Symmetry approach for differential-difference equations

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Based on: VN, Jing Ping Wang, CMP, 406, 1, 1-36 (2025);
A.V. Mikhailov, VN, Jing Ping Wang, CMP, 393, 2, (2022).

Geometry and Integrability
ICMS, Edinburgh, 12-16 May 2025

Introduction

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$$\begin{aligned} \mathrm{mKdV}^1: \quad u_t &= u_{xxx} - 3u^2 u_x - 3u_x u^2, \\ \mathrm{mKdV}^2: \quad u_t &= u_{xxx} + 3[u, u_{xx}] - 6uu_x u, \\ \text{Burgers equation}: \quad u_t &= u_{xx} + 2uu_x, \end{aligned}$$

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5. Sokolov *et al*: nonabelian Painlevé equations.

Examples of integrable nonabelian differential-difference equations

1. Nonabelian Volterra equation

$$u_{n,t} = u_n u_{n+1} - u_{n-1} u_n, \quad u = u(n,t) \in \mathrm{Mat}(N, \mathbb{C}), \quad n \in \mathbb{Z}.$$

Notation: $u := u_n,\ u_1 := u_{n+1},\ u_{-1} := u_{n-1}\ \dots .$
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2. Other examples:

$$\begin{aligned}
\mathrm{mVL}^1: &\quad u_t = (u-\alpha)u_1(u+\alpha) - (u+\alpha)u_{-1}(u-\alpha),\\
\mathrm{mVL}^2: &\quad u_t = u^2 u_1 - u_{-1} u^2,\\
\text{Bogoyavlensky}^1: &\quad u_t = u \sum_{i=1}^{n} u_i - \sum_{i=1}^{n} u_{-i} u,\\
\text{Bogoyavlensky}^2: &\quad u_t = u u_1 \cdots u_n - u_{-n} \cdots u_{-1} u,\\
\text{Ablowitz-Ladik}: &\quad u_t = u_1(1 - vu), \quad v_t = (vu - 1)v_{-1}.
\end{aligned}$$

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2. Classification of nonabelian differential-difference equations.

1. Let

$$\mathfrak{A} = \langle e;\ u_n,\ n \in \mathbb{Z} \rangle$$

be a free associative unital algebra over $\mathbb{C}$ generated by $u_n,\ n \in \mathbb{Z}$ and a unit element e.

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Free associative difference algebra

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   Notation: $\alpha \cdot e \equiv \alpha,\ \ \alpha \in \mathbb{C}$.

2. Shift operator $S : \mathfrak{A} \to \mathfrak{A}$: an automorphism of $\mathfrak{A}$ defined as

$$S(u_n) = u_{n+1}, \quad S(e) = e,$$

   so $S(f(u_p, \ldots, u_q)) = f(u_{p+1}, \ldots, u_{q+1}),\ \ f(u_p, \ldots, u_q) \in \mathfrak{A}$.

Natural grading

1. Natural grading on $\mathfrak{A}$:

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2. $\pi_k : \mathfrak{A} \to \mathfrak{A}_k$.

A derivation $\mathcal{D}$ is a $\mathbb{C}$-linear map satisfying the Leibnitz's rule

$$\mathcal{D}(fg) = \mathcal{D}(f)g + f\mathcal{D}(g), \quad f, g \in \mathfrak{A}, \quad \alpha, \beta \in \mathbb{C}.$$

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3. Every $f \in \mathfrak{A}$ can be uniquely written as $f = \sum_{k \geq 0}^{M} f_k, \ f_k = \pi_k(f) \in \mathfrak{A}_k$.

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Evolutionary derivations

Definition

A derivation is called evolutionary if it commutes with the shift operator S.

An evolutionary derivation is completely defined by its action on the generator u:

$$\mathcal{D}(u) = a, \quad \mathcal{D}(u_n) = S^n(a), \quad a \in \mathfrak{A}.$$

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1. The element a is called the characteristic of an evolutionary derivation. We shall denote an evolutionary derivation with characteristic a by $\mathcal{D}_a$.
2. Commutator of two evolutionary derivations is also an evolutionary derivation:

$$\mathcal{D}_c = [\mathcal{D}_a, \mathcal{D}_b], \quad c = \mathcal{D}_a(b) - \mathcal{D}_b(a).$$

Operators and formal series

1. Denote by $\mathcal{L}_a, \mathcal{R}_a$ left and right multiplication operators:

$$\mathcal{L}_a(f) = af, \quad \mathcal{R}_a(f) = fa, \quad f \in \mathfrak{A}.$$

Relations:

$$[\mathcal{L}_a, \mathcal{R}_b] = 0, \quad \mathcal{L}_a \mathcal{L}_b = \mathcal{L}_{ab}, \quad \mathcal{R}_a \mathcal{R}_b = \mathcal{R}_{ba}.$$

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$$\mathcal{M} = \bigoplus_{p,q\geq 0} \mathcal{M}_{p,q}, \quad \mathcal{M}_{p,q} = \{\mathcal{L}_a\mathcal{R}_b \,|\, a \in \mathfrak{A}_p,\ b \in \mathfrak{A}_q\}.$$

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3. Local difference operator: $B = \sum_{i=p}^{q} b_i S^i,\ \ b_i \in \mathcal{M}$,
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4. Local formal series: $B = \sum_{i \leq N} \sum_{p,q \geq 0} b_{ipq} S^i, \quad b_{ipq} \in \mathcal{M}_{p,q}$.

Lie algebra structure on $\mathfrak{A}$

Definition

A Fréchet derivative of an element $f \in \mathfrak{A}$ is defined as a local difference operator f_*:

$$f_*(a) = \frac{d}{d\epsilon} f(u_p + \epsilon S^p(a), \ldots, u_q + \epsilon S^q(a)), \quad \forall a \in \mathfrak{A}.$$

Example:

$$f = uu_1 - u_{-1}u, \quad f_* = \mathcal{R}_{u_1} + \mathcal{L}_u S - \mathcal{R}_u S^{-1} - \mathcal{L}_{u_{-1}}.$$

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   $[a, b] := \mathcal{D}_a(b) - \mathcal{D}_b(a) = b_*(a) - a_*(b)$.
3. Natural grading: $[\mathfrak{A}_n, \mathfrak{A}_m] \in \mathfrak{A}_{n+m-1}$.

Assume now that the generators u_k depend on $t \in \mathbb{C}$: $u_k = u_k(t)$. With every evolutionary derivation $\mathcal{D}_f$ we can identify a differential-difference equation on $\mathfrak{A}$:

$$u_t = f.$$

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3. Order of the equation is defined as order of f_*.

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Symmetry algebra

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2. If we assume that the generators u_k depend on $\tau \in \mathbb{C}$ then the common way of representing a symmetry is associating with $\mathcal{D}_g$ another differential-difference equation
$$u_\tau = g.$$

Example: Consider the nonabelian Volterra equation

$$u_t = uu_1 - u_{-1}u.$$

The nonabelian Volterra equation has the order $(-1, 1)$.
Symmetry of order $(-2, 2)$:

$$u_\tau = uu_1u_2 - u_{-2}u_{-1}u + u(u + u_1)u_1 - u_{-1}(u + u_{-1})u$$

Infinitely many symmetries of orders $(-n, n)$, $n \in \mathbb{N}$.

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2. The Lie subalgebra of symmetries of $u_t = f$ is the centraliser of f $\mathcal{C}_f = \{g \in \mathfrak{A} \mid [g, f] = 0\}$.

Definition

A nonabelian differential-difference equation $u_t = f$ is called integrable if its Lie algebra of symmetries $\mathcal{C}_f$ is infinite dimensional and contains symmetries of arbitrary large order, i.e. for every $N \in \mathbb{N}$ there exists a symmetry g of the total order $Ord(g) \geq N$.

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3. Due to natural grading

$$[f, g] = 0 \quad \Longleftrightarrow \quad \sum_{k=1}^{p} [f_k, g_{p-k}] = 0,\ p = 1, \ldots N + M.$$

Symbolic representation

We construct the symbolic representation $\hat{\mathfrak{A}} = \bigoplus_{p\geq 0} \hat{\mathfrak{A}}_p$ of the naturally graded difference algebra $\mathfrak{A} = \bigoplus_{p\geq 0}$.

1. $\phi : \mathfrak{A}_p \to \hat{\mathfrak{A}}_p$:

$$\phi: \ \alpha \mapsto \alpha, \qquad \phi: \ u_{i_1} u_{i_2} \cdots u_{i_n} \mapsto \hat{u}^n \xi_1^{i_1} \xi_2^{i_2} \cdots \xi_n^{i_n}.$$

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2. Examples:

$$u_k \overset{\phi}{\mapsto} \hat{u}\xi_1^k, \ \ u^m \overset{\phi}{\mapsto} \hat{u}^m, \ \ u_1 u_{-1} \overset{\phi}{\mapsto} \hat{u}^2 \xi_1 \xi_2^{-1},$$

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3.

$$f \in \mathfrak{A}_n \stackrel{\phi}{\mapsto} \hat{f} = \hat{u}^n a(\xi_1, \ldots, \xi_n), \quad a(\xi_1, \ldots, \xi_n) \in \mathbb{C}[\xi_1^{\pm 1}, \ldots, \xi_n^{\pm 1}].$$

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4.

$$S(f) \overset{\phi}{\mapsto} \hat{u}^n a(\xi_1, \ldots, \xi_n)(\xi_1 \cdots \xi_n).$$

Symmetries in the symbolic representation

Equation in the symbolic representation:

$$u_t = f, \quad f \stackrel{\phi}{\mapsto} \hat{u}\omega(\xi_1) + \hat{u}^2 a_2(\xi_1, \xi_2) + \hat{u}^3 a_3(\xi_1, \xi_2, \xi_3) + \cdots .$$

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Symmetry in the symbolic representation:

$$u_\tau = g, \quad g \overset{\phi}{\mapsto} \hat{u}\Omega(\xi_1) + \hat{u}^2 A_2(\xi_1, \xi_2) + \hat{u}^3 A_3(\xi_1, \xi_2, \xi_3) + \cdots .$$

Symbolic representation

Theorem

If $[f, g] = 0$ *then* A_k *can be found recursively:*

$$A_2(\xi_1, \xi_2) = \frac{\Omega(\xi_1\xi_2) - \Omega(\xi_1) - \Omega(\xi_2)}{\omega(\xi_1\xi_2) - \omega(\xi_1) - \omega(\xi_2)} a_2(\xi_1, \xi_2),$$

$$A_s(\xi_1, \ldots, \xi_s) = \frac{\Omega(\xi_1 \cdots \xi_s) - \Omega(\xi_1) - \cdots - \Omega(\xi_s)}{\omega(\xi_1 \cdots \xi_s) - \omega(\xi_1) - \cdots - \omega(\xi_s)} a_s(\xi_1, \ldots, \xi_s) +$$

$$+ R_s(a_2, \ldots, a_{s-1}, A_2, \ldots, A_{s-1}), \quad s = 3, \ldots.$$

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$$A_s(\xi_1,\ldots,\xi_s)=\frac{\Omega(\xi_1\cdots\xi_s)-\Omega(\xi_1)-\cdots-\Omega(\xi_s)}{\omega(\xi_1\cdots\xi_s)-\omega(\xi_1)-\cdots-\omega(\xi_s)}a_s(\xi_1,\ldots,\xi_s)+$$
$$+R_s(a_2,\ldots,a_{s-1},A_2,\ldots,A_{s-1}),\quad s=3,\ldots.$$

- We call $\Omega(\xi)\in\mathbb{C}[\xi,\xi^{-1}]$ admissible if A_s are Laurent polynomials in $\xi_1,\ldots,\xi_s$.
- The set of admissible $\Omega(\xi)$ is a vector space and if the equation is integrable then this vector space is infinite dimensional.
- The algebra of symmetries is completely parametrised by the set of admissible $\Omega(\xi)$.

Example: Volterra chain

Consider the Volterra equation $u_t = uu_1 - u_{-1}u$.
We introduce the linear term by $u \to u + 1$:

$$u_t = u_1 - u_{-1} + uu_1 - u_{-1}u =: f.$$

Symbolic representation:

$$f \overset{\phi}{\mapsto} \hat{u}\omega(\xi_1) + \hat{u}^2 a_2(\xi_1, \xi_2), \quad \omega(\xi_1) = \xi_1 - \xi_1^{-1}, \quad a_2(\xi_1, \xi_2) = \xi_2 - \xi_1^{-1}.$$

Let us seek a symmetry with $\Omega(\xi_1) = \xi_1^2 - \xi_1^{-2}$. Then:

$$A_2(\xi_1, \xi_2) = \frac{(1 + \xi_1)(1 + \xi_2)(\xi_1\xi_2 - 1)(\xi_1\xi_2 + 1)}{\xi_1^2 \xi_2},$$

$$A_3(\xi_1, \xi_2, \xi_3) = \frac{(\xi_1\xi_2\xi_3 - 1)(1 + \xi_1 + \xi_1\xi_2 + \xi_1\xi_2\xi_3)}{\xi_1^2 \xi_2},$$

and $A_s = 0$, $s > 3$. This corresponds to a symmetry

$$u_\tau = u_2 - u_{-2} + (u + u_1)(u_1 + u_2) - (u_{-2} + u_{-1})(u_{-1} + u) +$$

$$+ uu_1 u_2 - u_{-2}u_{-1}u + u(u + u_1)u_1 - u_{-1}(u_{-1} + u)u.$$

Admissible linear terms: $\Omega_k(\xi) = \xi^k - \xi^{-k}$.

1. Consider algebra of formal series $\mathfrak{A}((S))$ consisting of formal series of the form

$$\sum_{i\leq n}\sum_{p+q\geq 0} a_{ipq}S^i, \quad a_{ipq}\in\mathcal{M}_{p,q}, \quad n\in\mathbb{Z}.$$

Quasi-locality and roots of formal series

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2. Root extraction problem: for a generic formal series

$$A = S^n + \sum_{i \leq n'} \sum_{p+q \geq 1} a_{ipq} S^i, \quad a_{ipq} \in \mathcal{M}_{p,q}, n \in \mathbb{N}.$$

   there is *no local* formal series $B = S + \sum_{i \leq n'} \sum_{p+q \geq 1} b_{ipq} S^i$, such that $B^n = A$.

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   there is *no local* formal series $B = S + \sum_{i\leq n'}\sum_{p+q\geq 1} b_{ipq}S^i$, such that $B^n = A$.

3. The root exists in the extended algebra.

Quasi-locality and roots of formal series

1. Define $\theta_k = \frac{S-1}{S^{k+1}-1} = (1 + S + \cdots + S^k)^{-1},\ k = 1, 2, \ldots$.

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1. Define $\theta_k = \frac{S-1}{S^{k+1}-1} = (1+S+\cdots+S^k)^{-1},\ k = 1, 2, \ldots$.
2. Define the operator algebra $\mathcal{M}$ extension as

$$\begin{aligned}
\mathcal{M}^{(0)} &= \mathcal{M}, \\
\mathcal{M}^{(i+1)} &= \overline{\mathcal{M}^{(i)} \bigcup_{k=1}^{\infty} \theta_k(\mathcal{M}^{(i)})}, \\
\mathcal{M}^{(s)} &\xrightarrow{s\to\infty} \mathcal{M}(\theta).
\end{aligned}$$

Quasi-locality and roots of formal series

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3. We shall call the algebra $\mathfrak{A}((S))$ with coefficients in $\mathcal{M}(\theta)$ the algebra of quasi-local formal series.

Quasi-locality and roots of formal series

Theorem

Let

$$A = S^n + \sum_{p+q\geq 1} \sum_{i\leq m(p,q)} a_{ipq} S^i \quad a_{ipq} \in \mathcal{M}_{p,q}, n \in \mathbb{N},$$

be a local formal series. Then there exists a unique quasi-local formal series

$$B = S + \sum_{p+q\geq 1} \sum_{i\leq m(p,q)} b_{ipq} S^i, \quad b_{ipq} \in \mathcal{M}(\theta),$$

such that $B^n = A$.

Formal recursion operator

Definition

A quasi-local formal series

$$\Lambda = \varphi(S) + \sum_{p+q\geq 1} \sum_{i\leq m(p,q)} \varphi_{ipq} S^i, \quad \varphi(x) \in \mathbb{C}[x, x^{-1}],$$

is called a formal recursion operator for the equation $u_t = f,\ \pi_0(f) = 0,\ \pi_1(f) \neq 0$, *if*

$$\Lambda_t = f_* \circ \Lambda - \Lambda \circ f_*.$$

Formal recursion operator

Definition

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$$\Lambda_t = f_* \circ \Lambda - \Lambda \circ f_*.$$

Theorem

If the differential-difference equation $u_t = f,\ \pi_1(f) \neq 0$ *possesses an infinite dimensional algebra of symmetries, then it possesses a formal recursion operator of the form*

$$\Lambda = S + \sum_{p+q\geq 1} \sum_{i\leq m(p,q)} \varphi_{ipq} S^i.$$

Idea of the proof

1. $u_\tau = g(u_M, \ldots, u_N)$ - a symmetry, such that

$$\pi_1(g) = u_n + \sum_{k=M}^{N-1} c_k u_k, \quad N >> n.$$

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3. The first $N - n$ terms of g_* can be found from $g_{*,t} = f_* \circ g_* - g_* \circ f_*$.
4. $\Lambda \approx g_*^{\frac{1}{N}}$.

Integrability test

The above theorems provides the following integrability test for a given nonabelian differential-difference equation $u_t = f$:

- For a given f the equation $\Lambda_t = f_* \circ \Lambda - \Lambda \circ f_*$ can be formally resolved for Λ, i.e. its coefficients can be explicitly found in terms of f (in the symbolic representation);
- The requirement of quasi-locality of these coefficients provides the necessary integrability conditions independent on the symmetry structure.

Classification results

We classify integrable nonabelian differential-difference equations

$$u_t = f(u_{-n}, u_{-n+1}, \dots, u_n), \quad f \in \mathfrak{A}, \quad n = 1, 2, 3,$$

satisfying the following conditions

- Non-zero linear term: $\pi_1(f) \neq 0$ and $\pi_1(f)$ depends of u_n, where π_1 is the projection $\pi_1 : \mathfrak{A} \to \mathfrak{A}_1$;
- Quasi-linearity: $\frac{d^2}{d\epsilon^2} f(\epsilon u_{-n}, u_{-n+1}, \dots, u_{n-1}, \epsilon u_n) = 0$;
- Skew-symmetry: $\mathcal{T}(f) = -f$, where $\mathcal{T} : \mathfrak{A} \to \mathfrak{A}$ is defined as $\mathcal{T}(u_n) = u_{-n},\ \mathcal{T}(ab) = ba$;
- No lower order symmetries.

Classification of skew-symmetric quasi-linear equations: order $(-1, 1)$

Theorem

Up to re-scaling and shift transformations, every skew-symmetric quasi-linear integrable differential-difference equation of the form

$$u_t = f(u_{-1}, u, u_1)$$

can be obtained from one of the equations in the following list:

$$\begin{aligned} u_t &= (\alpha u^2 + \beta u + \gamma)u_1 - u_{-1}(\alpha u^2 + \beta u + \gamma), \\ u_t &= (u - \alpha)u_1(u + \alpha) - (u + \alpha)u_{-1}(u - \alpha) \end{aligned}$$

Classification of skew-symmetric quasi-linear equations: order $(-2, 2)$

Theorem

Up to re-scaling and shift transformations, every skew-symmetric quasi-linear integrable differential-difference equation of the form

$$u_t = f(u_{-2}, u_{-1}, u, u_1, u_2)$$

can be obtained from one of the equations in the following list:

Classification of skew-symmetric quasi-linear equations: order $(-2, 2)$

Volterra type:

$$\begin{aligned}
u_t &= (\alpha u^2 + \beta u + \gamma)u_2 - u_{-2}(\alpha u^2 + \beta u + \gamma),\\
u_t &= (u - \alpha)u_2(u + \alpha) - (u + \alpha)u_{-2}(u - \alpha),\\
u_t &= uu_1u_2 - u_{-2}u_{-1}u + u(u_{-1} - u_1)u,\\
u_t &= (u_{-1} + u)u_2 - u_{-2}(u_1 + u) + uu_1 - u_{-1}u,\\
u_t &= uu_1u_2 - u_{-2}u_{-1}u + u(u - u_1)u_1 - u_{-1}(u - u_{-1})u,\\
u_t &= (u + u_{-1})(u_1 + u)u_2 - u_{-2}(u_{-1} + u)(u_1 + u) + u(u + u_{-1})u_1\\
&\quad - u_{-1}(u + u_1)u - u(u_1 - u_{-1})u,\\
u_t &= u(u_1u_2u_1 - u_{-1}u_{-2}u_{-1} - u_1uu_1 + u_{-1}uu_{-1})u,\\
u_t &= (uu_{-1} + 1)(uu_1 + 1)u_2 - u_{-2}(u_{-1}u + 1)(u_1u + 1),\\
u_t &= uu_1u_2u_{-1}u - uu_1u_{-2}u_{-1}u + \alpha(uu_1u_2 - u_{-2}u_{-1}u + u(u_{-1} - u_1)u),\\
u_t &= uu_{-1}uu_1u_2 - u_{-2}u_{-1}uu_1u + \alpha(uu_1u_2 - u_{-2}u_{-1}u + u(u_{-1} - u_1)u),\\
u_t &= (u^2 - \alpha^2)(u_1^2 - \alpha^2)u_2 - u_{-2}(u_{-1}^2 - \alpha^2)(u^2 - \alpha^2) - (u^2 - \alpha^2)u_1uu_1\\
&\quad + u_{-1}uu_{-1}(u^2 - \alpha^2) + uu_{-1}(u^2 - \alpha^2)u_1 - u_{-1}(u^2 - \alpha^2)u_1u.
\end{aligned}$$

Classification of skew-symmetric quasi-linear equations: order $(-2, 2)$

Bogoyavlensky type:

$$
\begin{aligned}
u_t &= u(u_1 + u_2) - (u_{-1} + u_{-2})u, \\
u_t &= uu_1u_2 - u_{-2}u_{-1}u, \\
u_t &= uu_1u_2 - u_{-2}u_{-1}u + u[u_1, u] - [u, u_{-1}]u, \\
u_t &= (u + u_{-1})(u + u_1)(u_1 + u_2) - (u_{-2} + u_{-1})(u + u_{-1})(u + u_1), \\
u_t &= uu_1u_2(u + \alpha) - (u + \alpha)u_{-2}u_{-1}u, \\
u_t &= u(u + \alpha)u_1(u_2 + \alpha) - (u_{-2} + \alpha)u_{-1}(u + \alpha)u \\
&\quad + (u + \alpha)u_{-1}u(u_1 + \alpha) - (u_{-1} + \alpha)uu_1(u + \alpha), \\
u_t &= (uu_{-1} + \alpha)(uu_1 + \alpha)uu_1u_2 - u_{-2}u_{-1}u(u_{-1}u + \alpha)(u_1u + \alpha), \\
u_t &= (uu_{-1} + \alpha)(u_1u + \alpha)(u_2u_1 + \alpha)u \\
&\quad - u(u_{-1}u_{-2} + \alpha)(uu_{-1} + \alpha)(u_1u + \alpha).
\end{aligned}
$$

Classification of skew-symmetric quasi-linear equations: order $(-3, 3)$

Theorem

Up to re-scaling and shift transformations, every skew-symmetric quasi-linear integrable differential-difference equation of the form

$$u_t = f(u_{-3}, u_{-2}, u_{-1}, u, u_1, u_2, u_3)$$

can be obtained from one of the equations in the following list:

Classification of skew-symmetric quasi-linear equations: order $(-3, 3)$

Volterra type:

$$
\begin{aligned}
u_t &= (\alpha u^2 + \beta u + 1)u_3 - u_{-3}(\alpha u^2 + \beta u + 1),\\
u_t &= (u - \alpha)u_3(u + \alpha) - (u + \alpha)u_{-3}(u - \alpha),\\
u_t &= uu_{-1}u_1u_3 - u_{-3}u_{-1}u_1u,\\
u_t &= uu_1u_2u_3 - u_{-3}u_{-2}u_{-1}u - u(u_1u_2 - u_{-2}u_{-1})u,\\
u_t &= (u + u_{-1} + u_{-2})u_3 - u_{-3}(u + u_1 + u_2) + u(u_1 + u_2) - (u_{-1} + u_{-2})u,\\
u_t &= (uu_{-2}u_{-1} + \alpha)uu_1u_2u_3 - u_{-3}u_{-2}u_{-1}u(u_1u_2u + \alpha) - \alpha u(u_1u_2 - u_{-2}u_{-1})\\
u_t &= uu_1u_2u_3(u_{-2}u_{-1}u + \alpha) - (uu_1u_2 + \alpha)u_{-3}u_{-2}u_{-1}u - \alpha u(u_1u_2 - u_{-2}u_{-1})
\end{aligned}
$$

Classification of skew-symmetric quasi-linear equations: order $(-3, 3)$

Bogoyavlensky type:

$$
\begin{aligned}
u_t &= u(u_1+u_2+u_3)-(u_{-1}+u_{-2}+u_{-3})u,\\
u_t &= uu_1u_2u_3-u_{-3}u_{-2}u_{-1}u,\\
u_t &= uu_1u_3-u_{-3}u_{-1}u+u^2u_2-u_{-2}u^2+[u,u_{-1}u_1],\\
u_t &= uu_1u_2u_3(u+\alpha)-(u+\alpha)u_{-3}u_{-2}u_{-1}u,\\
u_t &= uu_{-1}u_1uu_2u_1u_3-u_{-3}u_{-1}u_{-2}uu_{-1}u_1u,\\
u_t &= uu_1u_2u_3-u_{-3}u_{-2}u_{-1}u+u[u,u_1u_2]-[u_{-2}u_{-1},u]u+[u,u_{-1}uu_1],\\
u_t &= (u_{-2}+u_{-1}+u)(u_{-1}+u+u_1)(u+u_1+u_2)(u_1+u_2+u_3)\\
&\quad -(u_{-3}+u_{-2}+u_{-1})(u_{-2}+u_{-1}+u)(u_{-1}+u+u_1)(u+u_1+u_2),\\
u_t &= (u+\alpha)uu_1u_2u_3-u_{-3}u_{-2}u_{-1}u(u+\alpha)+u(u_{-1}+\alpha)uu_1u_2\\
&\quad -(u_{-1}+\alpha)uu_1u_2u-u_{-2}u_{-1}u(u_1+\alpha)u+uu_{-2}u_{-1}u(u_1+\alpha)\\
&\quad +\alpha(uu_{-1}uu_1-u_{-1}uu_1u),\\
u_t &= (uu_{-2}u_{-1}+\alpha)(uu_1u_{-1}+\alpha)(uu_1u_2+\alpha)uu_1u_2u_3\\
&\quad -u_{-3}u_{-2}u_{-1}u(u_{-2}u_{-1}u+\alpha)(u_1u_{-1}u+\alpha)(u_1u_2u+\alpha),
\end{aligned}
$$

Classification of skew-symmetric quasi-linear equations: order $(-3,3)$

Bogoyavlensky type:

$$\begin{aligned} u_t &= (uu_{-1}u_{-2}+\alpha)(u_1uu_{-1}+\alpha)(u_2u_1u+\alpha)(u_3u_2u_1+\alpha)u \\ &\quad -u(u_{-1}u_{-2}u_{-3}+\alpha)(uu_{-1}u_{-2}+\alpha)(u_1uu_{-1}+\alpha)(u_2u_1u+\alpha). \end{aligned}$$

Abelian equations

1. Classifiaction of Volterra type equations - Yamilov (1983)

$$u_t = f(u_{-1}, u, u_1);$$

Abelian equations

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Abelian equations

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$$u_t = f(u_{-1}, u, u_1);$$

2. Classification of Toda type equations - Adler, Shabat, Yamilov (1990s);
3. Classification of quasi-linear equations of order $(-2, 2)$ - Garifullin, Levi, Yamilov (‘17-‘18)

$$u_t = A(u_{-1}, u, u_1)u_2 + B(u_{-1}, u, u_1)u_{-2} + C(u_{-1}, u, u_1).$$

Abelian equations

1. $\mathcal{A} = \bigoplus_{p>0} \mathcal{A}_p$ - abelian associative algebra of formal series. Each $\mathcal{A}_p$ consists of elements $u_{i_1} \cdots u_{i_p}$.

Abelian equations

1. $\mathcal{A} = \bigoplus_{p>0} \mathcal{A}_p$ - abelian associative algebra of formal series. Each $\mathcal{A}_p$ consists of elements $u_{i_1} \cdots u_{i_p}$.
2. Differential-difference equation: $u_t = f, \; f \in \mathcal{A}$.

Formal recursion operator

Definition

A quasi-local formal series

$$\Lambda = \varphi(S) + \sum_{p+q\geq 1} \sum_{i\leq m(p,q)} \varphi_{ipq} S^i, \quad \varphi(x) \in \mathbb{C}[x, x^{-1}],$$

is called a formal recursion operator for the equation $u_t = f, \ \pi_0(f) = 0, \ \pi_1(f) \neq 0$, *if*

$$\Lambda_t = f_* \circ \Lambda - \Lambda \circ f_*.$$

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$$\Lambda = \varphi(S) + \sum_{p+q\geq 1} \sum_{i\leq m(p,q)} \varphi_{ipq} S^i, \quad \varphi(x) \in \mathbb{C}[x, x^{-1}],$$

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Theorem

If the differential-difference equation $u_t = f,\ \pi_1(f) \neq 0$ *possesses an infinite dimensional algebra of symmetries, then it possesses a formal recursion operator of the form*

$$\Lambda = S + \sum_{p+q\geq 1} \sum_{i\leq m(p,q)} \varphi_{ipq} S^i.$$

Formal recursion operator - Adler's version

Theorem (V.E. Adler)

If a differential-difference equation $u_t = f$ *admits an infinite hierarchy of symmetries* $u_\tau = g(u_p, \ldots, u_q)$ *with* q *arbitrary large then there exists a* **local** *formal recursion operator of the form*

$$\Lambda = f_* + \sum_{k \geq 0} a_k S^{-k}.$$

Classification results

We classify integrable nonabelian differential-difference equations

$$u_t = f(u_{-3}, u_{-2}, u_{-1}, u, u_1, u_2, u_3), \quad f \in \mathcal{A},$$

satisfying the following conditions

- Non-zero linear term: $\pi_1(f) \neq 0$ and $\pi_1(f)$ depends of u_3, where π_1 is the projection $\pi_1 : \mathcal{A} \to \mathcal{A}_1$;
- Quasi-linearity: $\frac{d^2}{d\epsilon^2} f(\epsilon u_{-3}, u_{-2}, u_{-1}, u, u_1, u_2, \epsilon u_3) = 0$;
- Skew-symmetry: $\mathcal{T}(f) = -f$, where $\mathcal{T} : \mathfrak{A} \to \mathfrak{A}$ is defined as $\mathcal{T}(u_n) = u_{-n}$;
- No lower order symmetries.

Theorem

Up to re-scaling and shift transformations, every skew-symmetric quasi-linear integrable differential-difference equation of the form

$$u_t = f(u_{-3}, u_{-2}, u_{-1}, u, u_1, u_2, u_3)$$

can be obtained from one of the equations in the following list:

Volterra type:

$$u_t = u(u_3 - u_{-3}),$$
$$u_t = u^2(u_3 - u_{-3}),$$
$$u_t = (u^2 + u)(u_3 - u_{-3}),$$
$$u_t = u_{-1} u u_1 (u_3 - u_{-3}),$$
$$u_t = u(u_2 u_3 - u_1 u_2 + u u_1 - u u_{-1} + u_{-1} u_{-2} - u_{-2} u_{-3}),$$
$$u_t = u\left(\frac{u_3 u_1}{u_2} - \frac{u_{-3} u_{-1}}{u_{-2}}\right) + u^2\left(\frac{u_2}{u_1} - \frac{u_{-2}}{u_{-1}}\right),$$
$$u_t = u(u_1 u_2 u_3 - u u_1 u_2 + u u_{-1} u_{-2} - u_{-1} u_{-2} u_{-3}),$$

Classification results

$$u_t = u(\frac{u_3}{u_2} - \frac{u_{-3}}{u_{-2}}) + u(\frac{u_2}{u_1} - \frac{u_{-2}}{u_{-1}}) + u_1 - u_{-1},$$

$$u_t = u^2\left(u_1^2 u_2^2 u_3 - u_{-1}^2 u_{-2}^2 u_{-3} - 2uu_1 u_{-1}(u_1 u_2 - u_{-1}u_{-2})\right),$$

$$u_t = (u + u_{-1} + u_{-2})u_3 - (u + u_1 + u_2)u_{-3} + u(u_1 + u_2 - u_{-1} - u_{-2}),$$

$$u_t = (uu_{-1}u_{-2} + \alpha)uu_1u_2u_3 - (uu_1u_2 + \alpha)uu_{-1}u_{-2}u_{-3} - \alpha u^2(u_1u_2 - u_{-1}u_{-2}).$$

Classification results

Bogoyavlensky type:

$$
\begin{aligned}
u_t &= u(u_1 + u_2 + u_3 - u_{-1} - u_{-2} + u_{-3}),\\
u_t &= u(u_1 u_2 u_3 - u_{-1} u_{-2} u_{-3}),\\
u_t &= u^2(u_1 u_2 u_3 - u_{-1} u_{-2} u_{-3}),\\
u_t &= (u^2 + u)(u_1 u_2 u_3 - u_{-1} u_{-2} u_{-3}),\\
u_t &= u(u_1 u_3 + u u_2 - u u_{-2} - u_{-1} u_{-3}),\\
u_t &= u(u_2 u_3 + u u_1 - u u_{-1} - u_{-2} u_{-3}),\\
u_t &= u^2 u_1 u_{-1}(u_1 u_2 u_3 - u_{-1} u_{-2} u_{-3}),\\
u_t &= u^4 u_1^2 u_{-1}^2 u_2 u_{-2}(u_1 u_2 u_3 - u_{-1} u_{-2} u_{-3}),\\
u_t &= u^2(u_1^2 u_2^2 u_3 - u_{-1} u u_1^2 u_2 + u_1 u u_{-1}^2 u_{-2} - u_{-1}^2 u_{-2}^2 u_{-3}),\\
u_t &= (u + u_{-1} + u_{-2})(u + u_1 + u_2)(u + u_1 + u_{-1})\\
&\quad (u_3 + u_2 + u_1 - u_{-1} - u_{-2} - u_{-3}),\\
u_t &= u(u_{-2} u_{-1} u + \alpha)(u u_1 u_2 + \alpha)(u_{-1} u u_1 + \alpha)(u_1 u_2 u_3 - u_{-1} u_{-2} u_{-3}).
\end{aligned}
$$

Sawada-Kotera type:

$$u_t = u(u_1 u_3 + u u_2 - u u_{-2} - u_{-1} u_{-3}) - u(u_2 + u_1 - u_{-1} - u_{-2}),$$
$$u_t = u(u_2 u_3 + u u_1 - u u_{-1} - u_{-2} u_{-3}) - u(u_2 + u_1 - u_{-1} - u_{-2}),$$
$$u_t = u^2(u_1 u_2 u_3 - u_{-1} u_{-2} u_{-3}) - u(u_1 u_2 - u_{-1} u_{-2}),$$
$$u_t = u^2 u_1 u_{-1}(u_1 u_2 u_3 - u_{-1} u_{-2} u_{-3}) - u^2(u_1 u_2 - u_{-1} u_{-2}),$$
$$u_t = u^4 u_1^2 u_{-1}^2 u_2 u_{-2}(u_1 u_2 u_3 - u_{-1} u_{-2} u_{-3}) - u^3 u_1 u_{-1}(u_1 u_2 - u_{-1} u_{-2}),$$
$$u_t = u^2(u_1^2 u_2^2 u_3 - u_{-1} u u_1^2 u_2 + u_1 u u_{-1}^2 u_{-2} - u_{-1}^2 u_{-2}^2 u_{-3}) - u^2(u_1 u_2 - u_{-1} u_{-2}),$$
$$u_t = (u^2 + 1)(u_3 \sqrt{u_1^2 + 1}\sqrt{u_2^2 + 1} - u_{-3}\sqrt{u_{-1}^2 + 1}\sqrt{u_{-2}^2 + 1}).$$