Multiplicities in GCT: What is in \#P and what is not?

Christian Ikenmeyer
1 Geometric Complexity Theory

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Valiant's conjecture (often phrased as determinant vs permanent)

A homogeneous algebraic branching program:

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\text{Computes } h = \sum_{s-t\text{-path } P} \prod_{e \in P} \ell(e)
\]

Let \( h \) be a homogeneous degree \( d \) polynomial. The **homogeneous ABP width complexity** \( c(h) \) is defined as the smallest \( n \) such that there exist a **homogeneous** width \( n \) ABP computing \( h \).
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Remark: The noncommutative version of $c$ is completely understood (Nisan 1991).
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Clearly, \( c(h) \) is always finite: Compute each monomial independently.

Remark: The noncommutative version of \( c \) is completely understood (Nisan 1991).

Let \( \text{per}_m := \sum_{\pi \in \mathfrak{S}_m} x_{\pi(1)}x_{\pi(2)} \cdots x_{\pi(m)} \).

Valiant’s conjecture

The sequence \( c(\text{per}_m) \) grows superpolynomially. In other words, \( \text{VBP} \neq \text{VNP} \).

Theorem (Grenet 2012): \( c(\text{per}_m) \leq \binom{m}{\lfloor m/2 \rfloor} \).
A simpler machine model: Waring rank

Every homogeneous degree \(d\) polynomial \(h\) can be written as a sum of \(d\)-th powers of homogeneous linear polynomials \(\ell_i\):

\[
h = \sum_{i=1}^{r} (\ell_i)^d.
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The smallest \(r\) possible is called the Waring rank \(\text{WR}(h)\) of \(h\).
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Waring rank measures the size of ABPs of special format:

$$\omega = \liminf_{n \to \infty} \frac{\log n}{\text{WR} \left( \sum_{i,j,k=1}^{n} x_{i,j} x_{j,k} x_{k,i} \right)}$$
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Example:

$$6x^2y = (x + y)^3 + (y - x)^3 - 2y^3,$$

hence $\text{WR}(x^2y) \leq 3$. In fact, $\text{WR}(x^2y) = 3$. 

\[\text{Waring rank for cubic polynomials: matrix multiplication exponent} \quad \omega = \liminf_{n \to \infty} \frac{\log n}{\text{WR} \left( \sum_{i,j,k=1}^{n} x_i x_j x_k \right)}\]
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![Diagram](image)

Example:

\( 6x^2y = (x + y)^3 + (y - x)^3 - 2y^3 \), hence \( WR(x^2y) \leq 3 \). In fact, \( WR(x^2y) = 3 \).

Waring rank for cubic polynomials: matrix multiplication exponent [Chiantini, Hauenstein, I, Landsberg, Ottaviani, 2017]

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\omega = \liminf_{n \to \infty} \left\{ \log_n \ WR \left( \sum_{i,j,k=1}^{n} x_{i,j} x_{j,k} x_{k,i} \right) \right\}
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Example:

\[
h = x^2 y = \frac{1}{6}((x + y)^3 + (y - x)^3 - 2y^3), \text{ WR}(h) = 3.
\]

\[
WR(h) \leq 1 \quad \text{WR}(h) \leq 2 \quad x^2 y \quad \text{WR}(h) \leq 3
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Main advantage:

\[
X_n := \{ h | \text{WR}(h) \leq n \} = \{ h | \text{WR}(h) \leq n \}
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is closed, so it is guaranteed that non-membership \( p \not\in X_n \) can be proved by finding a continuous function \( \Delta \) that vanishes on \( X_n \), but does not vanish on \( p \).

Example: Consider the 3-dim vector space \( \mathbb{C}[x,y]^2 \). Let

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A lower bound:

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\text{WR}(xy) \geq 2, \text{ because } 1^2 - 4 \cdot 0 \cdot 0 = 1 \neq 0.
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Such \( \Delta = b^2 - 4ac \) is sometimes called a polynomial obstruction or a separating polynomial. \n
\( X_k \) has more structure and we can obtain more information about polynomial obstructions.
Example:
\[ h = x^2 y = \frac{1}{6}((x + y)^3 + (y - x)^3 - 2y^3), \text{ WR}(h) = 3. \]

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\frac{1}{3\varepsilon} \left( (x + \varepsilon y)^3 - x^3 \right)
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This makes determining \( \text{WR}(x^2 y) \) subtle!
If a continuous function \( \Delta \) vanishes on all \( h \) with \( \text{WR}(h) \leq 2 \), then \( f \) also vanishes on \( x^2 y \).

**Definition (border Waring rank)**

The **border Waring rank** \( \text{WR}(h) \) is defined as the smallest \( n \) such that \( h \) can be approximated arbitrarily closely by polynomials of Waring rank \( \leq n \).

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Definition \((\Sigma \land \Sigma)\)

Let \(\Sigma \land \Sigma\) be the set of seq. of polynomials whose degree, number of variables, and Waring rank is polynomially bounded.
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Open questions: $\Sigma \wedge \Sigma \neq \Sigma \wedge \Sigma$. 
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  \item $\overline{\text{VF}} \supsetneq \text{VF}$.
  \item $\overline{\text{VBP}} \supsetneq \text{VBP}$.
  \item $\overline{\text{VP}} \supsetneq \text{VP}$.
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• [Nisan 1991] showed that $\text{VBP}_{\text{non-comm}} = \overline{\text{VBP}}_{\text{non-comm}}$. 

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Placing a closure of a class into a larger non-closed class (“de-bordering”):
- [Bläser Dörlfer I 2020] prove $c(p) \leq \text{WR}(p)$. Hence $\Sigma \land \Sigma \subseteq \text{VBP}$ (which was discovered earlier by Forbes).
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- [Dutta Dwivedi Saxena 2021] prove \(\Sigma^k \Pi \Sigma \subseteq \text{VBP}\) for any constant \(k\), using the logarithmic derivative.
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$\bullet$ [Dutta Dwivedi Saxena 2021] prove $\Sigma^k \prod \Sigma \subseteq \mathbf{V} \mathbf{B} \mathbf{P}$ for any constant $k$, using the logarithmic derivative.

Open question: Is $\overline{\mathbf{V} \mathbf{F}} \subseteq \mathbf{V} \mathbf{N} \mathbf{P}$ or not?

Mulmuley-Sohoni strengthening of Valiant’s conjecture:

**Valiant’s conjecture**

The sequence $c(\text{per}_m)$ grows superpolynomially. In other words, $\mathbf{V} \mathbf{B} \mathbf{P} \neq \mathbf{V} \mathbf{N} \mathbf{P}$.

**Mulmuley-Sohoni conjecture**

The sequence $c(\text{per}_m)$ grows superpolynomially. In other words, $\mathbf{V} \mathbf{N} \mathbf{P} \subsetneq \overline{\mathbf{V} \mathbf{B} \mathbf{P}}$ or equivalently $\overline{\mathbf{V} \mathbf{B} \mathbf{P}} \neq \mathbf{V} \mathbf{N} \mathbf{P}$.

Fundamental open question: Are the two conjectures equivalent?
Let $X_n := \{ h \mid c(h) \leq n \} = \{ h \mid c(h) \leq n \}$.

would imply $c(\text{per}_3) > 6$

[Bläser, I, Lysikov, Pandey, Schreyer, SODA21]: In general, orbit closure containment is NP-hard (minrank of tensors).
$X_n$ is a projective variety

Let $X_n := \{ h \mid c(h) \leq n \}$. 

A subset $\mathcal{X} \subseteq \mathbb{C}^N$ is a **projective variety** if there exist finitely many homogeneous polynomials $\Delta_1, \ldots, \Delta_k$ such that 

$$h \in \mathcal{X} \iff \Delta_1(h) = \Delta_2(h) = \cdots = \Delta_k(h) = 0.$$ 

$X_n$ is a projective variety (by Chevalley’s theorem).

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![Diagram](image)

Fundamental Conclusion: All border complexity lower bounds can be proved via **polynomials**

$\text{per}_m \notin X_n$ iff there exists a homogeneous polynomial $\Delta$ with

- $\Delta(h) = 0$ for all $h \in X_n$  
- $\Delta(\text{per}_m) \neq 0$.

Meta-complexity (algebraic natural proofs): What can be said about the complexity of the $\Delta_i$?
Complexity lower bounds via highest weight polynomials

**Definition (highest weight polynomial)**

A function $\Delta$ is called a **highest weight polynomial** of weight $\lambda = (\lambda_1, \ldots, \lambda_N)$, if

- $\Delta$ is invariant under the action of upper triangular matrices with 1s on the diagonal
- and $\Delta$ gets rescaled by $\alpha_1^{\lambda_1} \cdots \alpha_N^{\lambda_N}$ under the action of diagonal matrices $\text{diag}(\alpha_1, \ldots, \alpha_N)$. 

Recall: Want $\Delta$ vanishing on $X_n$ and $\Delta(\text{per}_m) \neq 0$.

**Theorem (representation theory)**

If $\text{per}_m \notin X_n$, then there exists a highest weight polynomial $\Delta$ such that $A\Delta$ vanishes on $X_n$ and $A\Delta(\text{per}_m) \neq 0$ for a generic matrix $A$. 

[Bläser, Dörfler, I, CCC21]: Given a highest weight polynomial $\Delta$, succinctly encoded as a semistandard Young tableau, then it is #P-hard to evaluate $\Delta$ at a fixed point of Waring rank 3.
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**Fundamental Conclusion: All border complexity lower bounds can be proved via highest weight polynomials**

If complexity lower bounds exist, then there exist highest weight polynomials proving them.

[Bläser, Dörfler, I, CCC21]: Given a highest weight polynomial $\Delta$, succinctly encoded as a semistandard Young tableau, then it is $\#P$-hard to evaluate $\Delta$ at a fixed point of Waring rank 3.
Proposition (a coarse technique for finding complexity lower bounds: occurrence obstructions)

If there exists $\lambda$ such that for a generic matrix $A$ we have

- for all (!) highest weight polynomials $\Delta$ of weight $\lambda$: $A\Delta$ vanishes on $X_n$
- there exists a highest weight polynomial $\Delta$ of weight $\lambda$ such that $A\Delta(\text{per}_m) \neq 0$

then $\text{per}_m \notin X_n$.

[I, Panova FOCS16] and [Bürgisser, I, Panova FOCS16]: In a non-homogeneous setting, no occurrence obstructions exist. Nothing is known about the homogeneous setting.
More general heuristic attempt: Multiplicity obstructions

Let $Y_m := \{ \text{per}_m(Ax) \}$

$\text{per}_m \in X_n \iff Y_m \subseteq X_n$

Not much is known about these multiplicities!

[Dörfler, I, Panova ICALP19] There are finite homogeneous settings where multiplicity obstructions are stronger than occurrence obstructions.

[I, Kandasamy STOC20] Multiplicity obstructions can be created in homogeneous toy settings without constructing the highest weight polynomials.
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Def.: The multiplicity $\text{mult}_\lambda(\mathbb{C}[X_n])$ is defined as the dimension of the space of highest weight polynomials of weight $\lambda$ restricted to $X_n$.
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If $\text{mult}_\lambda(\mathbb{C}[Y_m]) > \text{mult}_\lambda(\mathbb{C}[X_n])$, then $\text{per}_m \notin X_n$.

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- [Dörfler, I, Panova ICALP19] There are finite homogeneous settings where multiplicity obstructions are stronger than occurrence obstructions.
- [I, Kandasamy STOC20] Multiplicity obstructions can be created in homogeneous toy settings without constructing the highest weight polynomials.
The original GCT approach (Mulmuley-Sohoni):

1. The multiplicities are easier to study than the polynomials.
2. Oftentimes the multiplicities in representation theory have well-understood combinatorial descriptions (Littlewood-Richardson coefficient).
3. Positivity of the Littlewood-Richardson coefficient can be decided in polynomial time (using a combinatorial algorithm).
4. So maybe this works as well for $\text{mult}_\lambda(\mathbb{C}[Y_m])$ and $\text{mult}_\lambda(\mathbb{C}[X_n])$.
5. Then one could analyze the algorithms and construct an input $\lambda$ with $\text{mult}_\lambda(\mathbb{C}[Y_m]) > 0 = \text{mult}_\lambda(\mathbb{C}[X_n])$. 

We know by now that About 3.: the positivity of coefficients is often NP-hard [I Mulmuley Walter 2017], [Fischer I 2020].

About 2.: Connections to classical questions in algebraic combinatorics!

Christian Ikenmeyer
The original GCT approach (Mulmuley-Sohoni):

1. The multiplicities are easier to study than the polynomials.
2. Oftentimes the multiplicities in representation theory have well-understood combinatorial descriptions (Littlewood-Richardson coefficient).
3. Positivity of the Littlewood-Richardson coefficient can be decided in polynomial time (using a combinatorial algorithm).
4. So maybe this works as well for \( \text{mult}_\lambda(\mathbb{C}[Y_m]) \) and \( \text{mult}_\lambda(\mathbb{C}[X_n]) \).
5. Then one could analyze the algorithms and construct an input \( \lambda \) with \( \text{mult}_\lambda(\mathbb{C}[Y_m]) > 0 = \text{mult}_\lambda(\mathbb{C}[X_n]) \).

We know by now that

- About 3.: the positivity of coefficients is often NP-hard [I Mulmuley Walter 2017], [Fischer I 2020].
- About 2.: Connections to classical questions in algebraic combinatorics!
Geometric Complexity Theory

Multiplicities in GCT: What is in \#P and what is not?
Closely related multiplicities in Stanley’s “Positivity Problems and Conjectures in Algebraic Combinatorics” (2000):

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Recall that $f : \{0,1\}^* \to \mathbb{N}$ is in #P if there exists a polytime Turing machine $M$ with

$$\forall w \in \{0,1\}^* : \#\text{acc}_M(w) = f(w)$$
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Problems 9, 10, 11 are in $\text{GapP} = \#P - \#P$, and all these are nonnegative. Are they in $\#P$?
The problem \#Sperner

Input: A Boolean circuit $C$ with $3\lceil \log n \rceil$ inputs and 2-bit output.
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Input: A Boolean circuit $C$ with $3\lceil \log n \rceil$ inputs and 2-bit output. $C$ describes a coloring of a side length $n$ triangular grid.

- Positions are $(x, y, z)$ with $x + y + z = n$

- $\text{color}(x, y, z) = \begin{cases} 
    \text{red} & \text{if } C(x, y, z) = (0, 0) \\
    \text{blue} & \text{if } C(x, y, z) = (0, 1) \\
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We enforce constraints at the corners and sides:

- Corners:
  - $\text{color}(n, 0, 0) = \text{blue}$, 
  - $\text{color}(0, n, 0) = \text{green}$, 
  - $\text{color}(0, 0, n) = \text{red}$

- Sides:
  - $\text{color}(i, j, 0) \in \{\text{blue, green}\}$, 
  - $\text{color}(i, 0, j) \in \{\text{blue, red}\}$, 
  - $\text{color}(0, i, j) \in \{\text{green, red}\}$
#Sperner

\[ \#\text{Sperner}(C) = \text{the number of rainbow triangles in the coloring given by } C. \]
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Clearly \#Sperner \in \#P.
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Clearly \#Sperner \in \#P.

---

**Sperner’s Lemma**

\forall C \in \{0, 1\}^*: \ #Sperner(C) \geq 1.

Maybe \ #Sperner - 1 is another candidate for being in \text{GapP} \setminus \#P?
#Sperner − 1 ∈ #P?
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#Sperner $- 1 \in \#P$, because we can ignore the yellow triangles and count the purple triangles twice!
When does such a trick work? An easier version:

Given two \(#P\) functions \(f, g\) with the property that \(f + 1 = g\).
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**The functional closure properties of \(\#P\)**

Every polynomial \(\varphi \in \mathbb{Q}[f_1, \ldots, f_k]\) has a unique expression over the **binomial basis**:

\[
\varphi(f_1, \ldots, f_k) = \sum_{\alpha \in \mathbb{N}^k} c_{\alpha} (f_1^{a_1}) \cdots (f_k^{a_k}).
\]

If all \(c_{\alpha} \in \mathbb{N}\), then we say that \(\varphi\) is **binomial-good**.

For example, \(f^2 g = 2\binom{f}{2}\binom{g}{1} + \binom{f}{1}\binom{g}{1}\).
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Given an ideal \( I \), a coset \( \varphi + I \) is called **binomial-good** if \( \varphi + I \) contains a binomial-good representative.

- [I, Pak 2022]: For a large class of ideals, the Sperner trick works if and only if \( \varphi + I \) is binomial-good.
- If the ideal is generated by linear polynomials, then checking if \( \varphi + I \) is binomial-good reduces to integer programming.
This gives an algorithmic way of finding out when exactly “Sperner-type tricks” work. With some extra work (i.e., simulating #P instances in TFNP search problems) we get a TFNP-like inclusion diagram [I, Pak 2022]:

\[
\begin{align*}
#\text{COUNTGAP(BIPARTITEUNBALANCE)} & \rightarrow (#\text{PPA(LEAF)} - 1)/2 \rightarrow (#\text{PPA(PRELEAF)} - 1) \rightarrow (#\text{PPP(PIGEON)} - 1) \rightarrow (#\text{PLS(ITER)} - 1) \\
#\text{COUNTALL-PPA(LEAF)}/2 & \rightarrow (#\text{PPAD(SOURCEOREXCESS(2,1))} - 1) \rightarrow (#\text{CLS(EITHERSOLUTION(SOURCEOREXCESS(2,1),ITER))} - 1) \rightarrow \text{Thm} \\
#P & = (#\text{PPAD(SOURCEORSINK)} - 1)/2 \\
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All equalities with #P are shown via relativizing parsimonious reductions. A solid arrow represents a relativizing parsimonious reduction. An arrow with a ∆ represents a relativizing parsimonious reduction where there is an oracle separation in the other direction.
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Thank you for your attention!

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