TFNP:

- Collapses
- Separations
- Characterisations

Göös, Hollender, Jain, Maystre, Pires, Robere, Tao

TFNP = Total Function NP [Papa 80s]

Polytime $R(x,y)$

Input: x

Output: y: $R(x,y)=1$ & $|y| \leq |x|^{O(1)}$

Promise: R is TOTAL $\forall x \exists y : R(x,y)$

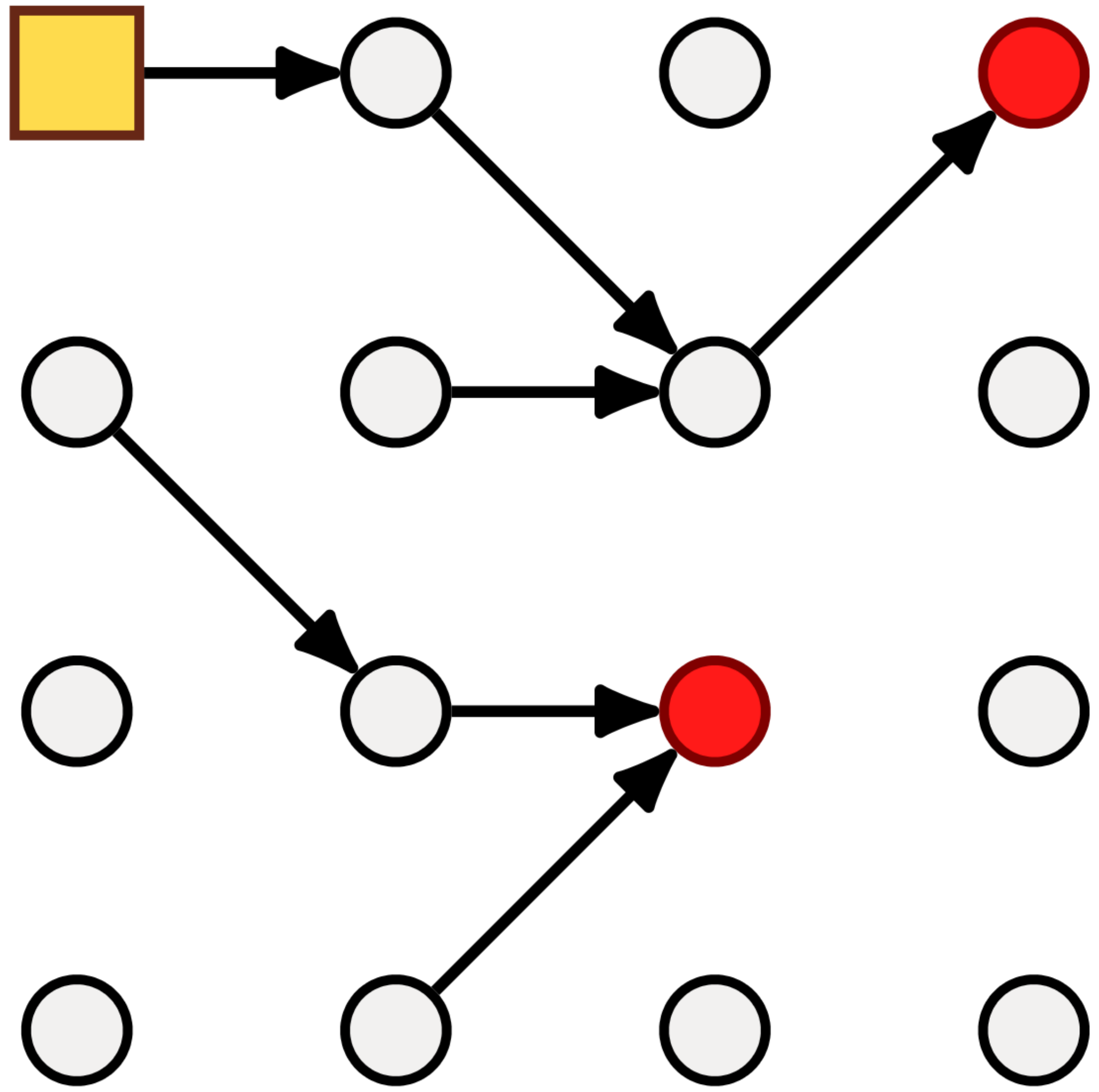
Sink-of-DAG

input Circuit $S: \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n \cup \{\perp\}$

Defines DAG on grid $[N] \times [N]$, $N := 2^n$

output sink node

(or $(1,1)$ is $\perp$)



Def: $\mathrm{PLS} = \{ P \in \mathrm{TFNP} : P \leq \mathrm{SoD} \}$

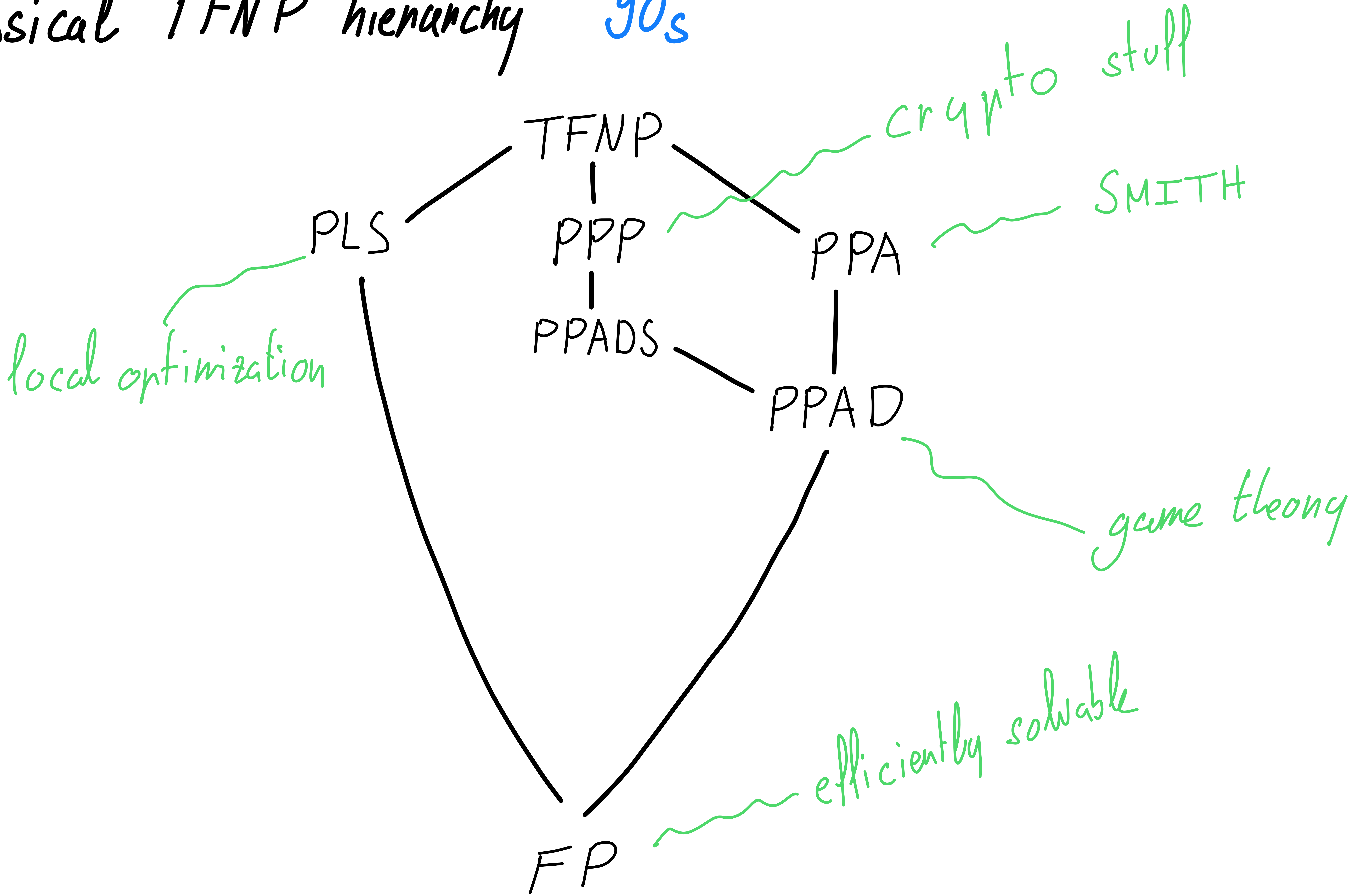
$A \leq B$ iff $\exists$ polytime f, g

input x to $A \xmapsto{\;f\;} f(x)$ to B

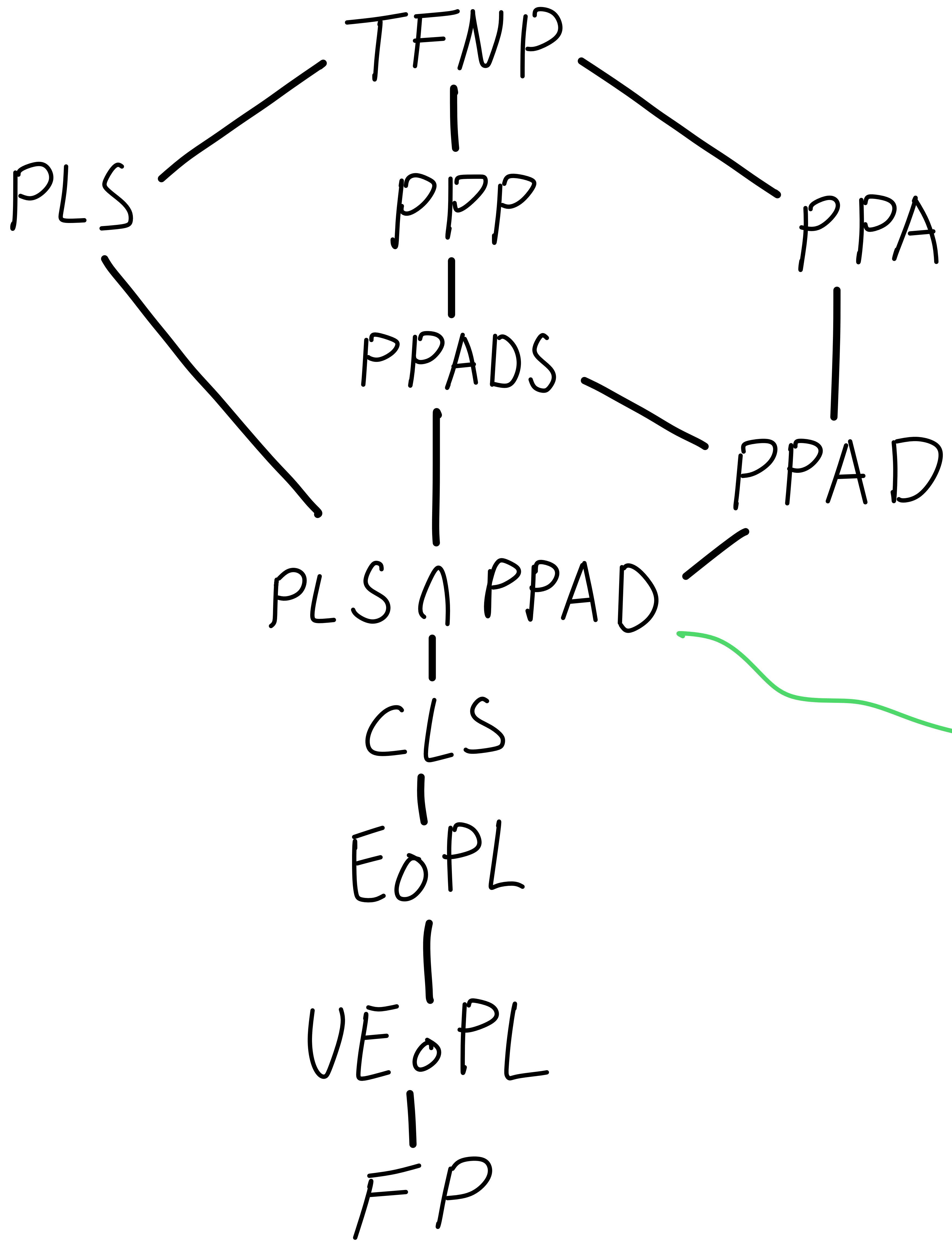
Alg for B

$A(x, g(x,y)) = 1 \xleftarrow{\;g\;} y : B(f(x), y) = 1$

The classical TFNP hierarchy 90s



New lower classes (10s)

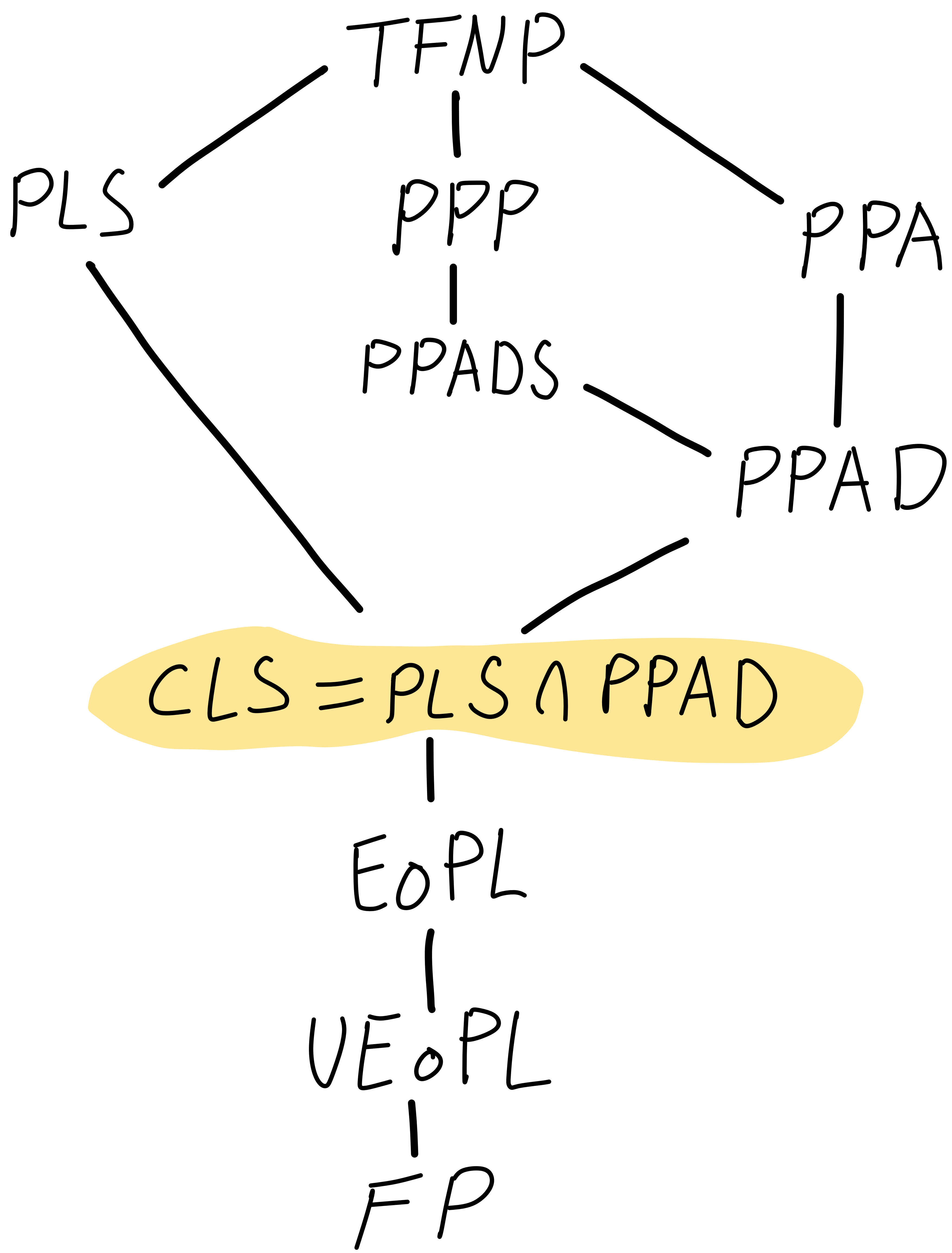


un-natural class that contains many interesting problems

A breakthrough collapse (2020)

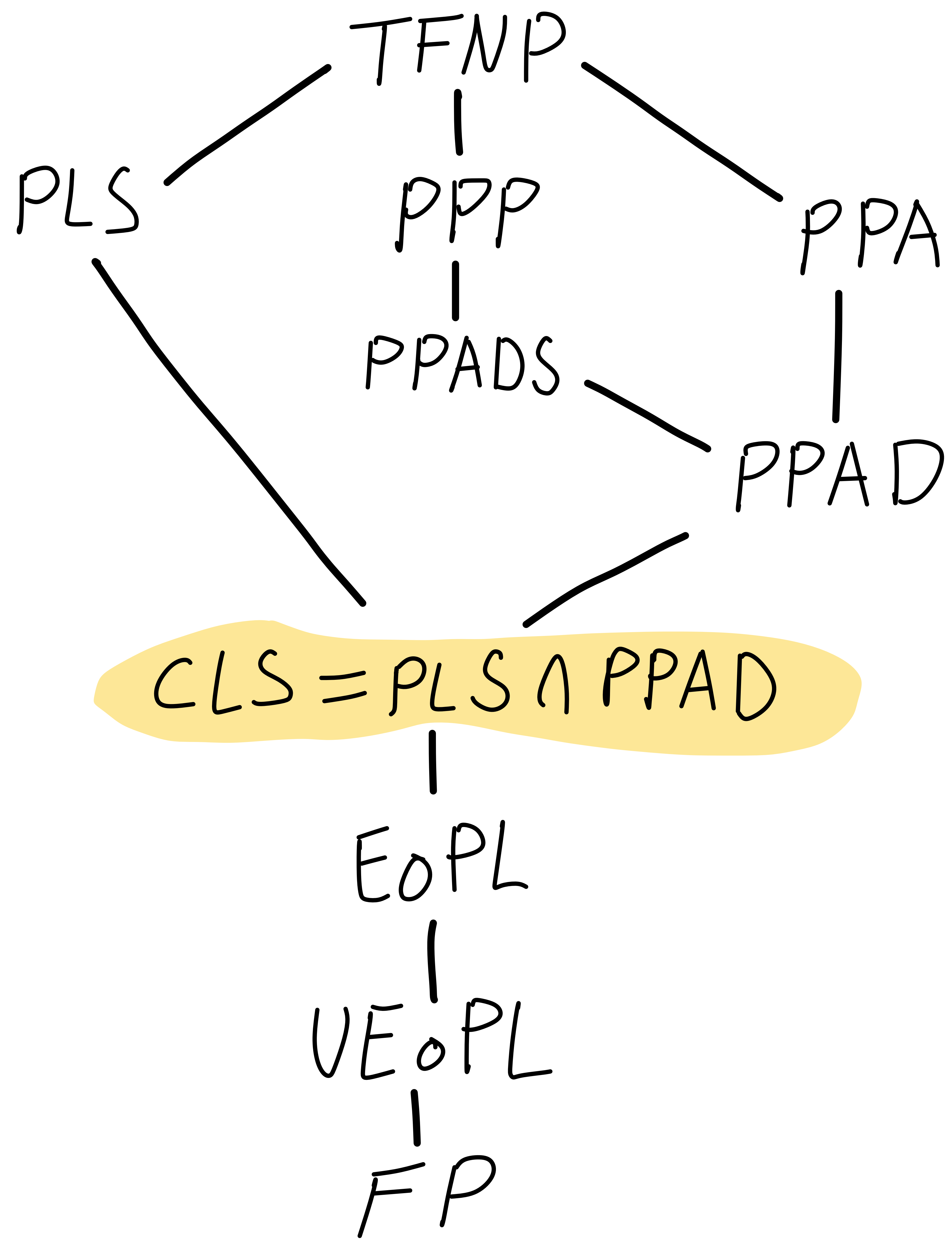
[FGHS 20']

Best Paper!



A breakthrough collapse (2020)

[FGHS 20']



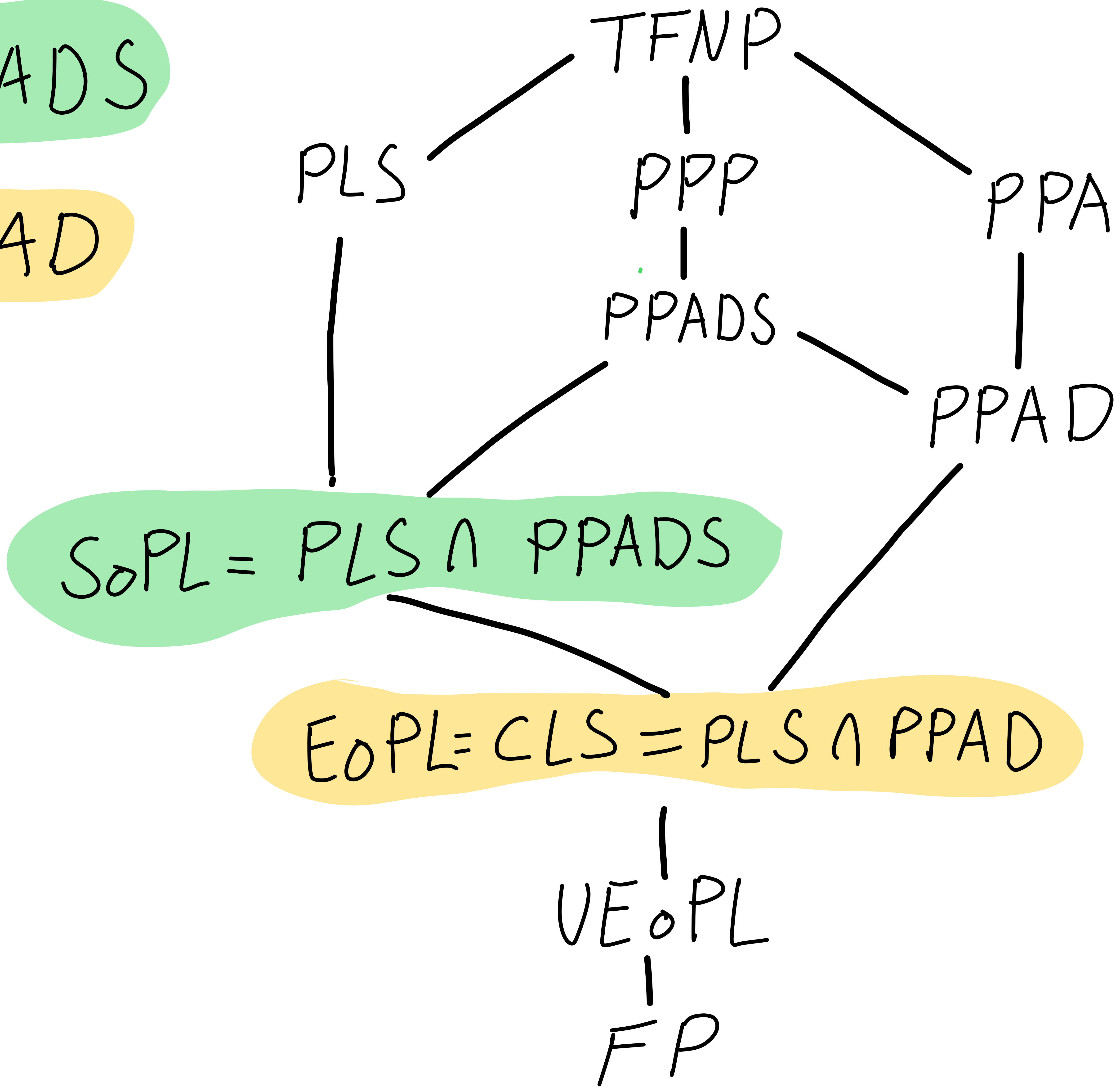
1. Are there any other collapses?

2. Can we argue against collapses?

Result I: two further collapses

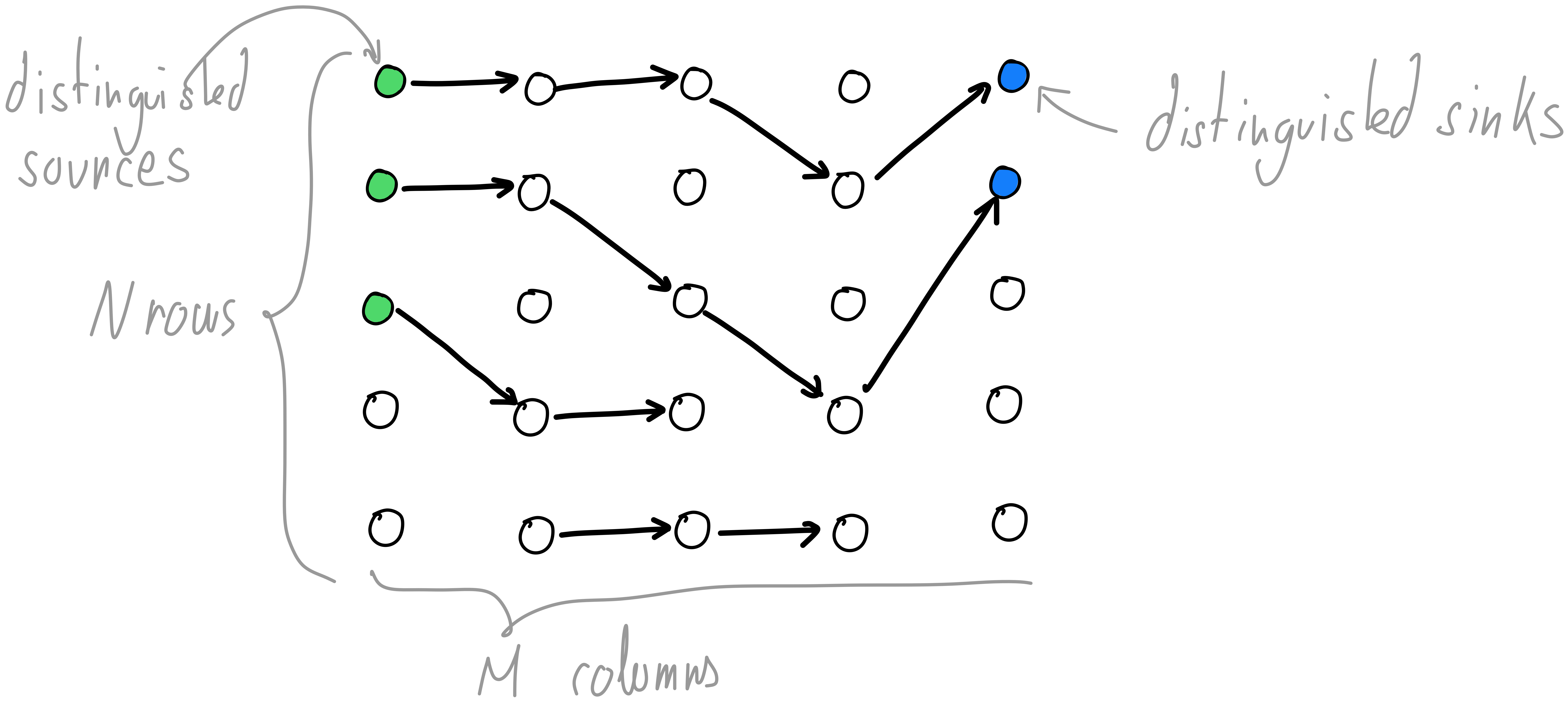
Theorem 1: $\mathrm{SoPL} = \mathrm{PLS} \cap \mathrm{PPADS}$

Theorem 2: $\mathrm{EoPL} = \mathrm{PLS} \cap \mathrm{PPAD}$



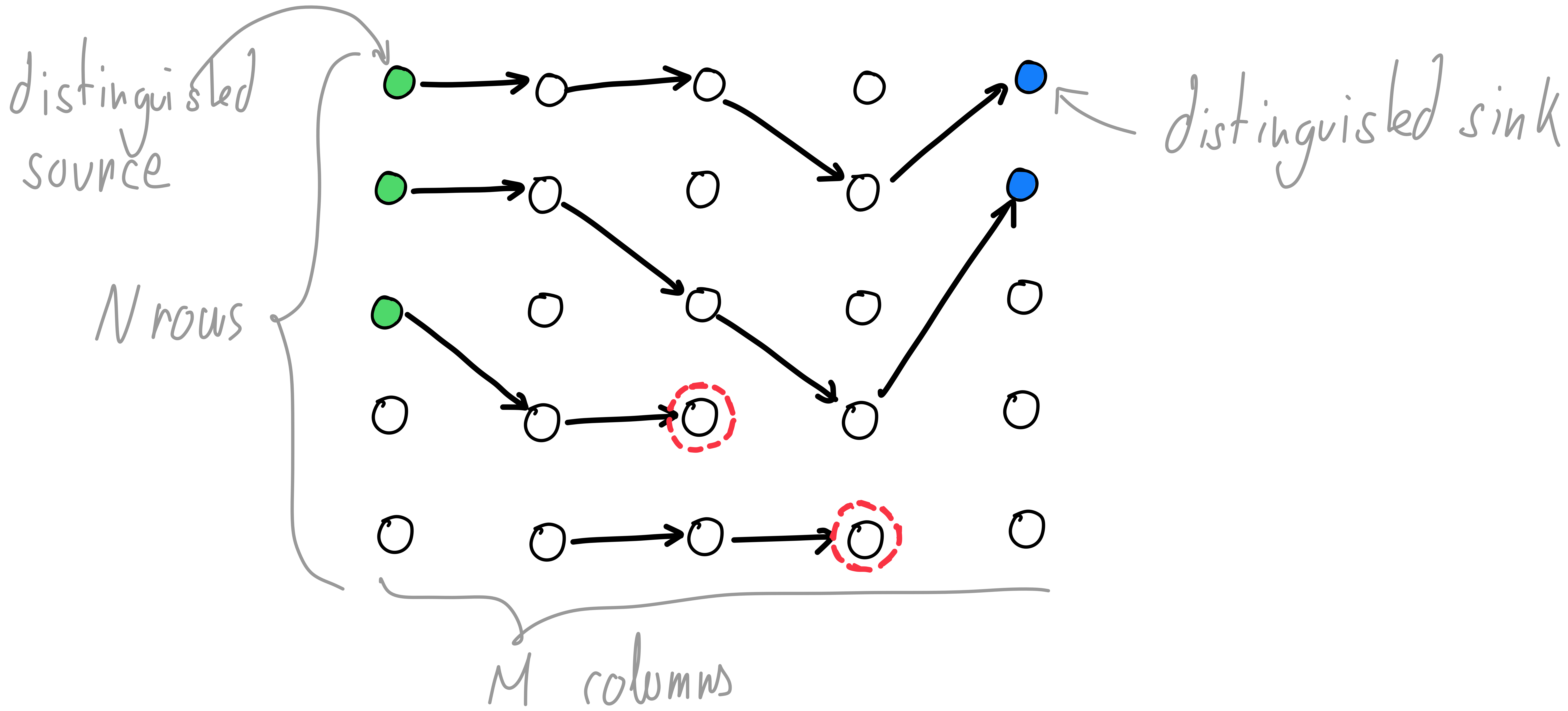
More on EoPL = PLS ∩ PPAD

Idea 1: a grid view of TFNP



More on EoPL = PLS ∩ PPAD

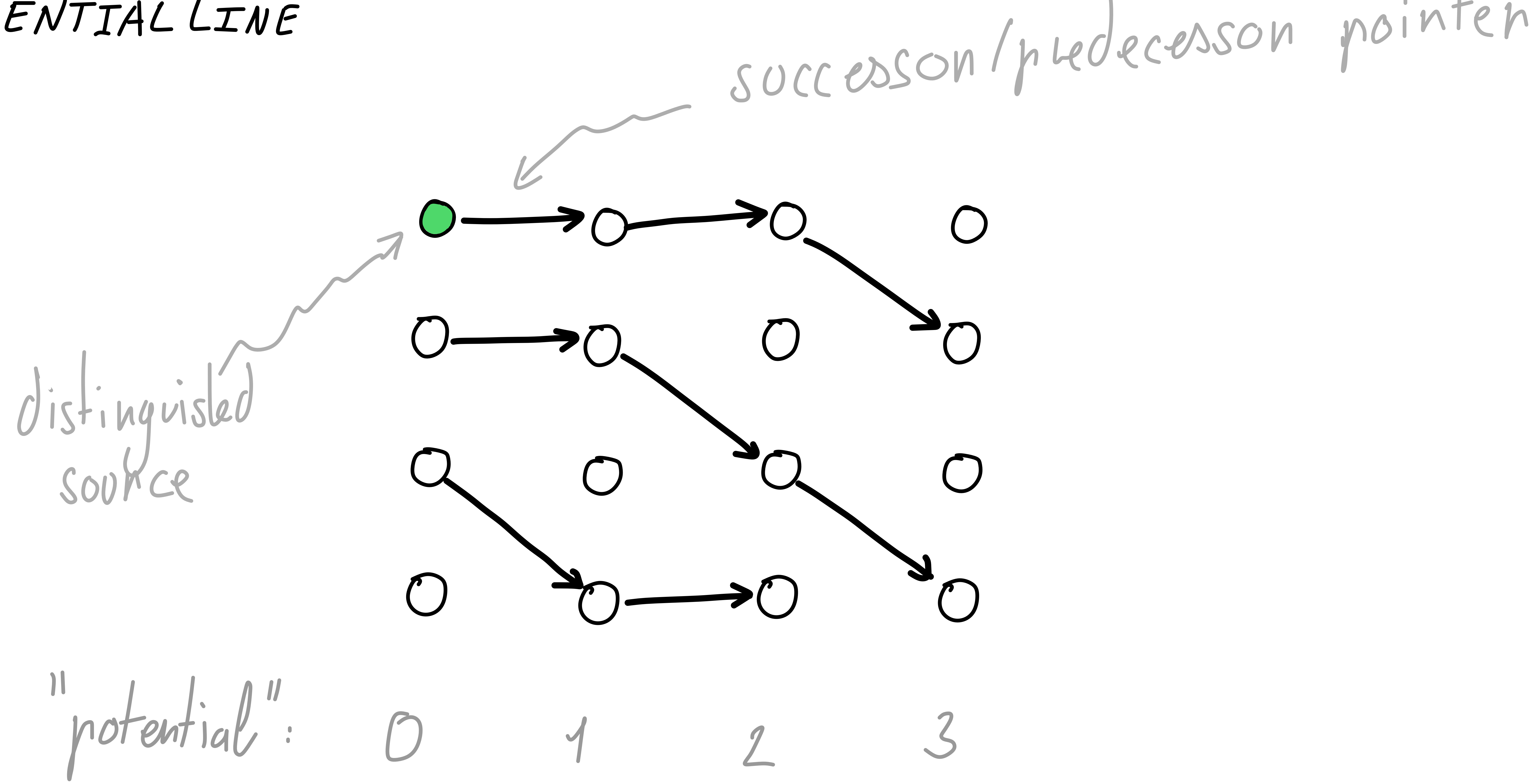
Idea 1: a grid view of TFNP



More on EoPL = PLS ∩ PPAD

Idea 1: a grid view of TFNP

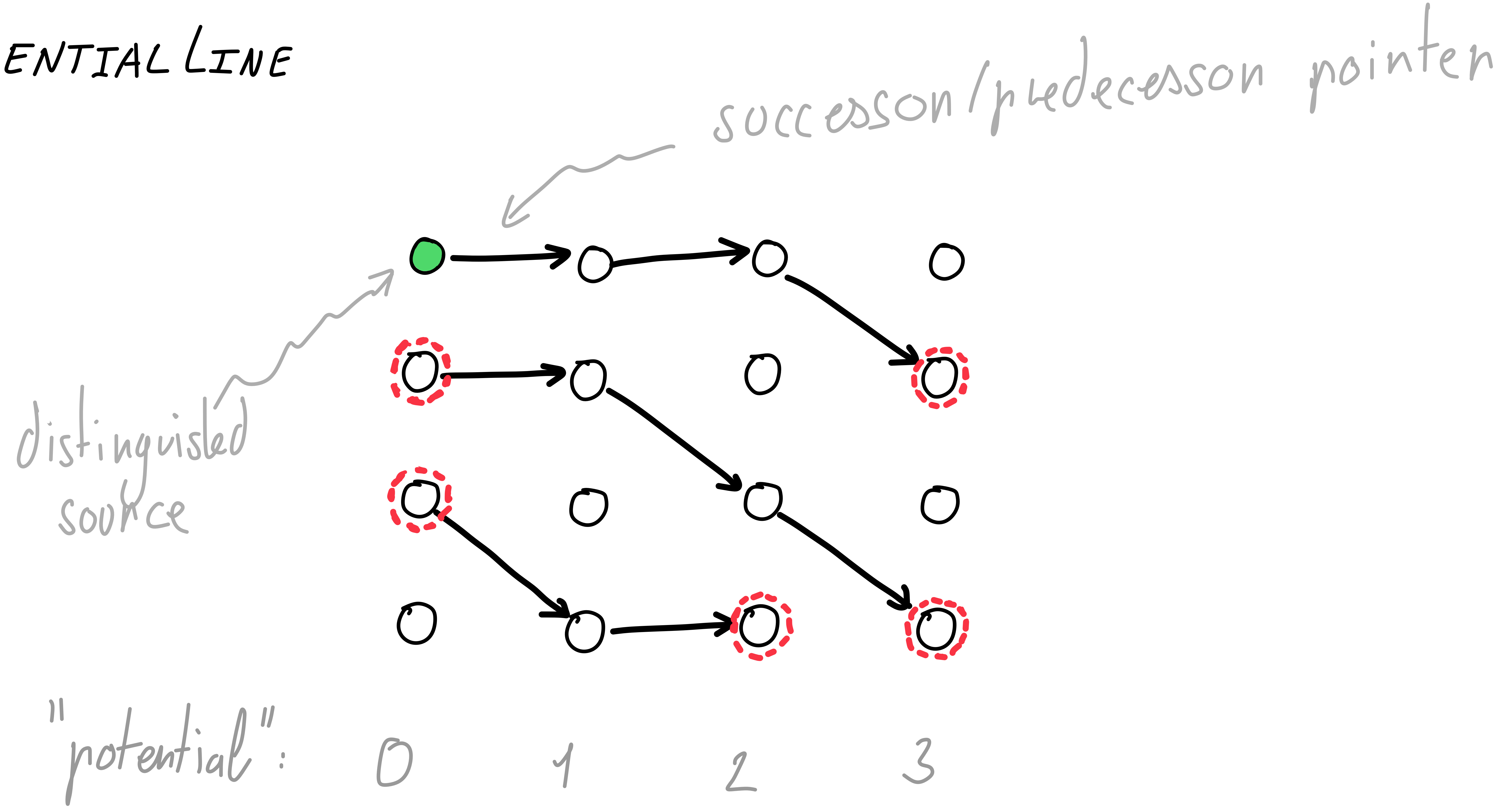
EoPL : END OF POTENTIAL LINE



More on EoPL = PLS ∩ PPAD

Idea 1: a grid view of TFNP

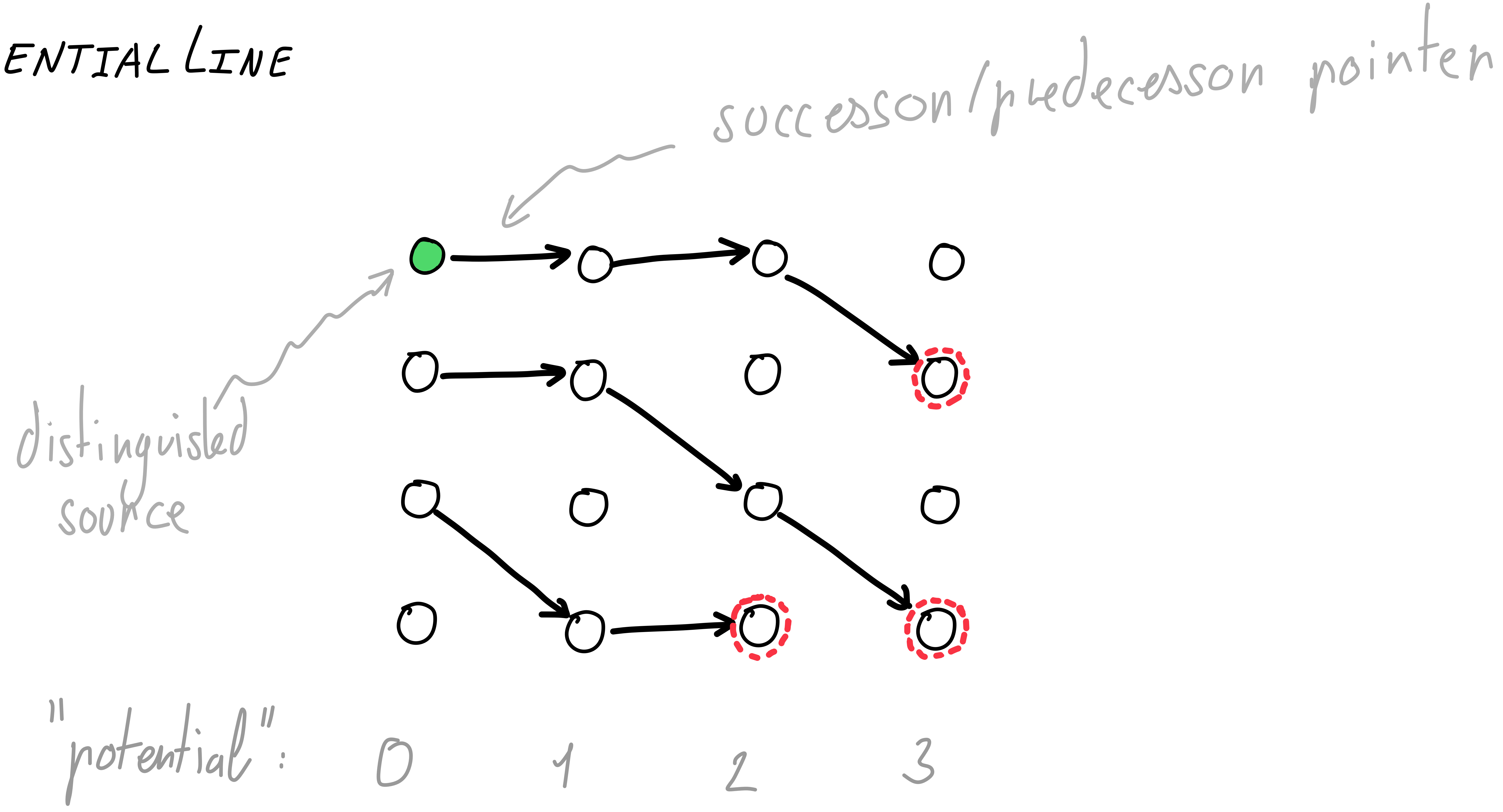
EoPL : End Of Potential Line



More on EoPL = PLS ∩ PPAD

Idea 1: a grid view of TFNP

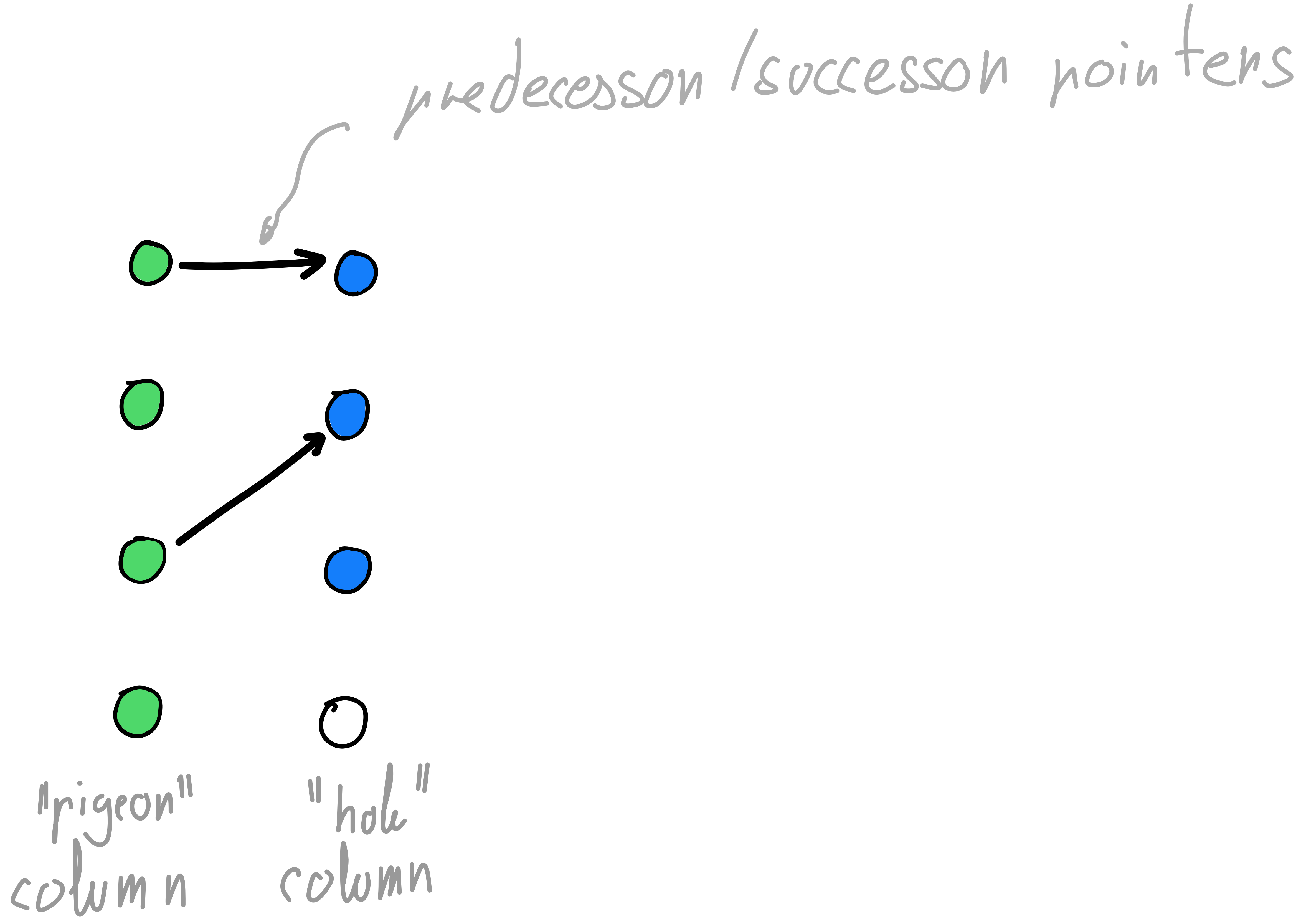
SoPL: SinkOfPotentialLine



More on EoPL = PLS ∩ PPAD

Idea 1: a grid view of TFNP

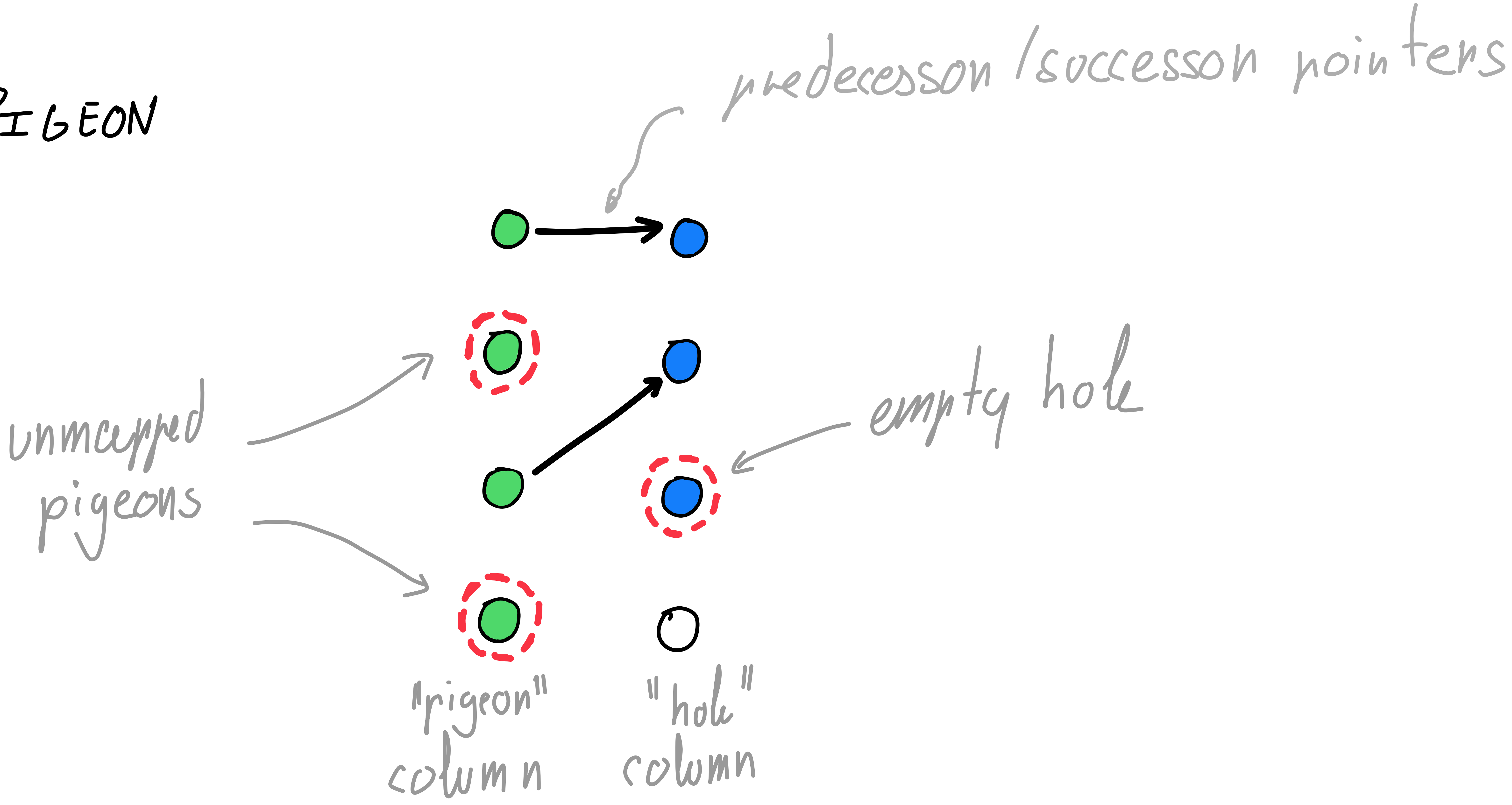
PPAD : BIJECTIVEPIGEON



More on EoPL = PLS ∩ PPAD

Idea 1: a grid view of TFNP

PPAD : BIJECTIVEPIGEON



More on EoPL = PLS ∩ PPAD

Idea 2: use SoPL = PLS ∩ PPADS

- Only need PLS ∩ PPAD ⊆ EoPL
- enough: SoPL ∩ PPAD ⊆ EoPL
- complete problem for SoPL ∩ PPAD:

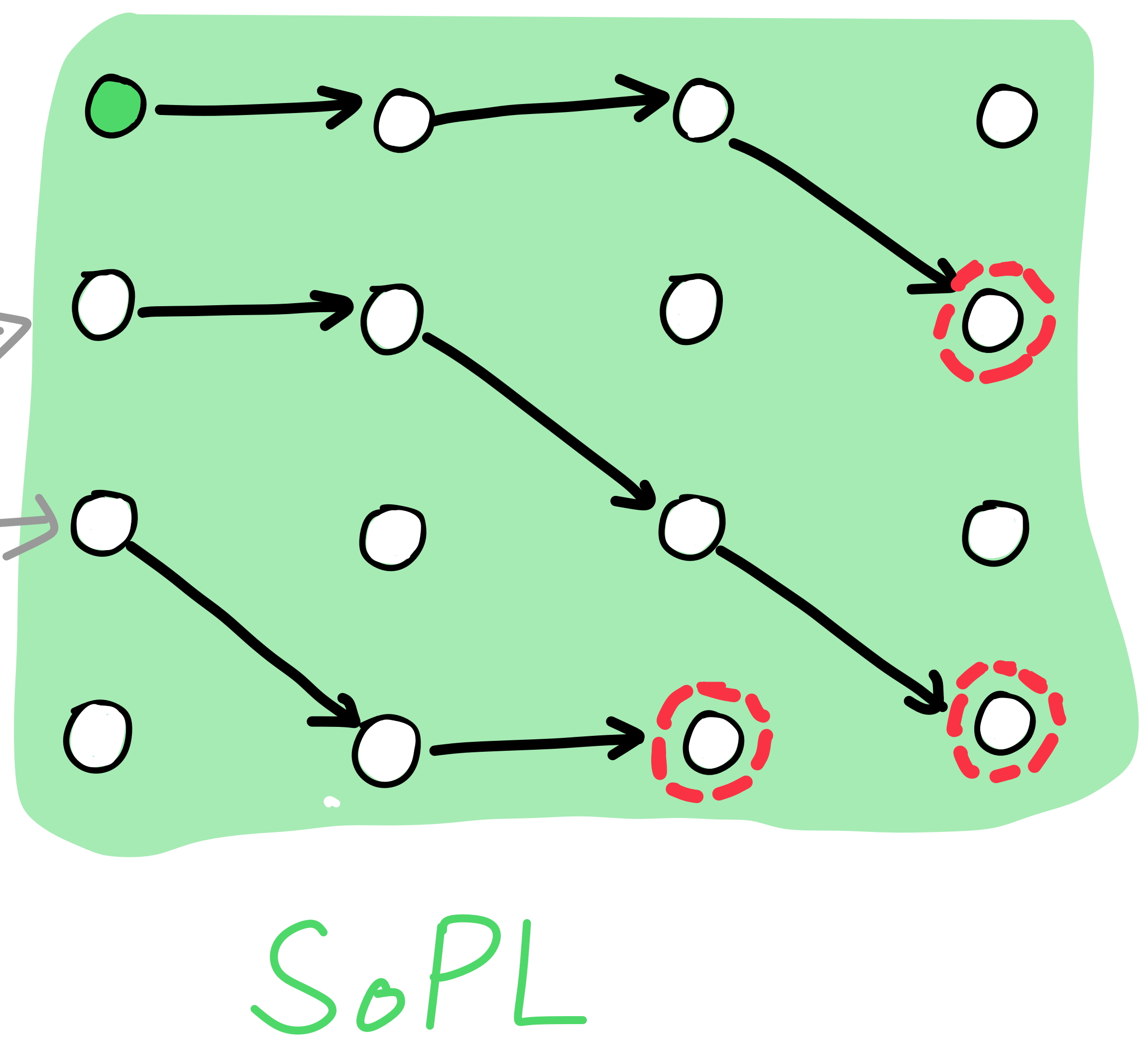
EITHER(SINKOFPOTENTIALLINE, BIJECTIVEPIGEON)

receive two instances, solve any of them

More on EoPL = PLS ∩ PPAD

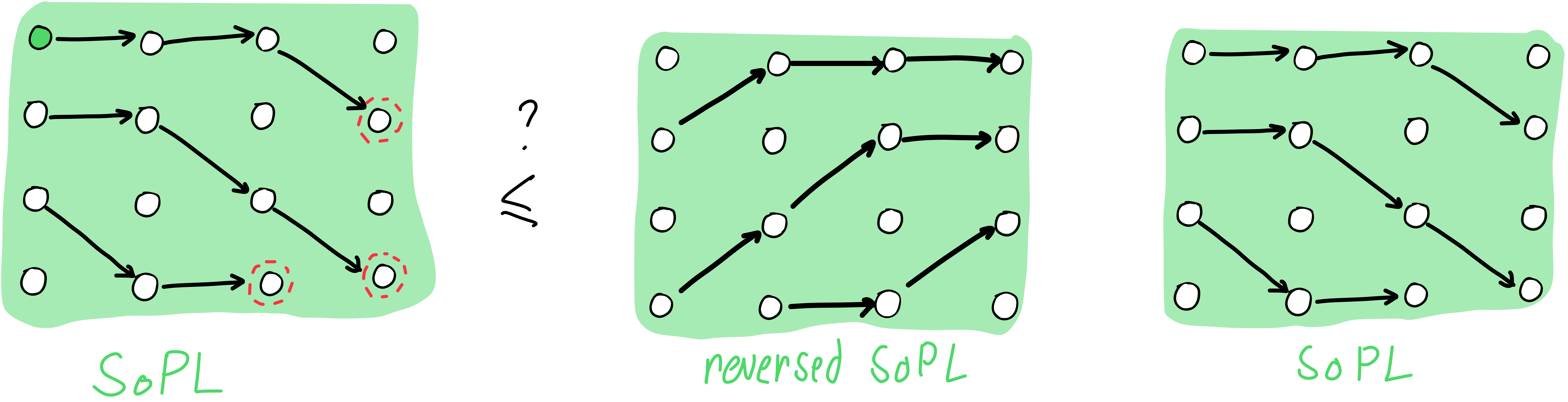
- Only need EITHER(SoPL, BigPigeon) ≤ END OF POTENTIAL LINE
- can we do SoPL ≤ END OF POTENTIAL LINE ?

sources are EoPL solution but not SoPL ones



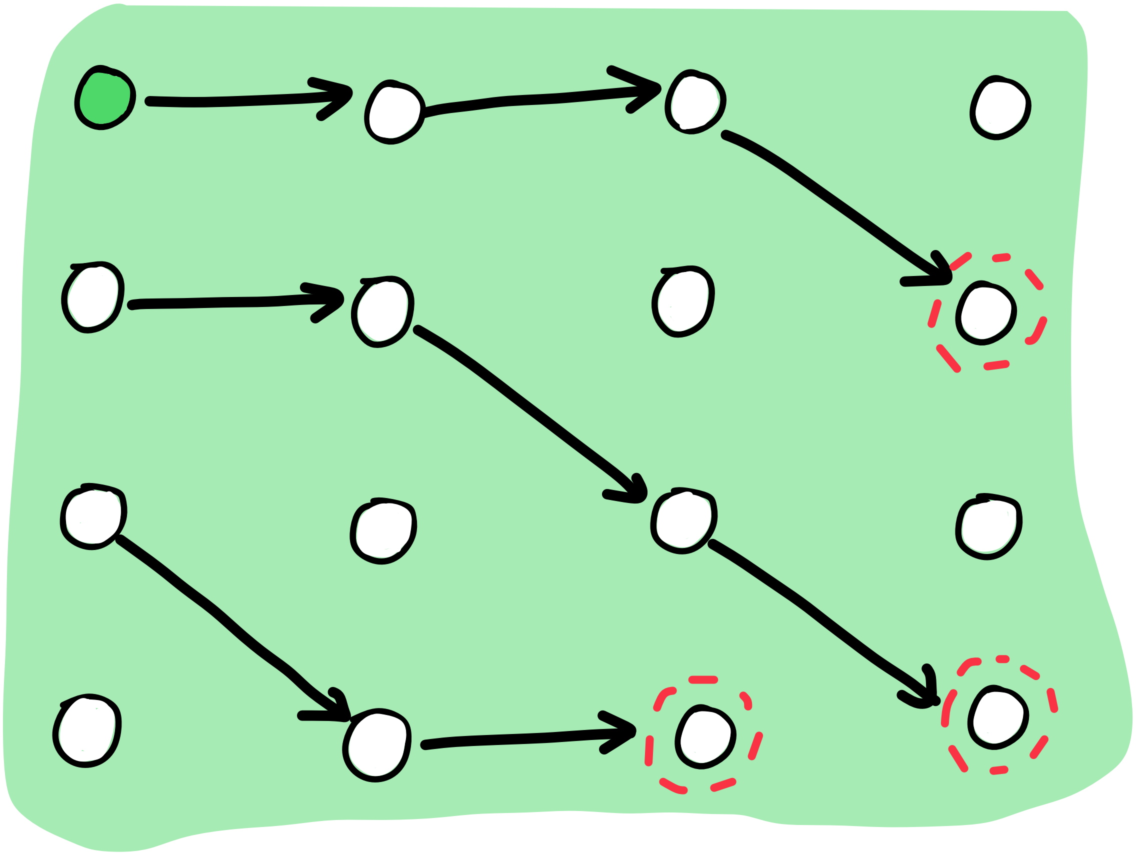
More on EoPL = PLS ∩ PPAD

- Only need EITHER(SoPL, BigPigeon) ≤ EndOfPotentialLine
- can we do SoPL ≤ EndOfPotentialLine ?

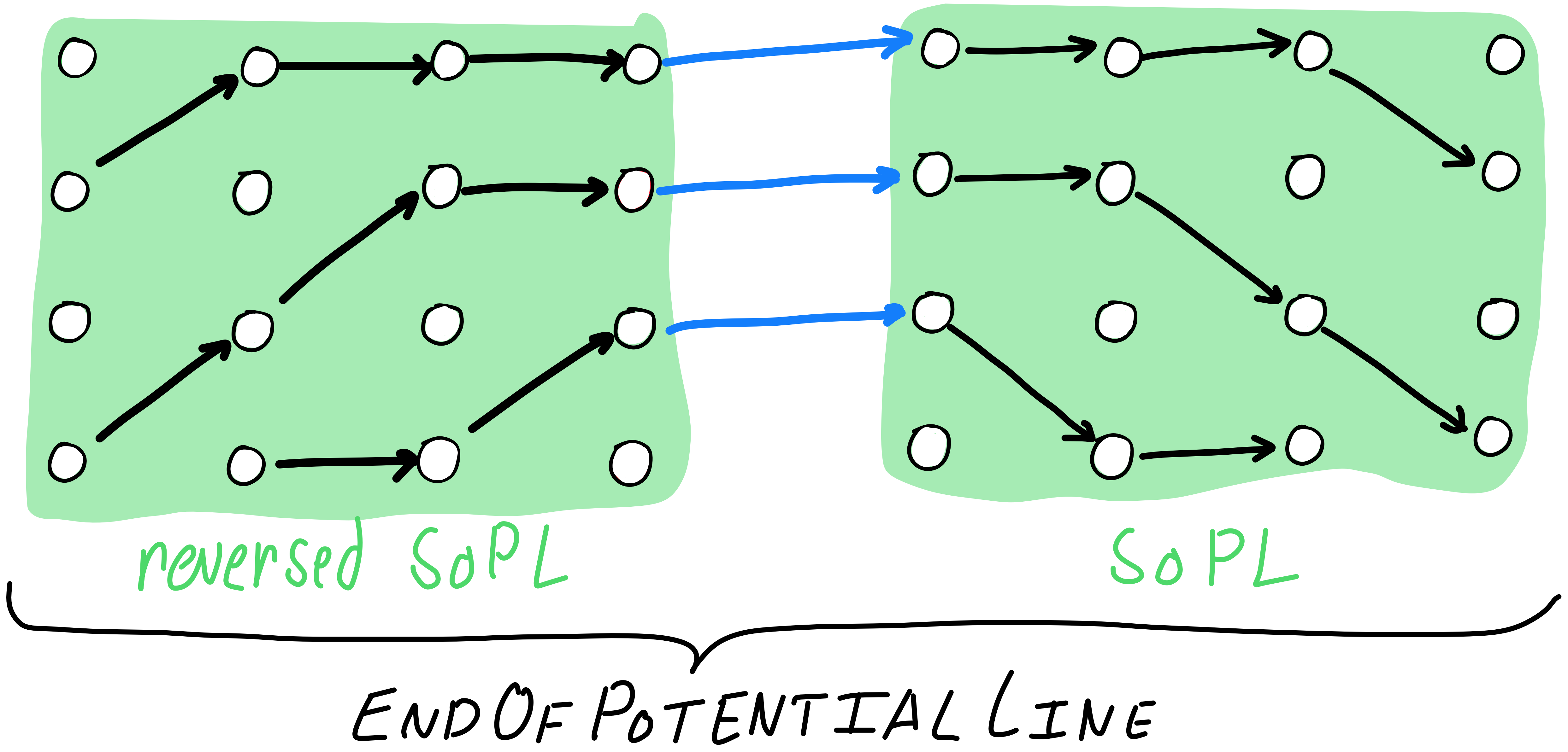


More on EoPL = PLS ∩ PPAD

- Only need EITHER(SoPL, BigPigeon) ≤ EndOfPotentialLine
- can we do SoPL ≤ EndOfPotentialLine ?



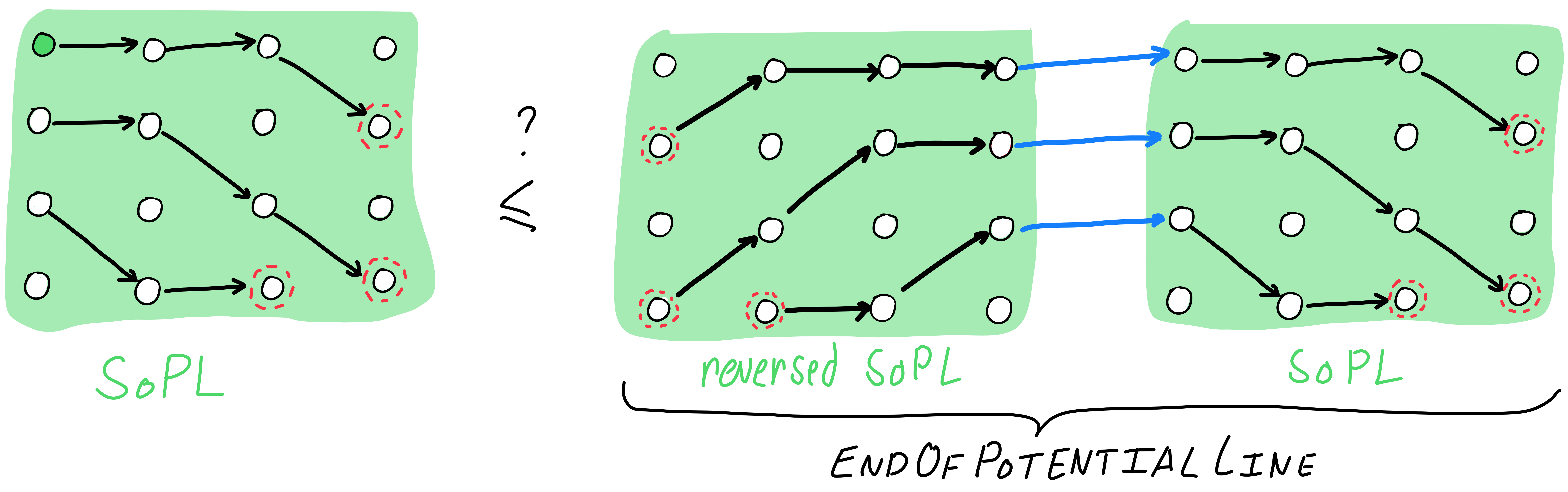
?
≤



More on EoPL = PLS ∩ PPAD

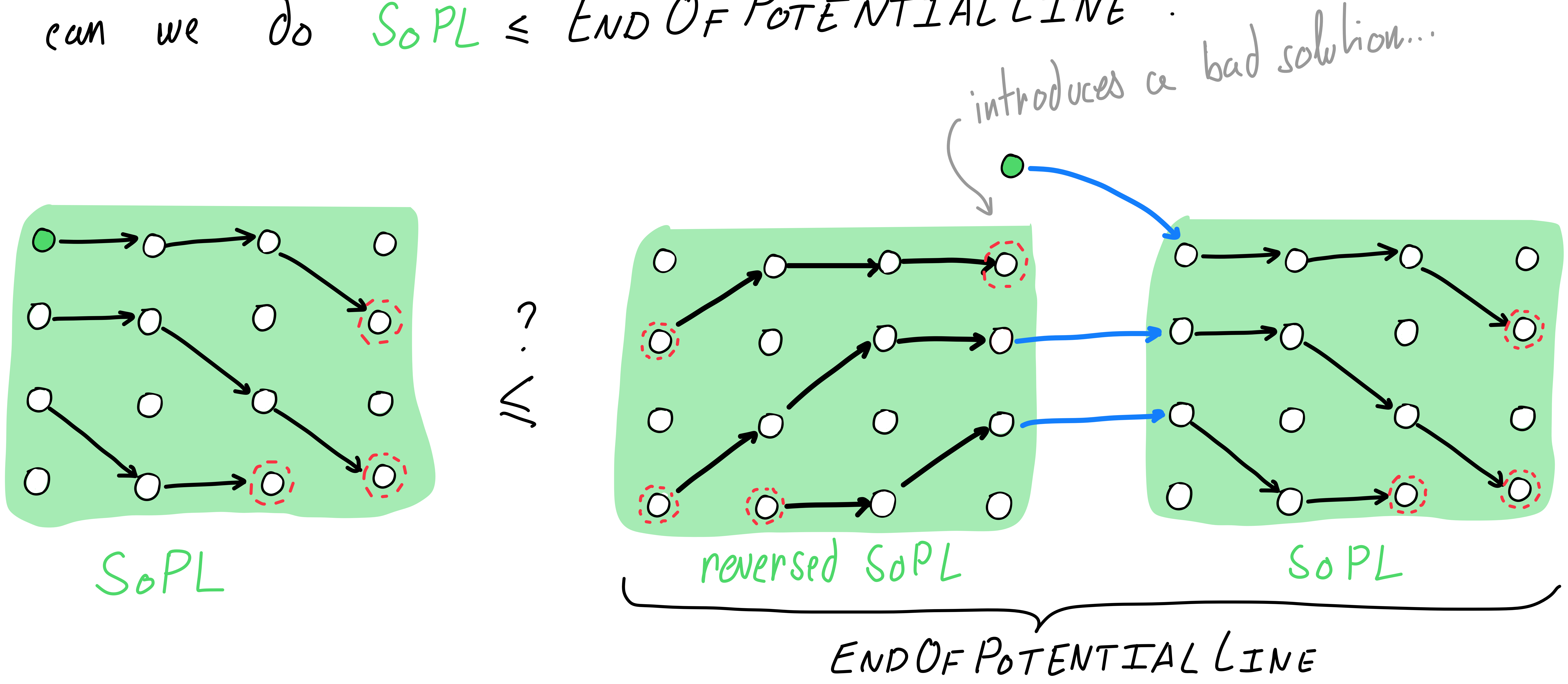
- Only need EITHER(SoPL, BigPigeon) ≤ EndOfPotentialLine
- can we do SoPL ≤ EndOfPotentialLine ?

where to put the source?



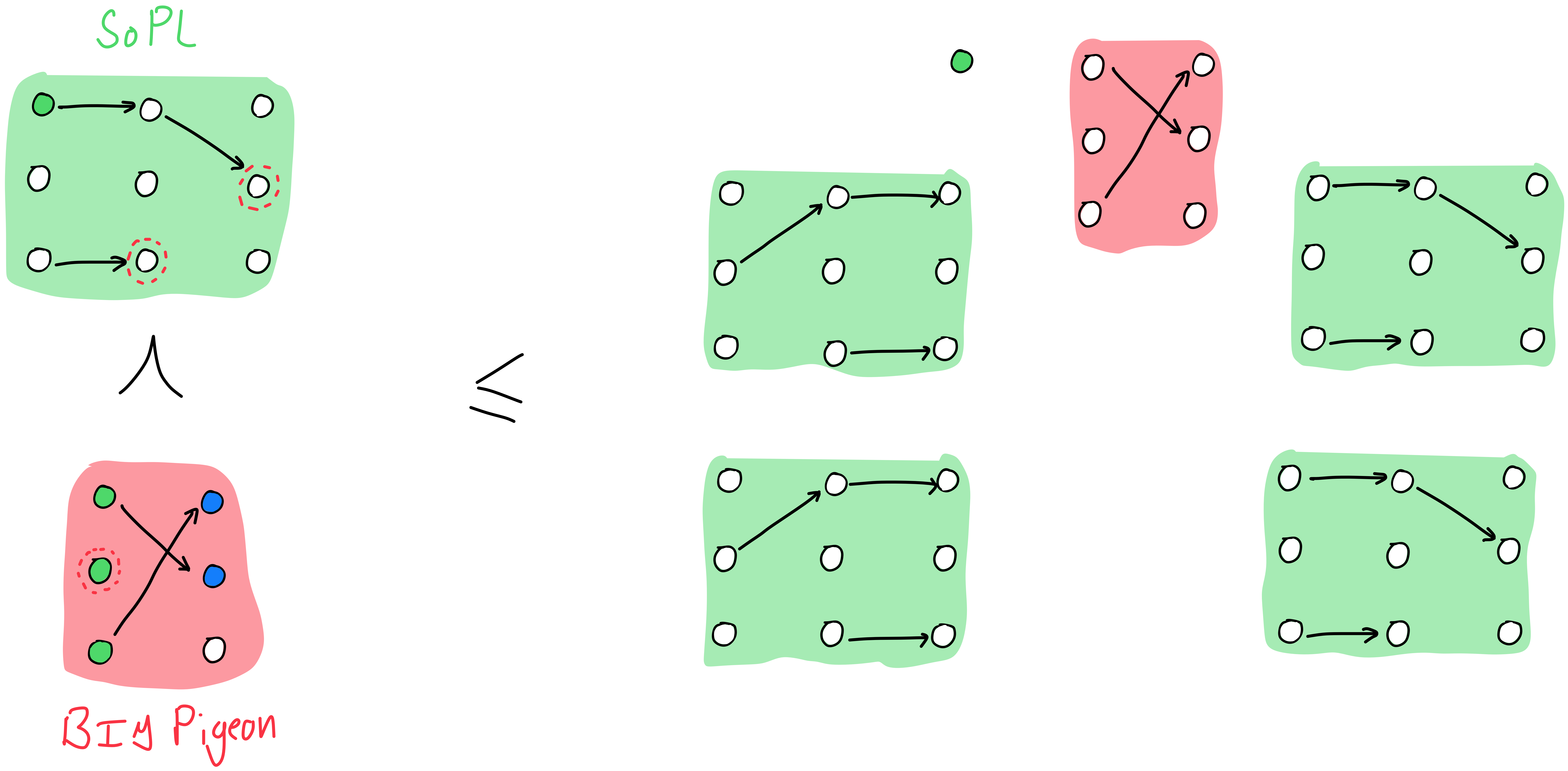
More on EoPL = PLS ∩ PPAD

- Only need EITHER(SoPL, BigPigeon) ≤ EndOfPotentialLine
- can we do SoPL ≤ EndOfPotentialLine ?



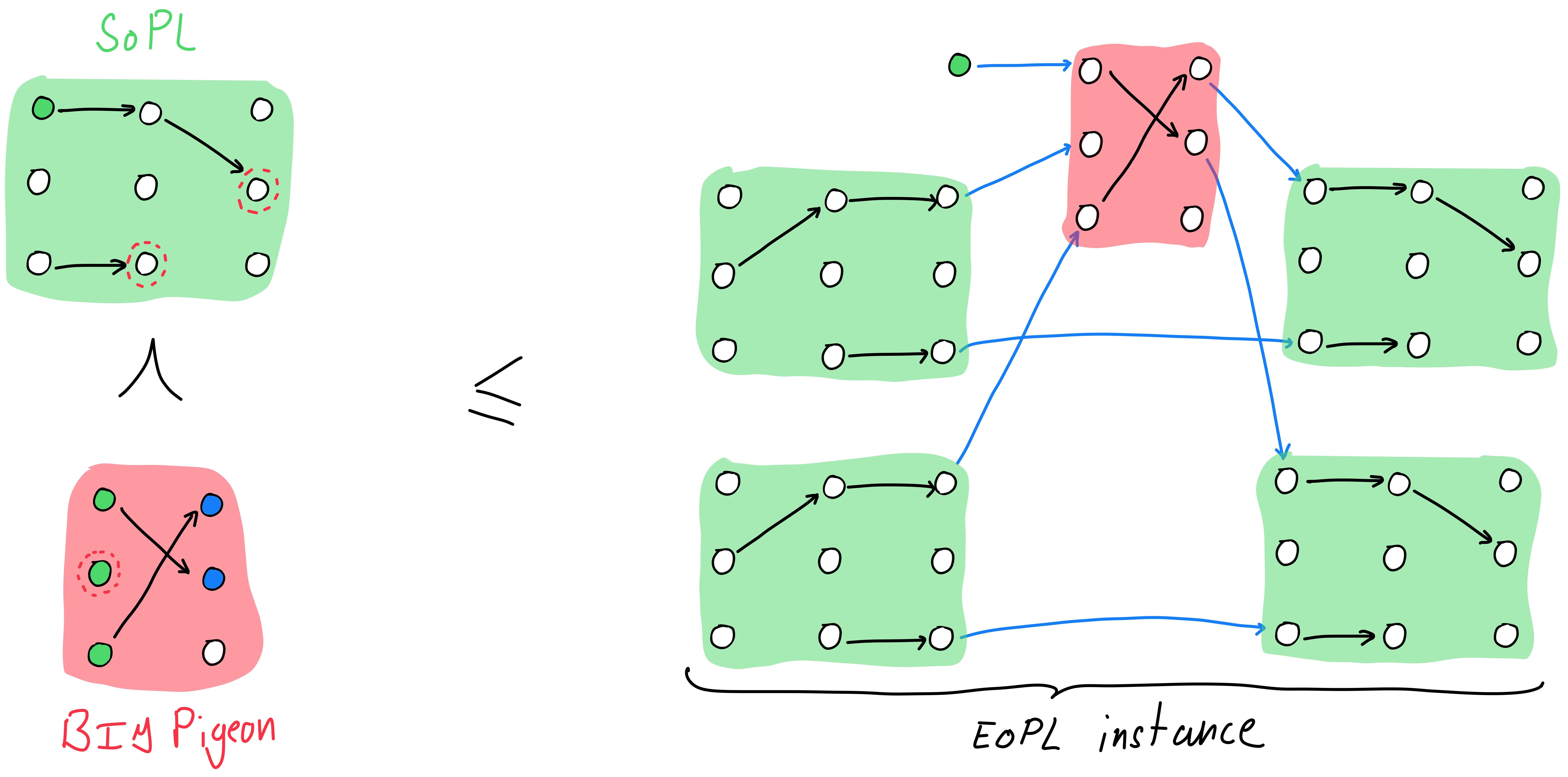
More on EoPL = PLS ∩ PPAD

Idea 3 : Use the BIG Pigeon instance to incorporate the source



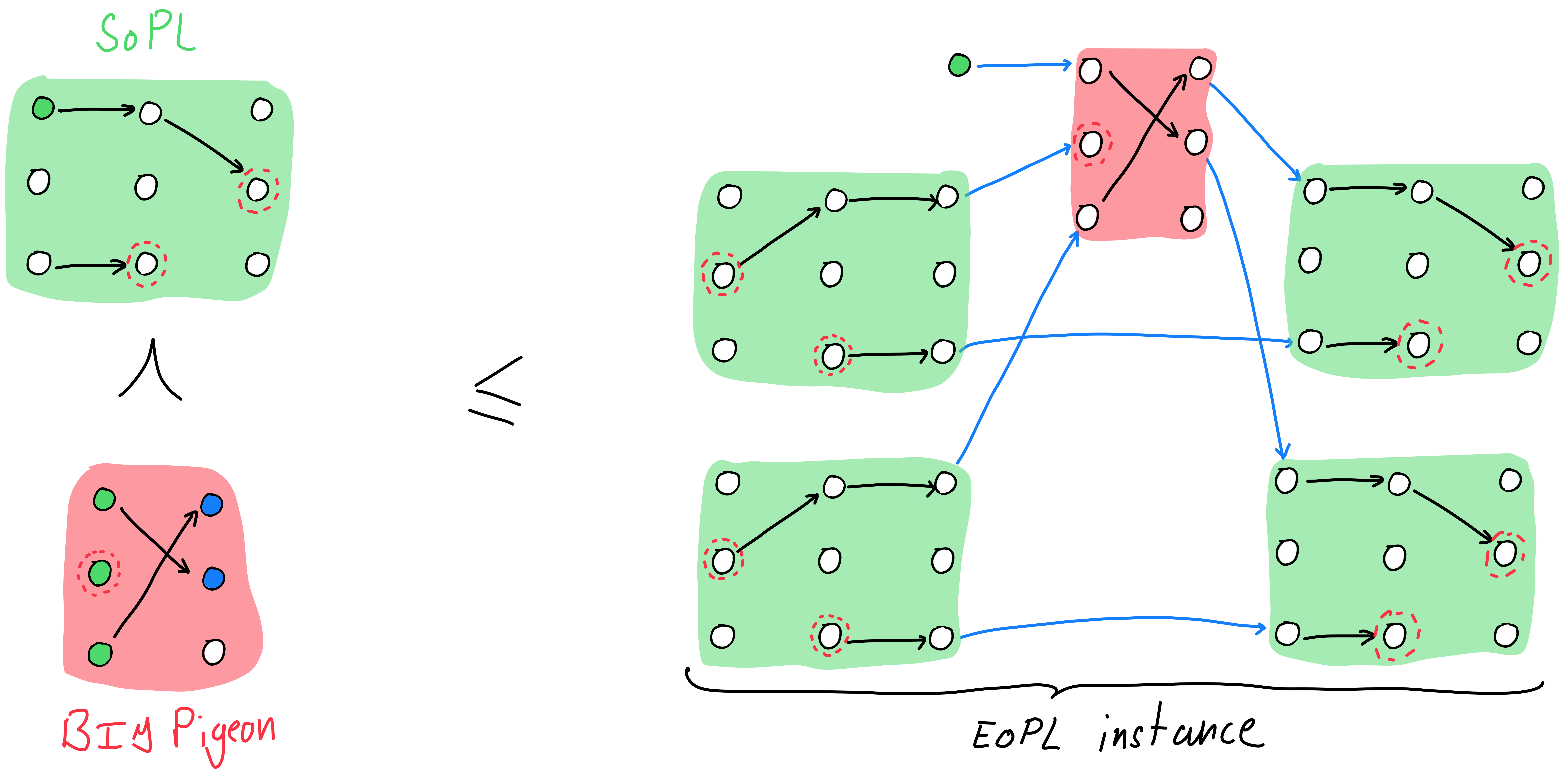
More on $\text{EoPL} = \text{PLS} \cap \text{PPAD}$

Idea 3: Use the BIJ Pigeon instance to incorporate the source

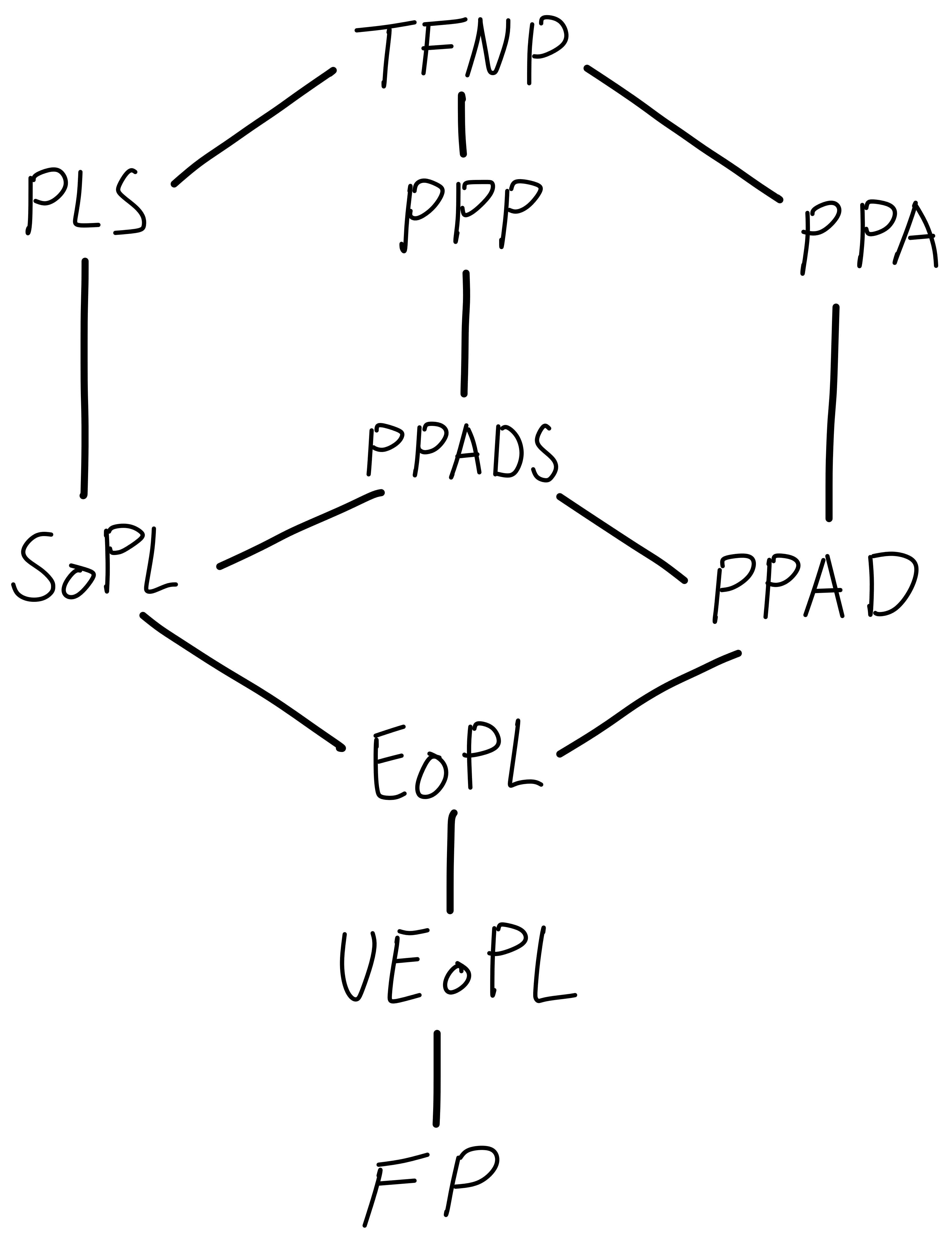


More on EoPL = PLS ∩ PPAD

Idea 3 : Use the Big Pigeon instance to incorporate the source

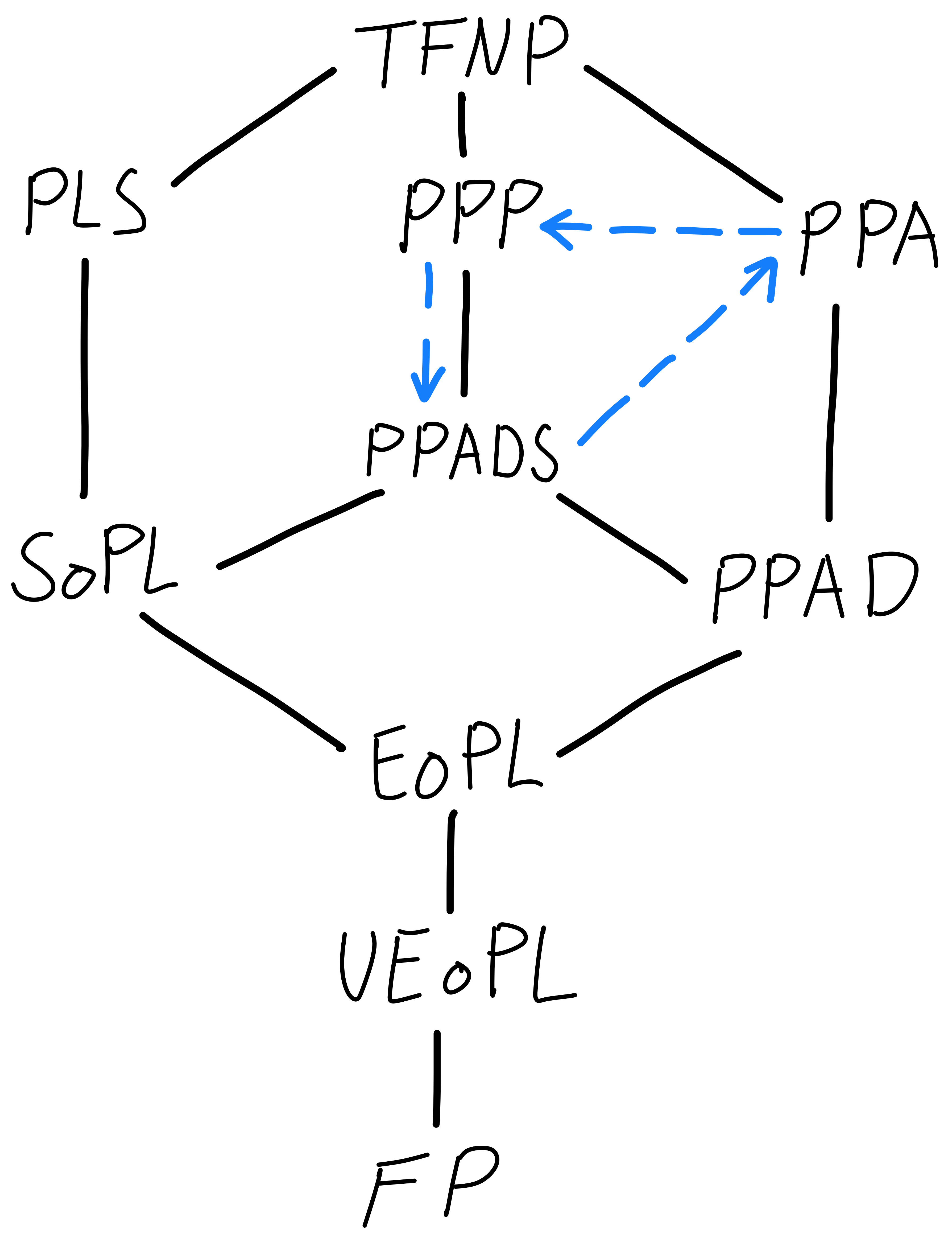


Result II: no further black-box collapses



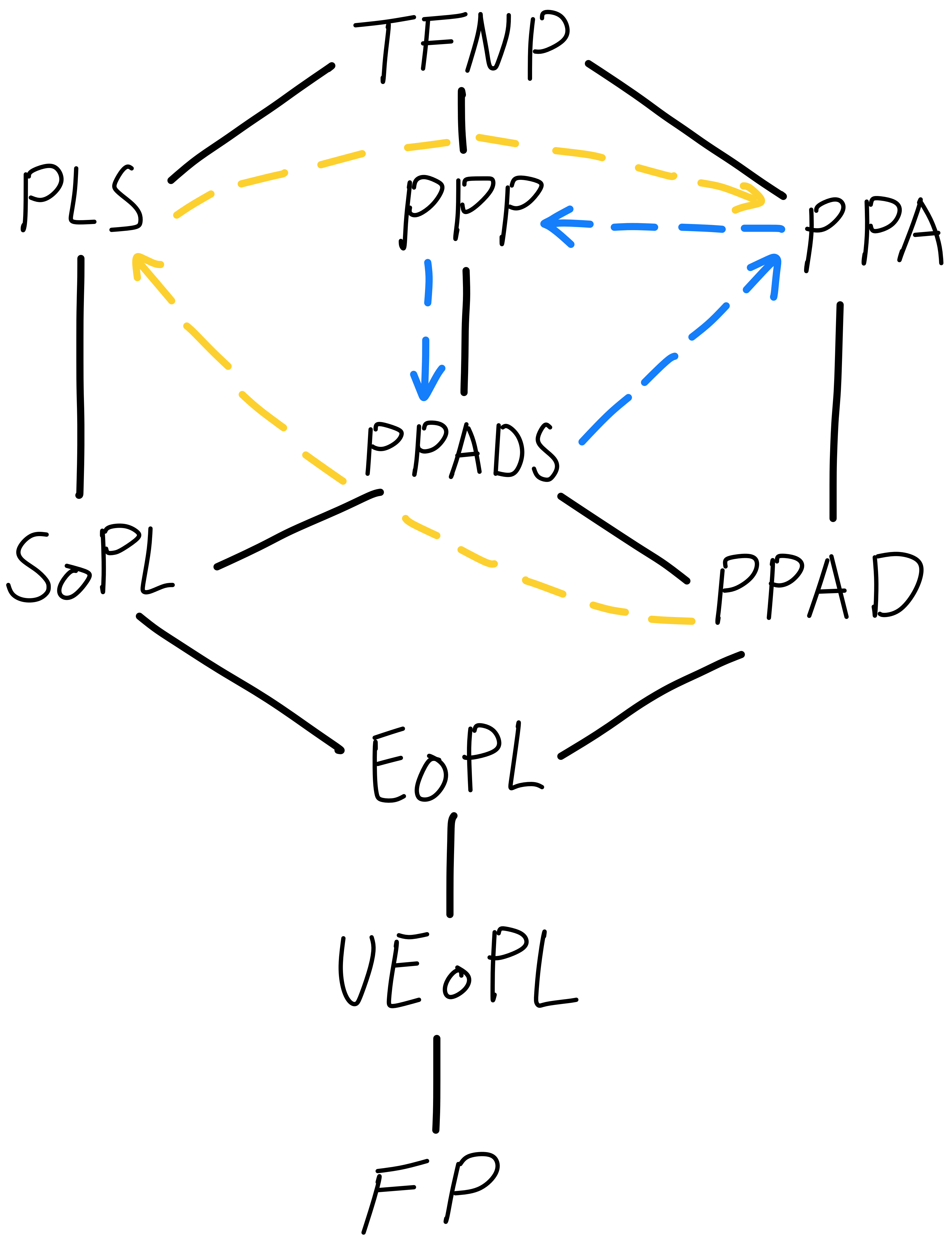
- unconditional separation $\Rightarrow$ P $\neq$ NP
- CLS, SoPL and EoPL collapses were black-box!
- black-box separation $\Rightarrow$ oracle separation
- initiated by Beame et. al. 98'

Result II: no further black-box collapses



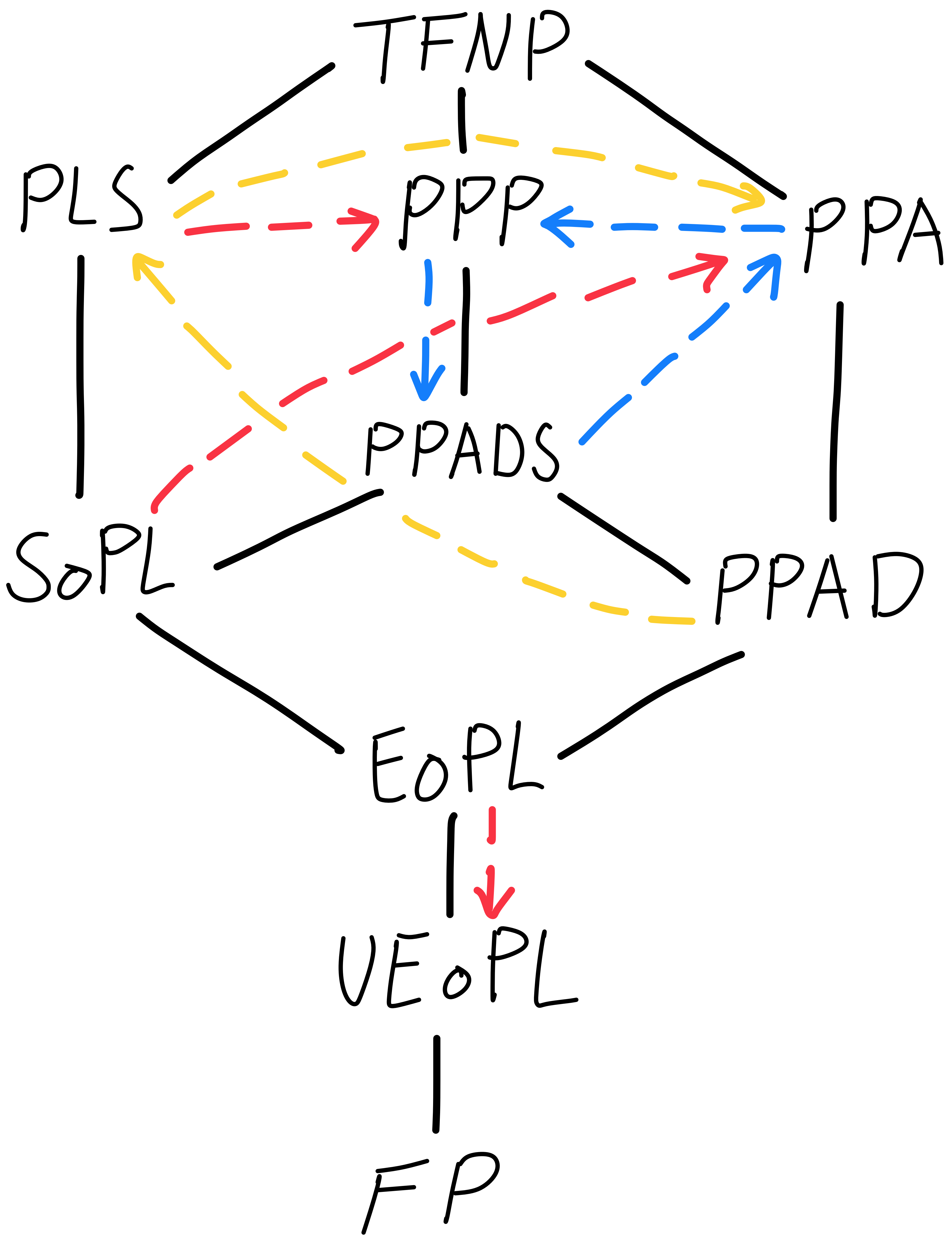
- unconditional separation $\Rightarrow$ $P \neq NP$
- CLS, SoPL and EoPL collapses were black-box!
- black-box separation $\Rightarrow$ oracle separation
- initiated by Beame et. al. 98' continued by Morioka & Buresh-Oppenheim 01+04'

Result II: no further black-box collapses



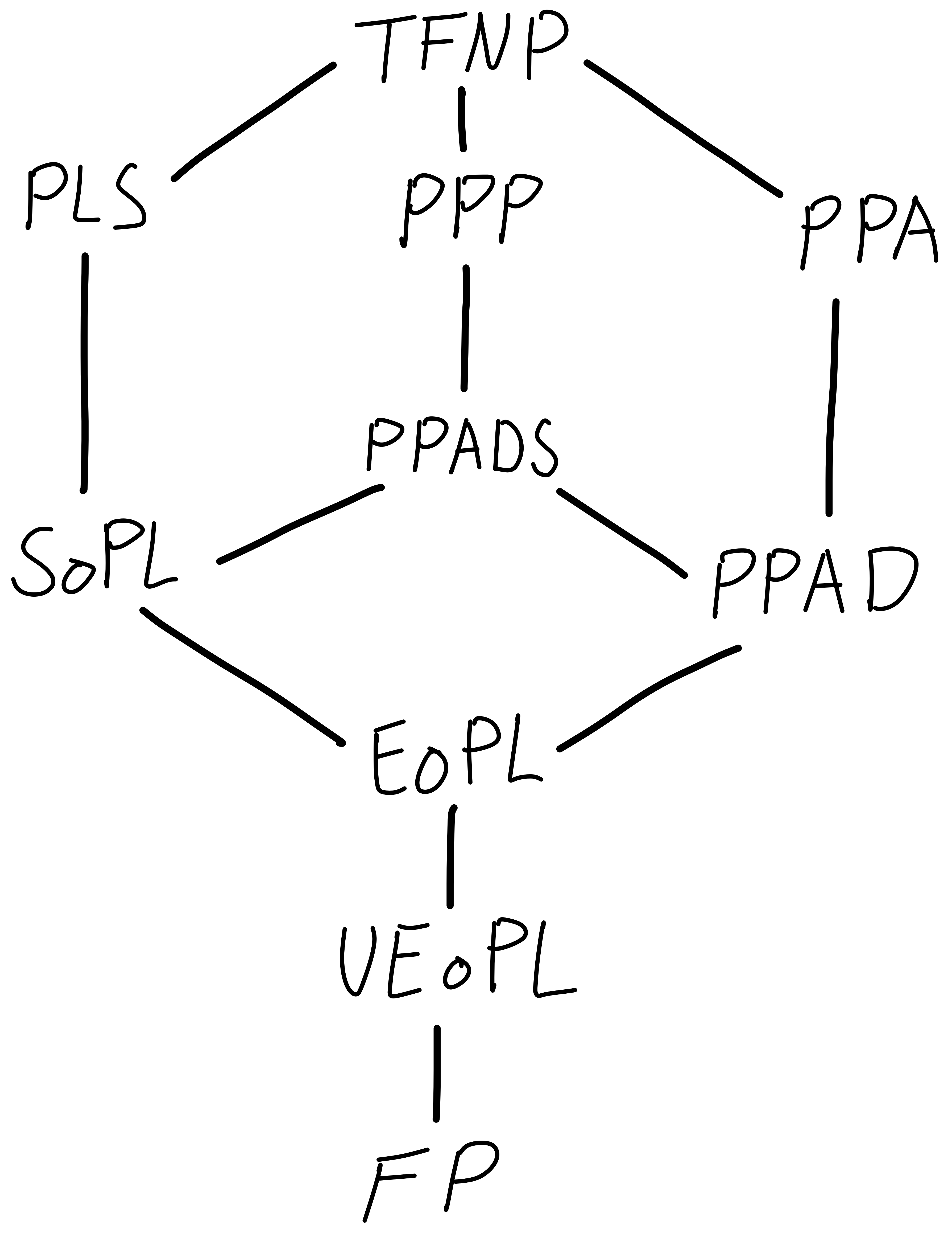
- unconditional separation $\Rightarrow$ $P \neq NP$
- CLS, SoPL and EoPL collapses were black-box!
- black-box separation $\Rightarrow$ oracle separation
- initiated by Beame et. al. 98' continued by Morioka & Buresh-Oppenheim 01'+04'

Result II: no further black-box collapses

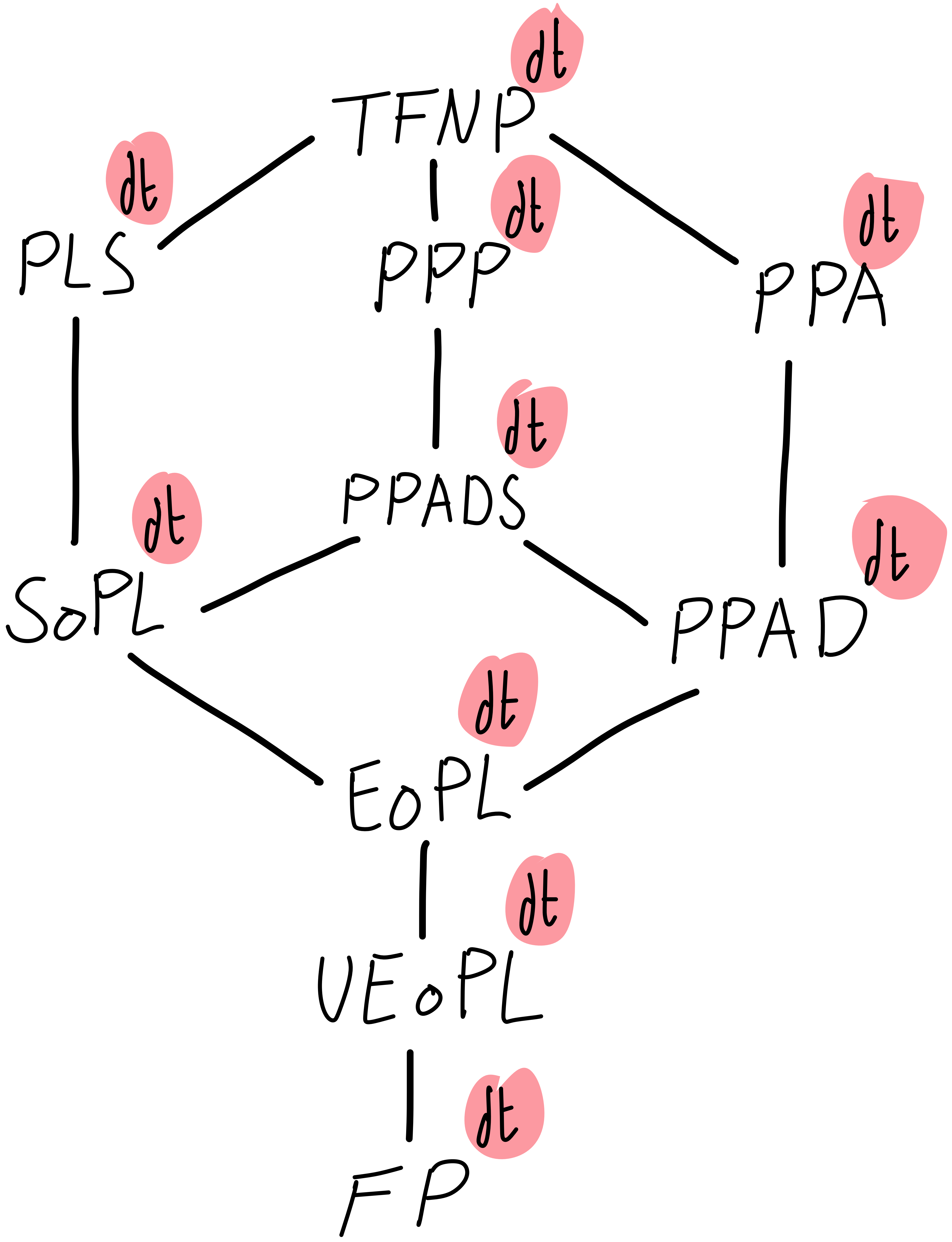


- unconditional separation $\Rightarrow$ $P \neq NP$
- CLS, SoPL and EoPL collapses were black-box!
- black-box separation $\Rightarrow$ oracle separation
- initiated by Beame et. al. 98' continued by Morioka & Buresh-Oppenheim 01+04'
- 2022

Tool 1: black-box classes



Tool 1: black-box classes

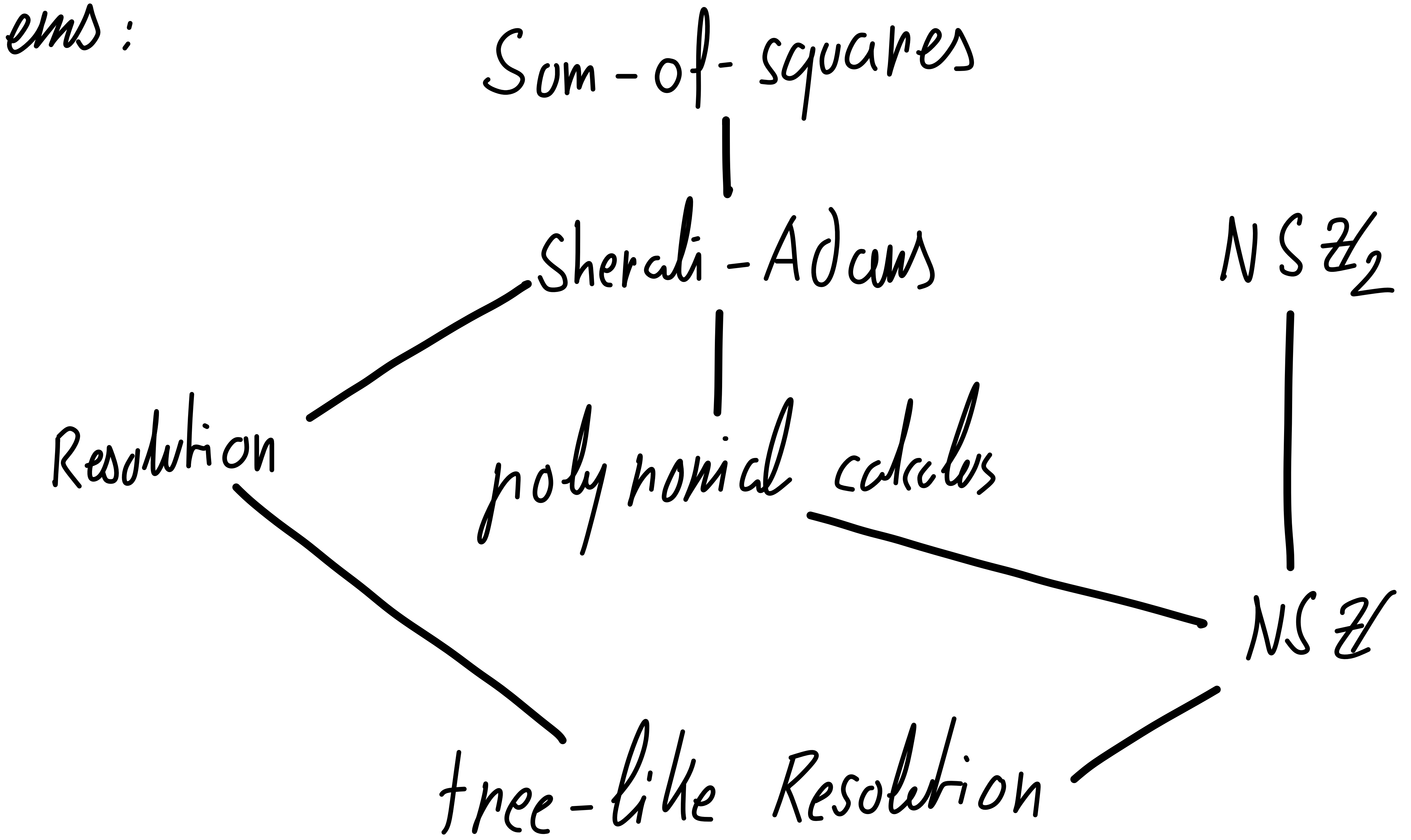


- study "query analogues" a.k.a type-2 complexity
- reduction carried out by shallow decision trees

Tool 2: proof systems

Is there a short way to show that a CNF is unsat?

- hierarchy of proof systems:



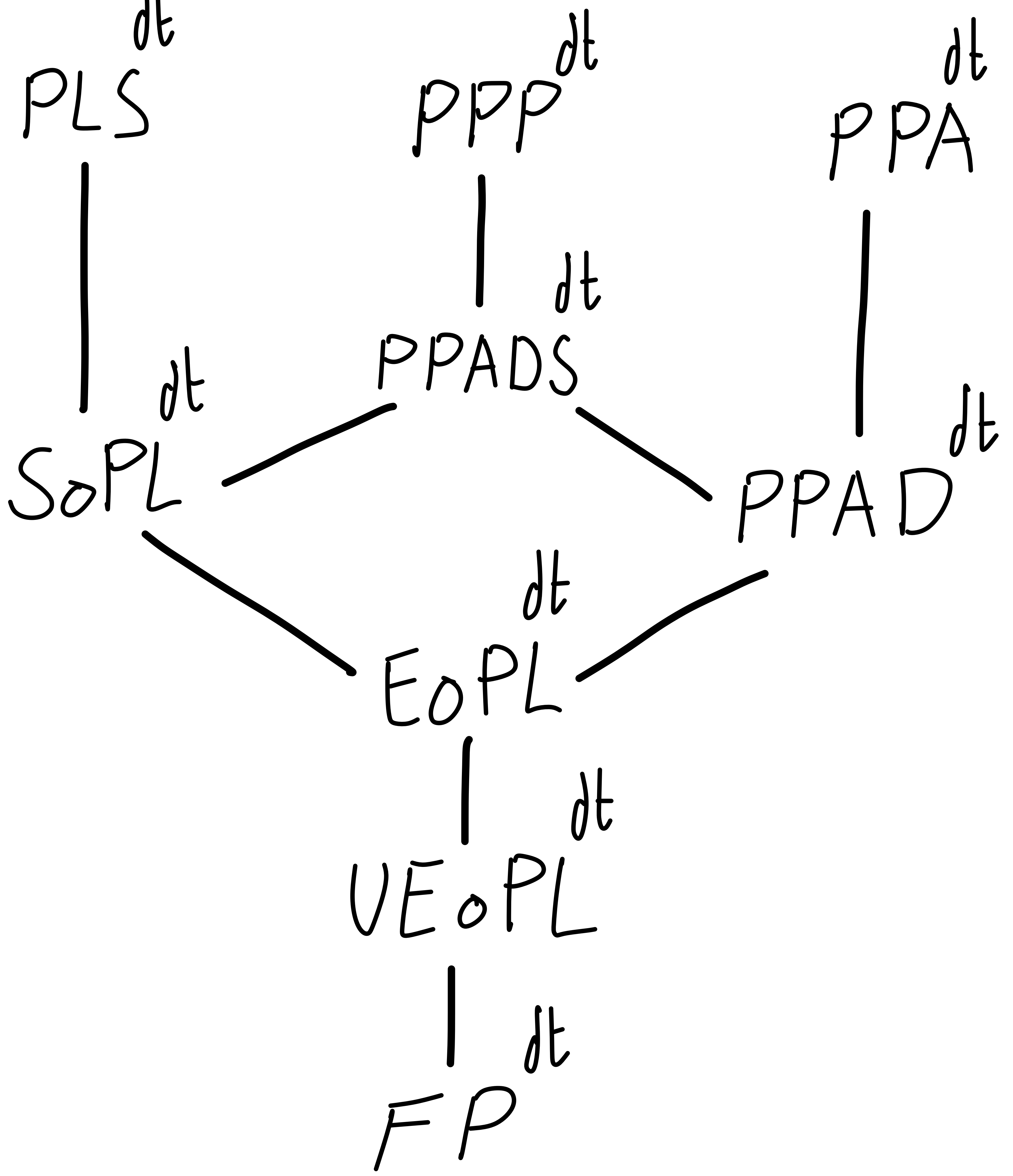
Tool 2: proof systems

Is there a short way to show that a CNF is unsat?

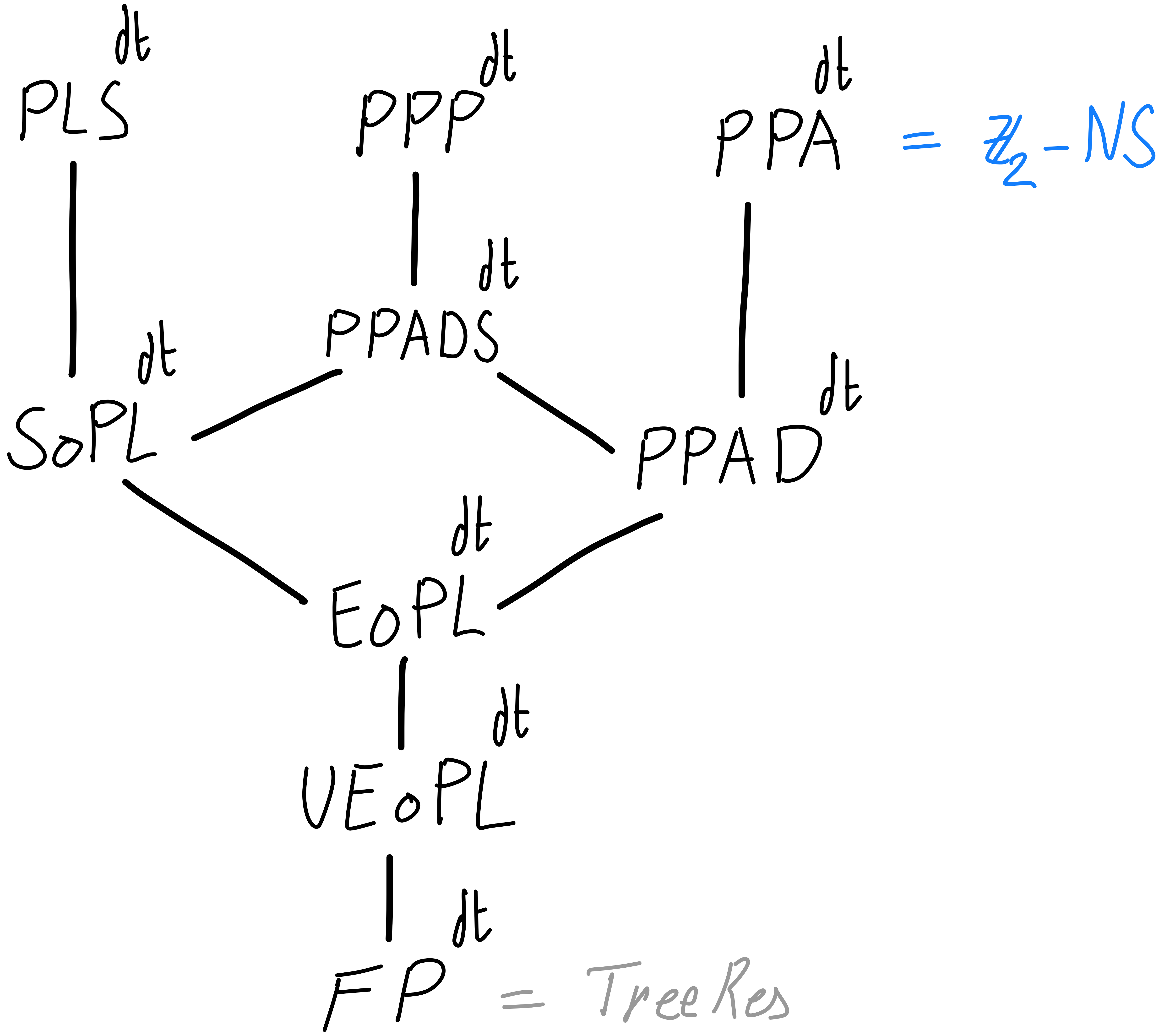
- hierarchy of proof systems
- TFNP^{dt} search problems can be translated to CNF contradictions

LONELY $\Rightarrow$ "there is no unmatched vertex in this odd-size graph"

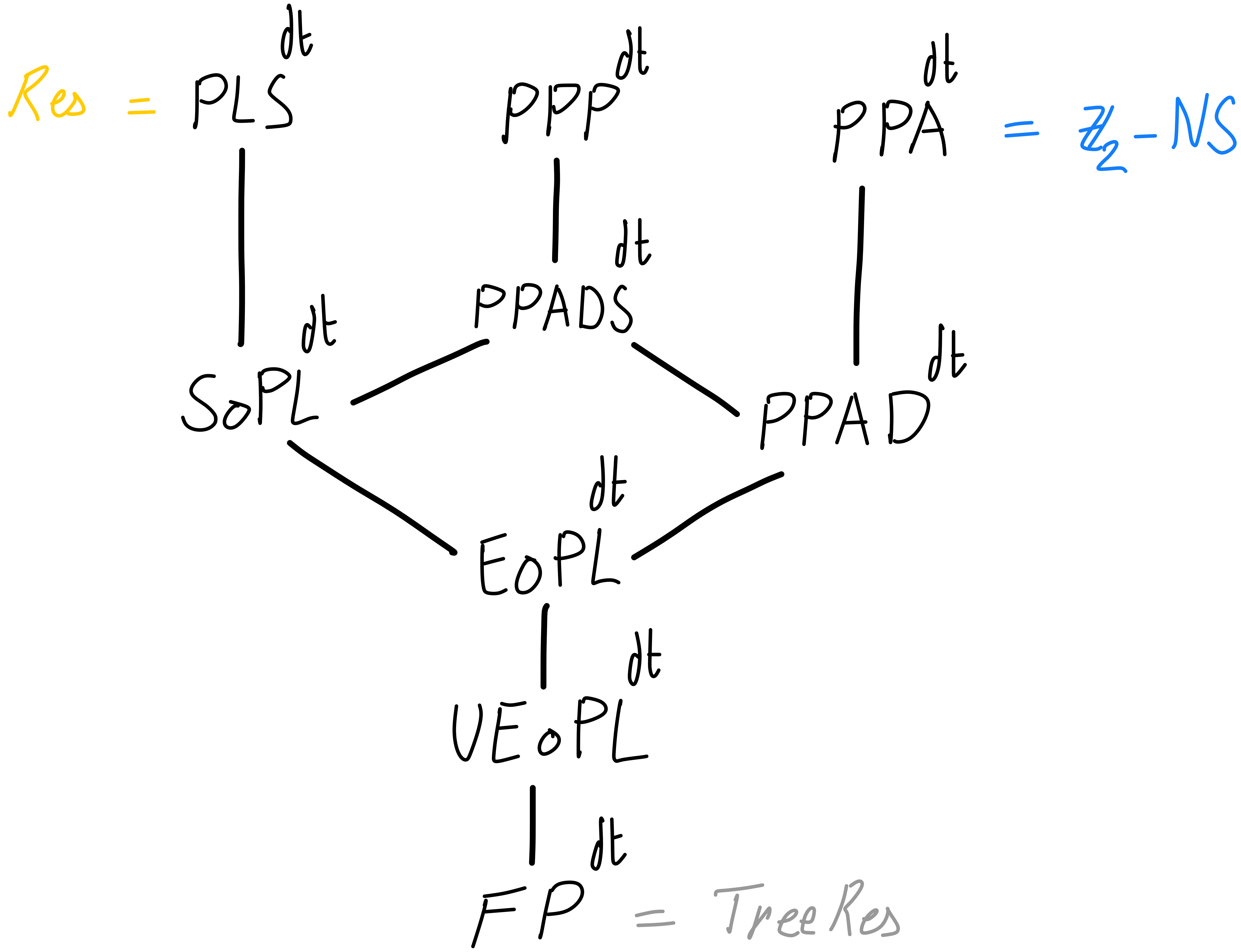
Tool 3: characterizations



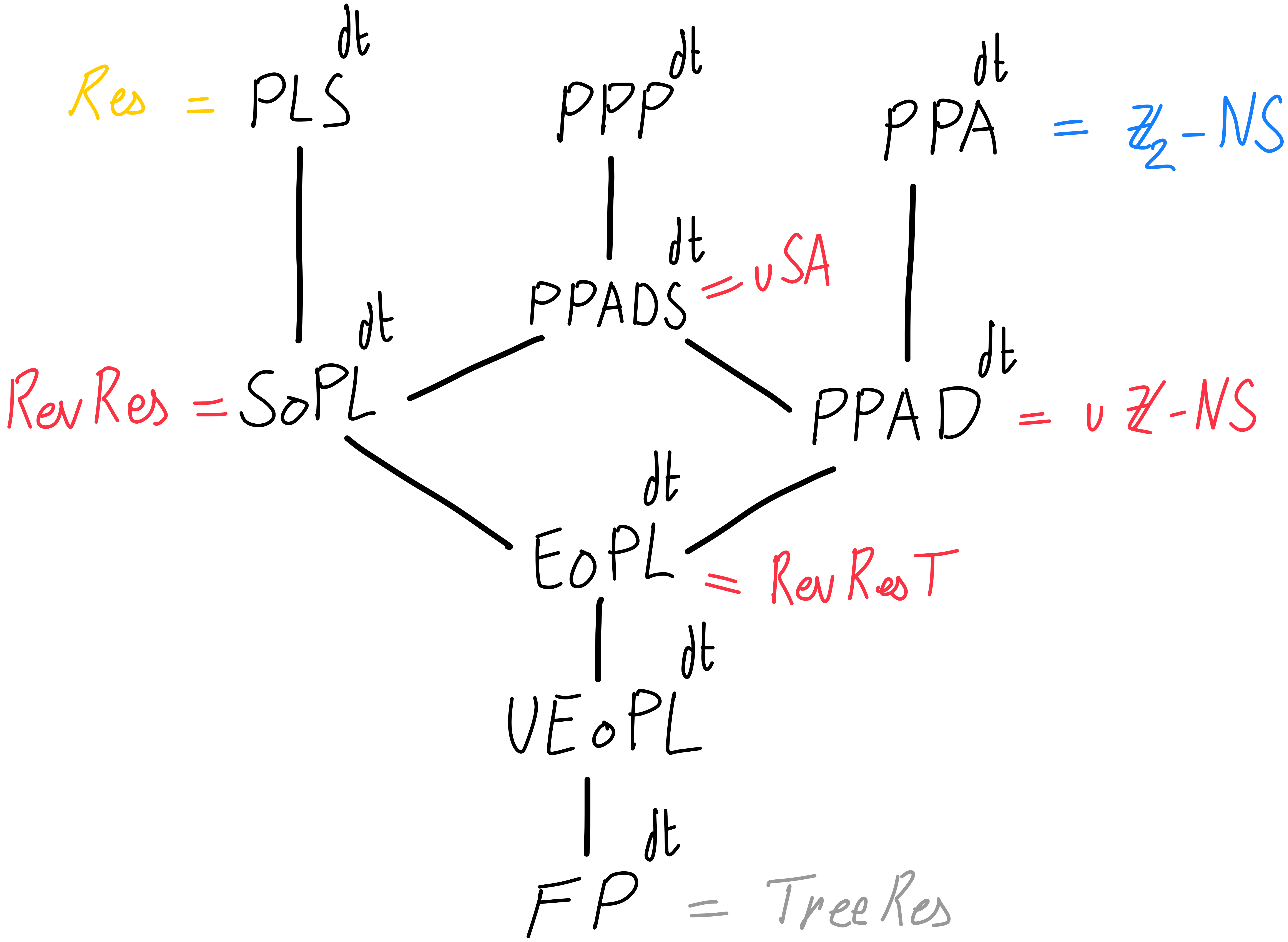
Tool 3: characterizations



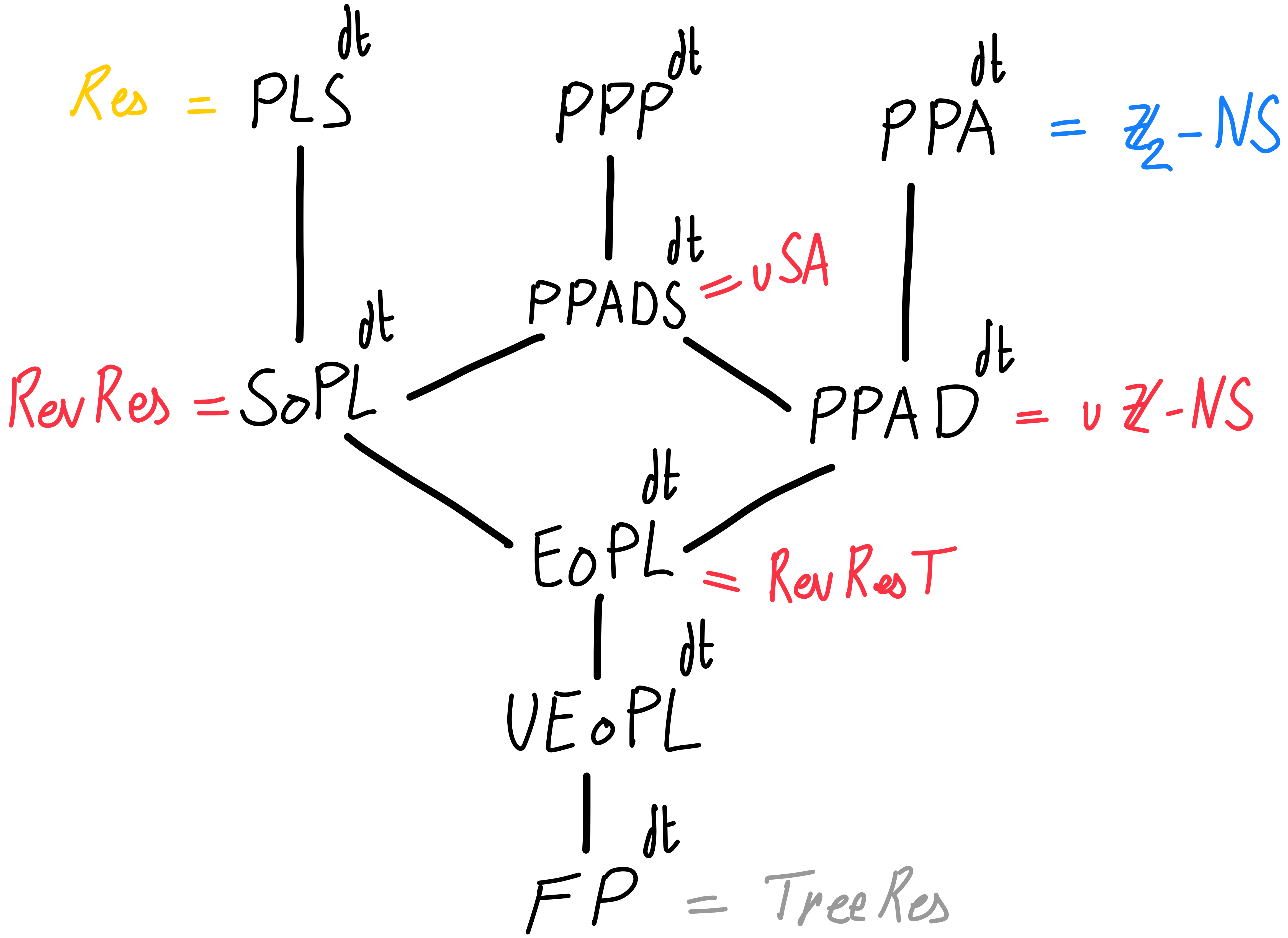
Tool 3: characterizations



Tool 3: characterizations

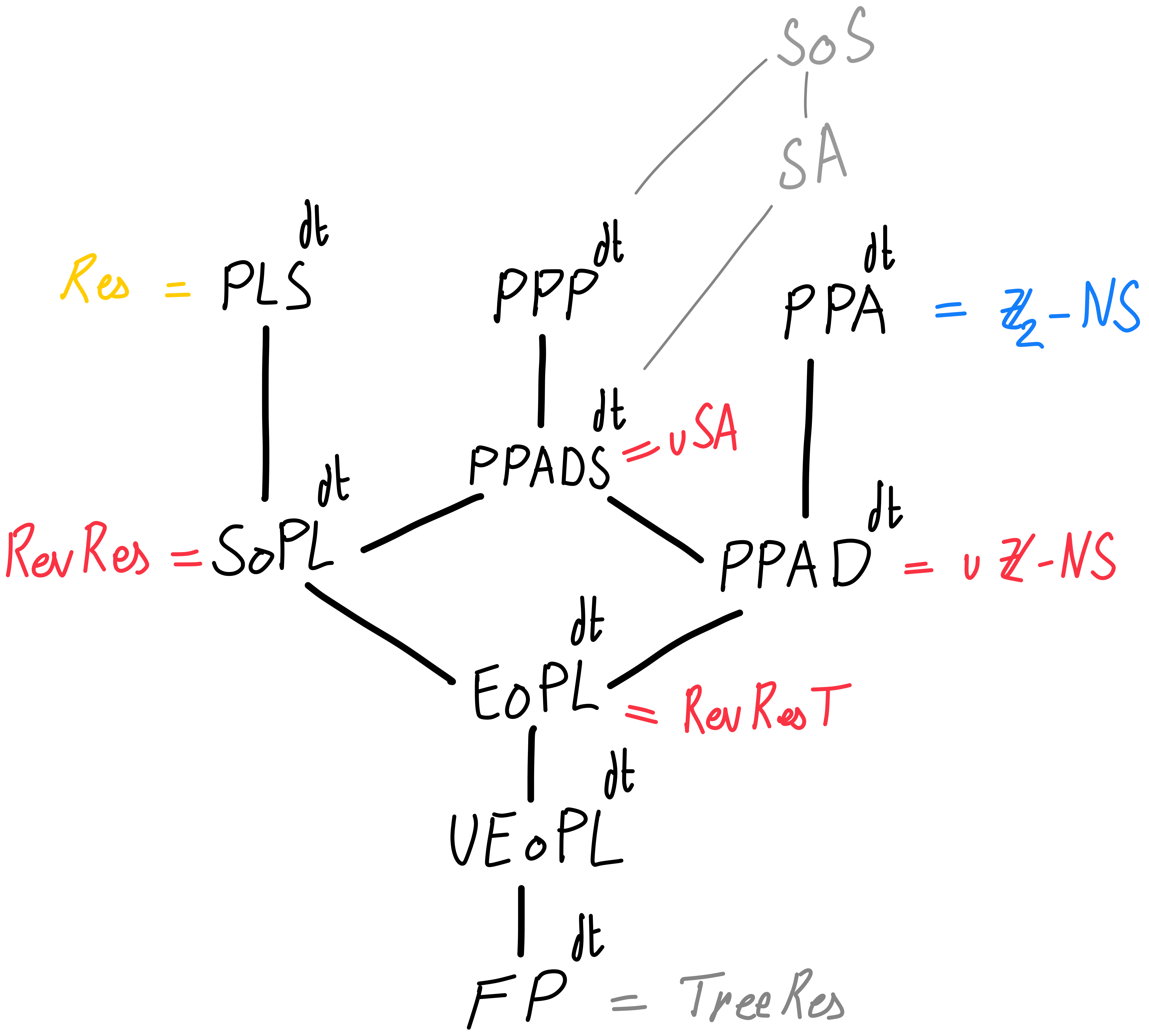


Tool 3: characterizations



- new view: separate proof systems
- for PPP^{dt} and $UEoPL^{dt}$: further combinatorial insights

Open problems



on characterizations

- a system for PPP?
- a system for UEoPL?
- a class for polynomial calculus?

on the structure of TFNP

- where lies PWPP? Ramsey? sunflower?