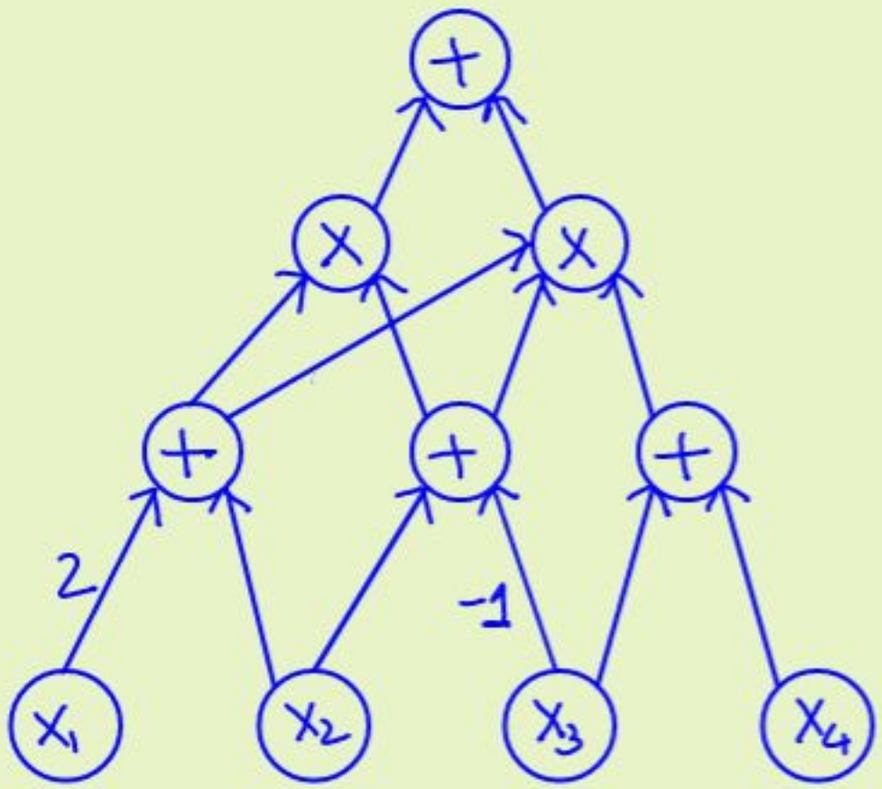

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VP = sequences of polynomials with circuits of polynomial size.

Eg: $IMM_{n,d} \in VP$

$det_n \in VP$

$per_n \notin VP$?

[VP vs. VNP]

VP vs. VNP

→ Special (syntactic) case of P/poly vs. NP
[Bür]

→ Concrete approaches
  → Syntactic simplification
  → Depth-reduction

Other Syntactic Restrictions

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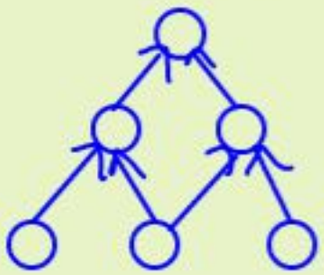
Homogeneous circuits: Each intermediate computation is homogeneous.



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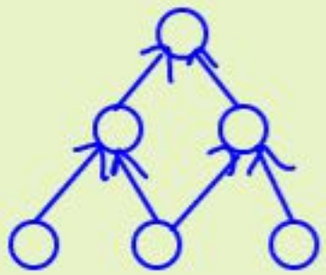


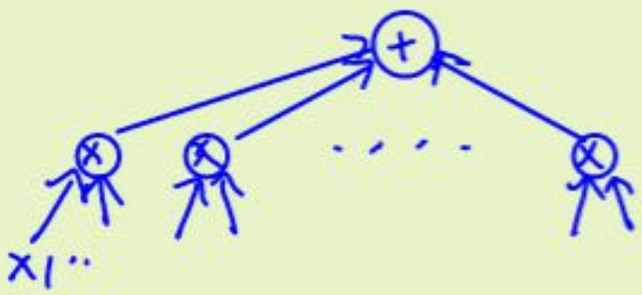
→ Sometimes, restricted circuits ≈ general circuits. [Good for lbds!]

→ Interesting situations where this may not be true.

Constant-depth Algebraic circuits

$$P(x_1, \ldots, x_N) = \sum_{e_i \geq 0 : \sum_i e_i \leq d} \alpha_{e_1, \ldots, e_N} x_1^{e_1} \cdots x_N^{e_N}$$

$\deg(P) = d$



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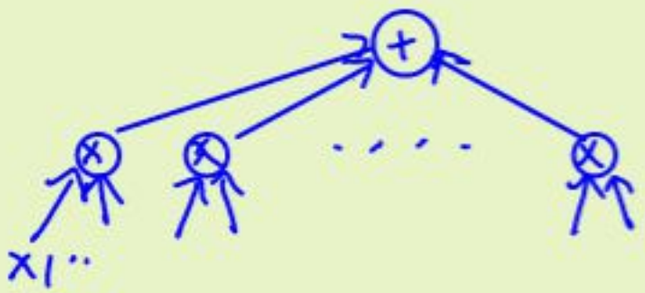

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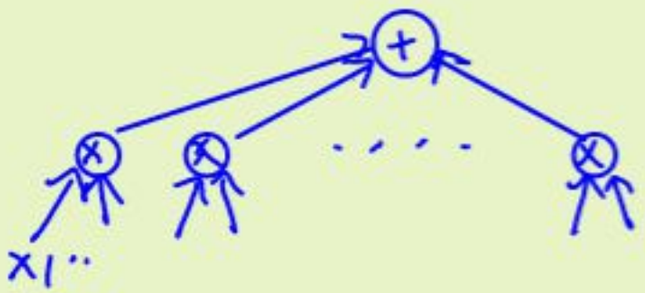

$\rightarrow$ $\Sigma\Pi$ circuit for P

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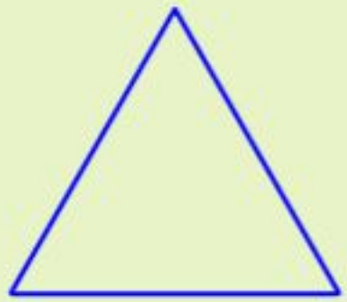
$\Sigma\Pi\Sigma$, $\Sigma\Pi\Sigma\Pi$, ...

↳ much harder to reason about!

Depth-Reduction [Brent, VSBR]



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General circuit $\Longrightarrow$ Const. depth circuit
poly. size subexponential size

Superpoly. lbds for general ckts $\Longleftarrow$ Exponential const. depth lower bounds

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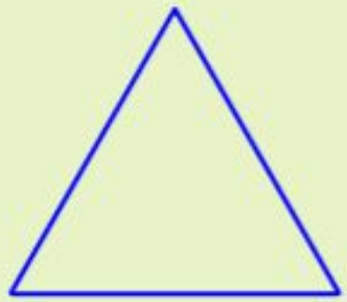

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$\Sigma\Pi\Sigma\Pi$ size $N^{O(\sqrt{d})}$

Poly-sized $\Longrightarrow$ homogeneous [AV, Koi, Tav]
ckt. for

$P(x_1, \ldots, x_N)$

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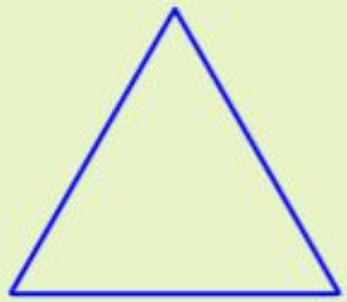

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inhomogeneous [akks] (*)

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① Introduction to Algebraic Complexity

② Lower bound Techniques

Constant-depth circuit lower bounds

[NW]: Partial Derivative Method

→ "Complexity of $P(x_1, \ldots, x_N)$" = rank of matrix M_P

Constant-depth circuit lower bounds

[NW]: Partial Derivative Method

→ "Complexity of $P(x_1, \ldots, x_N)$" = rank of matrix M_P

→ Small circuit ⇒ small rank

→ $rk(M_1 + M_2) \leq rk(M_1) + rk(M_2)$

Example : Tensor Rank

$$\{x_1^{(1)}, \ldots, x_n^{(1)}\}, \{x_1^{(2)}, \ldots, x_n^{(2)}\}, \ldots, \{x_1^{(d)}, \ldots, x_n^{(d)}\}$$

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$$\underbrace{Q(\bar{x}^{(1)}, \dots, \bar{x}^{(d)})}$$

$$\sum_{i_1, \dots, i_d} \alpha_{i_1, \dots, i_d} x_{i_1}^{(1)} \cdots x_{i_d}^{(d)}$$

set-multilinear poly.

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$\min s =$ Tensor-Rk(Q)

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set-multilinear poly.

$$M_p = x_{i_1}^{(1)} \cdots x_{i_{d/2}}^{(d/2)} \left(\text{coeff}\left(x_{i_1}^{(1)} \cdots x_{i_d}^{(d)}\right) x_{i_{d/2+1}}^{(d/2+1)} \cdots x_{i_d}^{(d)} \right)$$

$n^{d/2} \times n^{d/2}$

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– Columns are partial derivatives of P

Example : Tensor Rank

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$$Q(\vec{x}^{(1)}, \dots, \vec{x}^{(d)}) = l_1^{(1)} \cdots l_1^{(d)} + \cdots + \underbrace{l_s^{(1)} \cdots l_s^{(d)}}_{\text{rank} \leq 1}$$

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explicit Q

s.t. $rk(M_Q) \ge n^{d/2} \Rightarrow s \ge n^{d/2}$

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$l_s^{(1)} \cdots l_s^{(d)}$: rank ≤ 1

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$Q = \langle x_1, x_{d/2+1} \rangle \langle x_2, x_{d/2+2} \rangle \cdots \langle x_{d/2}, x_d \rangle$

"Product of Inner Products"

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"Product of Inner Products"

$M_Q = I_n \otimes I_n \otimes \cdots$

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$\Downarrow$

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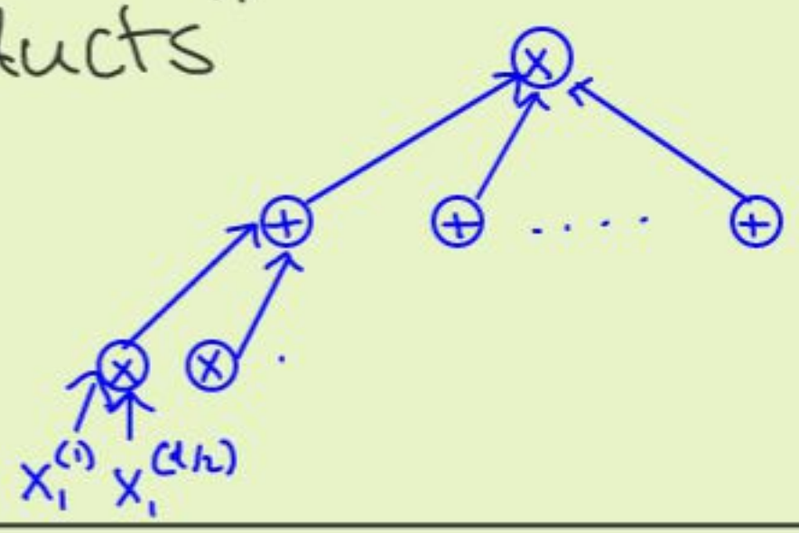

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→ Partial derivatives + Random Restrictions.

→ Necessary because Products of Inner Products is $\Sigma\Pi\Sigma\Pi$.

Shifted Partial Derivative Method [GKKS]

→ lower bounds against homogeneous $\Sigma\Pi\Sigma\Pi$ circuits

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→ derivative $\times$ monomial of degree $\leq \ell$

$$M_P = \left(\right)$$

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Hard polynomial = $IMM_{n,d}$

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→ Product of Inner Prods $(X^{(1)}, \ldots, X^{(d)})$

$$= \left(\sum_{i=1}^{n} X_i^{(1)} X_i^{(d_1+1)}\right)\left(\sum_{i=1}^{n} X_i^{(2)} X_i^{(d_1+2)}\right)\ldots$$

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→ Avoiding PIP: variable sets have diff-erent sizes!

Eg: $|X^{(1)}| = |X^{(2)}| = \cdots = |X^{(t)}| = n$

$|X^{(t+1)}| = \cdots = |X^{(d)}| = n^{\alpha}$

s.t. $|p-q\alpha|$ large for "small" p, q.

Lopsided Partial derivatives [LST]

Thm: Lbd. of $N^{-\Omega(\sqrt{d})}$ against $\Sigma\Pi\Sigma\Pi\Sigma$ set-multilinear circuits.

Lopsided Partial derivatives [LST]

Thm: Lbd. of $N^{-2(\sqrt{d})}$ against $\Sigma\Pi\Sigma\Pi\Sigma$ set-multilinear circuits.

Thm [SW, CKSV]: Depth-structure tradeoff.(*)

$\Sigma\Pi\Sigma$ size $s \Rightarrow$ $\Sigma\Pi\Sigma\Pi\Sigma$ s.m. size $s \cdot d^{O(d)}$

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Lopsided Partial derivatives [LST]

Thm: Lbd. of $N^{d^{\exp(-\Delta)}}$ against depth Δ set-multilinear circuits.

Thm [LST]: Depth-structure tradeoff$^{(*)}$.

Depth Δ, size s $\Rightarrow$ Depth $2\Delta+1$, s.m. size $s \cdot d^{O(d)}$

Cor: $d < \sqrt{\log N}$ $\Rightarrow$ $N^{d^{\exp(-\Delta)}}$ lbd. against depth Δ

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② Lower bound Techniques

③ Polynomial Identity Testing

Polynomial Identity Testing

$$(x_1^2+x_2^2+x_3^2+x_4^2)\cdot(y_1^2+y_2^2+y_3^2+y_4^2) = (x_1y_1-x_2y_2-x_3y_3-x_4y_4)^2 + (x_1y_2+x_2y_1+x_3y_4-x_4y_3)^2 + (x_1y_3-x_2y_4+x_3y_1+x_4y_2)^2 + (x_1y_4+x_2y_3-x_3y_2+x_4y_1)^2$$

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Input: $P(x_1,\dots,x_N)$ (as a circuit)

Output: Is $P=0$?

Deterministic Algorithms for PIT & Lower bounds

→ Rand algo : ① Fix $S \subseteq \mathbb{F}$. $|S| \geq 10d$.

② Pick $a \in_R \mathbb{F}^n$. Test if $P(a) = 0$.

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Thm $\begin{bmatrix} KI, \\ Agr \end{bmatrix}$ PIT algos $\Rightarrow$ Lower bounds

→ PIT algos. for weak circuit classes

→ Hardness - randomness tradeoffs.

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[GKSS]: Lower bounds ⟹ Poly-time PIT. for $O(1)$-variable polys

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Cor: Lower bounds for $IMM_{n,\sqrt{\log n}} \Rightarrow$ PIT

Talk Outline

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② Lower bound Techniques

③ Polynomial Identity Testing

④ Proof Complexity

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→ Lbds for restricted ckts [FSTW]

→ Lbds for $O(1)$-depth over char 0 [AF, GHT]

Talk Outline

1. Introduction to Algebraic Complexity
2. Lower bound Techniques
3. Polynomial Identity Testing
4. Proof Complexity
5. Possible Barriers to Lower Bounds

Barriers to rank-based techniques

→ Rank-based technique

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Thm: [EGOW] No rank-based technique can prove $\gg n^{d/2}$ - lbds on tensor rank.

[MGOW]: Extensions to higher notions of rank.

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→ Succinct hitting sets ⇒ no natural lower bound against VP

→ Evidence for [FSV, BIJL] & against [GKRST]

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What I didn't cover

→ Monotone Algebraic Circuits

→ GCT & Border Complexity

→ Symmetric Algebraic Circuits

→ Factorizations of Algebraic Circuits.

→ Reconstruction

⋮

Open Questions

→ Stronger lower bounds.

→ Lower bounds for depths $\gg \log \log_5 d$

→ char > 0 [GKKS, CICS, GKSS, LST, ...]

→ Lower bounds for syntactic simpli-
-fication

→ Hierarchy theorems

→ Optimal hardness-randomness tradeoffs

→ Other meta-complexity questions