The Minimum Circuit Size Problem is hard for Sum of Squares

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Joint work with Per Austrin

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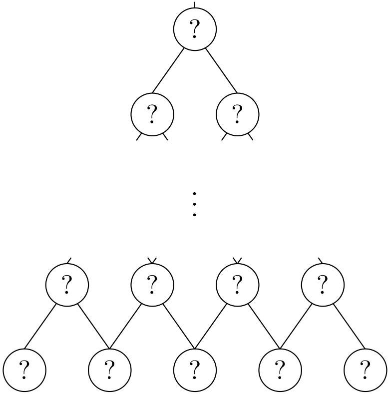

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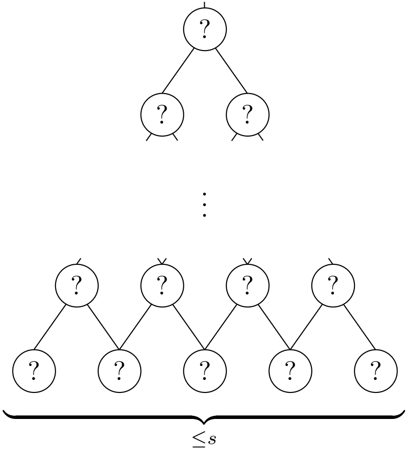

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Goal: show that classes of efficient algorithms do not solve $\mathrm{MCSP}(f, s)$.

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Open Problem ([Raz22])

Prove that SoS requires degree $s^{\Omega(1)}$ to refute $\mathrm{MCSP}(f, s)$.

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For all Boolean functions $f : \{0,1\}^n \to \{0,1\}$ *and* $s \geq n^{d(\varepsilon)}$*, it holds that SoS requires degree* $\Omega_\varepsilon(s^{1-\varepsilon})$ *to refute* $\mathrm{MCSP}(f, s)$*.*

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Similar results in the monotone setting for monotone slice functions.

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- Idea: restrict $\mathrm{Circuit}_s(y)$ by ρ s.t. sat assignments correspond to a "nice" set of circuits

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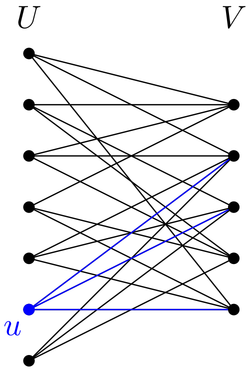

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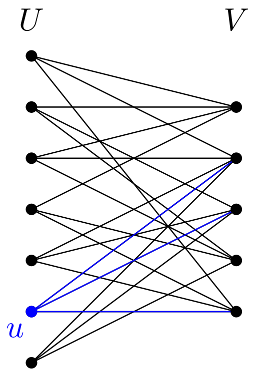

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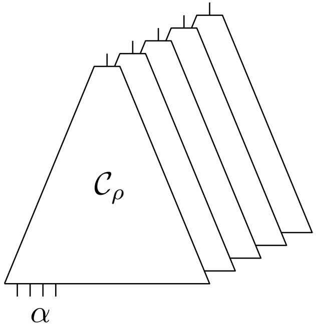

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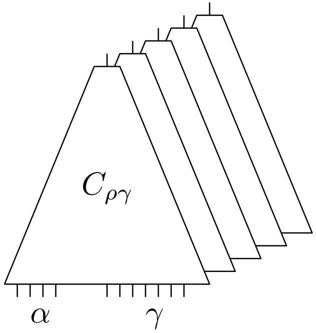

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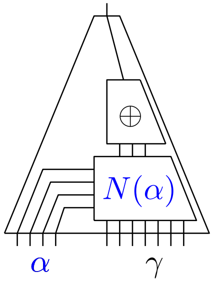

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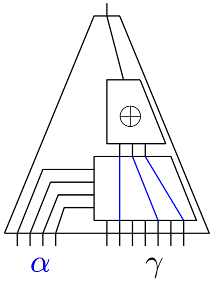

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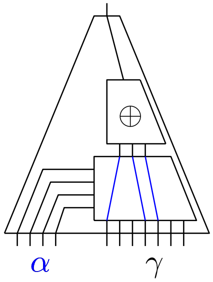

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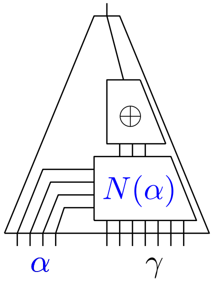

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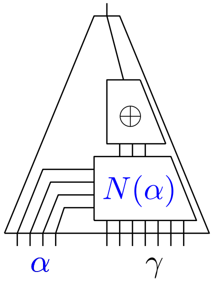

Use explicit expanders [GUV09] and known SoS lower bounds [Gri01] to obtain main theorem.

Conclusion and Open Problems

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- Unifies and simplifies these MCSP lower bounds
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