A tropical Edrei Theorem

Geometry and Integrability - Veselov birthday conference

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Konstanze Rietsch — King's College London

Infinite totally positive Toeplitz matrices

- A sequence $(c_i)_{i\in\mathbb{Z}_{\geq 0}}$ in $\mathbb{R}$ is called totally nonnegative (TNN) if the $\infty \times \infty$ Toeplitz matrix

$$M = \begin{pmatrix} 1 & c_1 & c_2 & c_3 & \cdots \\ & 1 & c_1 & c_2 & \ddots \\ & & 1 & c_1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

has all minors ≥ 0

"TNN matrix"

Gantmacher-Krein, Polya, Schoenberg, A. Whitney...

Conjecture of Schoenberg, proved by Edrei in 1951, then reproved and related to rep. thry of S_∞ by Thoma in 1960's, says:

Thm (Edrei 1951) Let $\Omega = \{((\alpha_i),(\beta_i)) \in \mathbb{R}^{\mathbb{N}}_{\geq 0} \times \mathbb{R}^{\mathbb{N}}_{\geq 0} \mid \alpha_1 \geq \alpha_2 \geq \dots,\ \beta_1 \geq \beta_2 \geq \dots,\ \sum \alpha_i + \beta_i < \infty\}$

Then (c_i) is a TNN sequence $\Longleftrightarrow$ $\sum c_i x^i = e^{\gamma x} \dfrac{\prod(1+\beta_i x)}{\prod(1-\alpha_i x)}$ for some $((\alpha_i),(\beta_i)) \in \Omega$, $\gamma \in \mathbb{R}_{\geq 0}$

Finite totally positive Toeplitz matrices

Thm (R '03 & '06)

Let $$\Delta_i\left(\left(\begin{matrix}1 & c_1 & c_2 & \cdots & c_n\\ & 1 & c_1 & \ddots & \vdots \\ & & 1 & \ddots & c_2\\ & & & \ddots & c_1 \\ & & & & 1\end{matrix}\right)\right) := \det\begin{pmatrix} \overbrace{c_{n-i+1} \quad \cdots}^{i} & c_n \\ c_{n-i} & \ddots & \vdots \\ \vdots & \ddots & \\ & c_{n-i} & c_{n-i+1}\end{pmatrix}.$$ Then we have a homeomorphism

$$T_{n+1}(\mathbb{R}_{>0}) = \left\{ \text{TP Toeplitz matrices } \begin{pmatrix}1 & c_1 & \cdots & c_n \\ & 1 & \ddots & \vdots \\ & & \ddots & c_1 \\ & & & 1\end{pmatrix} \in U_+ \right\} \xrightarrow{\Delta} \begin{pmatrix} \Delta_1/\Delta_0 & & & \\ & \Delta_2/\Delta_1 & & \\ & & \ddots & \\ & & & \Delta_{n+1}/\Delta_n\end{pmatrix} \in T_{SL_{n+1}}(\mathbb{R}_{>0})$$

$T_{SL_{n+1}}(\mathbb{R}_{>0}) \cong \mathbb{R}^n_{>0}$, so this is another parametrisation result.

Finite totally positive Toeplitz matrices

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↑ unipotent upper-triangular matrices

$T_{SL_{n+1}}(\mathbb{R}_{>0}) \cong \mathbb{R}^n_{>0}$, so this is another parametrisation result.

Geometric background

$$\begin{tikzcd} \mathcal{T}_{n+1}(\mathbb{R}_{>0}) \arrow[r, hook] \arrow[d, "\cong"] & \mathcal{T}^{\circ}_{n+1}(\mathbb{C}) = \left\{ \begin{matrix} \text{Toeplitz matrices } u(c) \\ \text{with minors } \Delta_i \neq 0 \end{matrix} \right\} = \mathrm{Spec}\left(qH^*\left(GL_{n+1}/B\right)_{(q)}\right) \arrow[d, "q=(q_1,\ldots,q_n)"] \\ \mathbb{R}^n_{>0} \arrow[r, hook] & (\mathbb{C}\setminus\{0\})^n \end{tikzcd}$$

D.Peterson '90's

Kostant: $q_i = \dfrac{\Delta_{i-1}\Delta_{i+1}}{\Delta_i^2}$

Finite totally positive Toeplitz matrices

Thm (R '03 & '06) Let $\Delta_i \left(\begin{pmatrix} 1 & c_1 & c_2 & \cdots & c_n \\ & 1 & c_1 & \ddots & \vdots \\ & & 1 & \ddots & c_2 \\ & & & \ddots & c_1 \\ & & & & 1 \end{pmatrix} \right) := \det \overbrace{\begin{pmatrix} c_{n-i+1} & \cdots & c_n \\ c_{n-i} & \ddots & \vdots \\ \vdots & \ddots & \\ \cdots & c_{n-i} & c_{n-i+1} \end{pmatrix}}^{i}$. Then we have a homeomorphism

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Proof via mirror symmetry (sketch)

- $\mathcal{B} = G^\vee / B^\vee$ has a "mirror dual" object:

  (Lie theoretic interp. of Laurent polynomial mirror of Givental inspired by Peterson's work on q. coh. also: Berenstein-Kazhdan geom. crystal)

$$\begin{tikzcd} B_+ \cap U_- T \dot{w}_0 U_- \arrow[r, "W"] & \mathbb{C} \\ b = u_1 t \dot{w}_0 u_2 \arrow[r, mapsto] \arrow[d, "\Delta"] & \sum f_i^*(u_1) + \sum f_i^*(u_2) \\ T & \end{tikzcd}$$

(R, '08)

- The critical points of $W|_{\Delta^{-1}(t)}$ are precisely the Toeplitz matrices in $\Delta^{-1}(t)$
- Show $W|_{\Delta^{-1}(t)}$ has a unique critical point in $B_+(\mathbb{R}_{>0})$. $\square$

Question : Is there a relation between these two parametrization theorems?

Thm (R '03, '06) Let $\Delta_i\left(\begin{pmatrix}1 & c_1 & c_2 & \cdots & c_n\\ & 1 & c_1 & & \vdots\\ & & 1 & \ddots & c_2\\ & & & \ddots & c_1\\ & & & & 1\end{pmatrix}\right) := \det\begin{pmatrix}\overbrace{c_{n-i+1} & \cdots & c_n}^{i}\\ c_{n-i} & \ddots & \vdots\\ \vdots & \cdots\; c_{n-i} & c_{n-i+1}\end{pmatrix}$ then

$$\Delta : \mathcal{T}_{n+1}(\mathbb{R}_{>0}) = \left\{\begin{array}{l}\text{TP Toeplitz matrices}\\ \begin{pmatrix}1 & c_1 & \cdots & c_n\\ & 1 & \ddots & \vdots\\ & & \ddots & c_1\\ & & & 1\end{pmatrix}\end{array}\right\} \longrightarrow \mathcal{T}_{SL_{n+1}}(\mathbb{R}_{>0}) \cong \mathbb{R}^n_{>0}$$

is a homeomorphism.

and

Thm (Edrei 1951) $\Omega = \left\{((\alpha_i),(\beta_i)) \in \mathbb{R}^{\mathbb{N}}_{\geq 0} \times \mathbb{R}^{\mathbb{N}}_{\geq 0} \;\middle|\; \begin{array}{l}\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots\\ \beta_1 \geq \beta_2 \geq \beta_3 \geq \cdots\end{array} \;\; \sum \alpha_i + \beta_i < \infty\right\}$

Then $\Omega \times \mathbb{R}_{\geq 0}$ parametrizes $\mathcal{T}_\infty(\mathbb{R}_{\geq 0})$ via

$$\Omega \times \mathbb{R}_{\geq 0} \longrightarrow \mathcal{T}_\infty(\mathbb{R}_{\geq 0}) = \left\{\begin{array}{l}\text{Infinite}\\ \text{TNN Toeplitz}\\ \text{matrices}\end{array} \begin{pmatrix}1 & c_1 & c_2 & c_3 & \cdots\\ & 1 & c_1 & c_2 & c_3 \cdots\\ & & 1 & c_1 & c_2 \\ & & & \ddots & \ddots & \ddots\end{pmatrix}\right\}$$

$$(\alpha_\cdot, \beta_\cdot, \gamma) \longmapsto e^{\gamma x} \frac{\prod (1 + \alpha_i x)}{\prod (1 - \beta_i x)} = \sum c_i x^i$$

ASIDE: Thoma and Vershik-Kerov interpretations of parameters

$S_n = \{\text{permutations of } \{1,\dots,n\}\}$, $\quad S_\infty = \bigcup_{n\in\mathbb{N}} S_n$ "infinite symmetric group"

Thm (Thoma) Let $\chi : S_\infty \longrightarrow \mathbb{R}$ be a "normalised character" of S_∞. If

$c_i :=$ the coefficient of the trivial character $\chi_{S_n}^{\text{triv}}$ in $\chi|_{S_n}$,

then the infinite Toeplitz matrix
$$\begin{pmatrix} 1 & 1 & c_2 & c_3 & \cdots & & \\ & 1 & 1 & c_2 & c_3 & \cdots & \\ & & 1 & 1 & c_2 & & \\ & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$
is totally positive,

and the character χ is given explicitly by

$$\chi((1,\dots,k)) = \sum \alpha_i^k + (-1)^{k+1} \sum \beta_i^k \qquad (k \geq 2)$$

in terms of the Schoenberg parameters α_i, β_i. $\left(1 + x + c_2x^2 + c_3x^3 + \dots = e^{\gamma x} \frac{\prod(1+\alpha_i x)}{\prod(1-\beta_i x)}\right)$

Thoma and Vershik-Kerov interpretation of parameters

S_n = n-th symmetric group , $S_\infty = \bigcup_{n\in\mathbb{N}} S_n$

for $\lambda \vdash n$ (Young diagram) write χ_λ for the corresponding normalised irreducible character of S_n.

↳ $\chi_\lambda(e)=1$

$\chi_{\square}$ on S_1

$\chi_{\substack{\square\\\square}}$ $\chi_{\square\square}$ on S_2

$\chi_{\substack{\square\\\square\\\square}}$ $\chi_{\substack{\square\square\\\square}}$ $\chi_{\square\square\square}$ on S_3

Thm (Vershik-Kerov) Consider a sequence λ_n of Young diagrams, $\lambda_n \vdash n$.

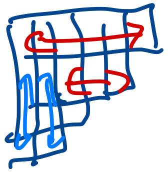
χ_{λ_n} converges pointwise to an irred character χ of S_∞

$\Updownarrow$

the normalised arm-lengths and the normalized leg lengths converge as $n \to \infty$.

↳ $\alpha_1 \geq \alpha_2 \geq ..$ ↳ $\beta_1 \geq \beta_2 \geq ..$

λ_n



The limiting character χ is the Thoma character $\chi((1..,k)) = \sum_i \alpha_i^k + (-1)^{k+1} \sum_i \beta_i^k$.

$\chi(e) = 1$

Total positivity in reductive groups and tropicalisation

Lusztig 1994
Progr. in Maths 123

G reductive alg group$/\mathbb{C}$ $\rightsquigarrow$ $G(\mathbb{R}_{>0})$ as well as $U_+(\mathbb{R}_{>0}), U_-(\mathbb{R}_{>0}), T(\mathbb{R}_{>0})\ldots$
with 'pinning'

$\mathcal{B}_{>0} \subset \mathcal{B} \cong G/B_+$ and also $\mathbb{R}_{\geq 0}$-analogues

$GL_n(\mathbb{R}_{>0}) = \{$"totally positive" matrices$\}$, matrices with all minors $\in \mathbb{R}_{>0}$.

- these "positive parts" come with coordinate systems 'positive charts'.

eg :
$$\phi^{212}_{>0}(a_1,a_2,a_3) = \begin{pmatrix} 1 & & \\ & 1 & a_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & a_3 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_2 & a_2a_3 \\ & 1 & a_1+a_3 \\ & & 1 \end{pmatrix} \in U_+(\mathbb{R}_{>0})$$
for $a_i \in \mathbb{R}_{>0}$

changing chart = birational transformation with $\mathbb{Z}_{>0}$ coeff's

- one can replace $\mathbb{R}_{>0}$ by other 'positive semifields' $K_{>0}$ to get $U_+(K_{>0})$ etc.

<u>in particular</u>: $K_{>0}$ = Laurent series $L(t)$ with leading term in $\mathbb{R}_{>0}$, so that

$$\boxed{\mathrm{val} : K_{>0} \longrightarrow (\mathbb{Z}, +, \min)} \quad \text{is semifield homomorphism}$$
$$t^A \longmapsto A$$

for $X(\mathcal{K}_{>0})$, totally positive part and

positive charts $\mathcal{K}_{>0}^{\dim X} \longrightarrow X(\mathcal{K}_{>0})$

with subtraction-free birat'l transformations

$$\begin{tikzcd}
(a_1,a_2,a_3) \arrow[r, mapsto] \arrow[d, dashed] & \phi_{212}(a_1,a_2,a_3) = \begin{pmatrix} 1 & a_2 & a_2a_3 \\ & 1 & a_1+a_3 \\ & & 1 \end{pmatrix} \\
(a_1',a_2',a_3') \arrow[r, mapsto] & \Phi_{121}(a_1,a_2,a_3) = \begin{pmatrix} 1 & a_1'+a_3' & a_1'a_2' \\ & 1 & a_2' \\ & & 1 \end{pmatrix}
\end{tikzcd}$$

$\overset{\text{tropicalise}}{\rightsquigarrow}$

Define $X(\mathbb{Z}_{min}) = X(\mathcal{K}_{>0})/\sim$

$\sim$: coordinates have same valuation

tropical charts $\mathbb{Z}^{\dim}_{min} \longrightarrow$ "$X(\mathbb{Z}_{min})$"

with piecewise linear transformations

$$\begin{tikzcd}
(A_1,A_2,A_3) \arrow[d, dashed] \\
(A_1',A_2',A_3')
\end{tikzcd}$$

$A_i = val(a_i)$, $A_i' = val(a_i')$

$A_1 = A_2 + A_3 - \min(A_1, A_3)$

$A_2' = \min(A_1, A_3)$

$A_3' = A_1 + A_2 - \min(A_1, A_3)$

Theorem (Lusztig)

- $U_+(\mathbb{Z}^{>0}_{min}) \subset U_+(\mathbb{Z}_{min})$ parametrizes the canonical basis of $\mathcal{U}^\vee_-$ ← Langlands dual

  (universal enveloping alg. of $U^\vee_-$)

- The element $[\phi^{\underline{i}}_{>0}(\underline{m})]$ has weight $\sum_{j=1}^{N} M_j \alpha^\vee_{(j)}$ where $M_j = val(m_j)$ and $(\alpha^\vee_{(1)}, \dots, \alpha^\vee_{(N)}) = (\alpha^\vee_{i_1}, s_{i_1}\alpha^\vee_{i_2}, \dots)$ positive roots ordered according to $\underline{i}$.

Lusztig's parametrization of U_+ and weight map

Theorem (Lusztig)

- $U_+(\mathbb{Z}^{>0}_{\min}) \subset U_+(\mathbb{Z}_{\min})$ parametrizes the canonical basis of $\mathcal{U}^\vee_-$

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- The element $[\phi^{\underline{i}}_{>0}(\underline{m})]$ has weight $\sum_{j=1}^{N} M_j \, \alpha^\vee_{(j)}$ where $M_j := \mathrm{val}(m_j)$

  and $(\alpha^\vee_{(1)}, \dots, \alpha^\vee_{(N)}) = (\alpha^\vee_{i_1}, s_{i_1}\alpha^\vee_{i_2}, \dots)$ positive roots ordered according to $\underline{i}$.

The type A Lusztig weight map coincides with the tropicalisation of Δ!

Here $U_+(\mathbb{R}_{>0}) \xrightarrow{\Delta} T(\mathbb{R}_{>0})$

$$u = \begin{array}{c} \begin{array}{ccccc} \Delta_{n+1} & \Delta_n & \cdots & \Delta_2 & \Delta_1 \end{array} \\ \begin{pmatrix} 1 & a_{12} & \cdots & & a_{1n} \\ & \ddots & \ddots & & \vdots \\ & & \ddots & & \vdots \\ & & & \ddots & a_{n-1,n} \\ & & & & 1 \end{pmatrix} \end{array} \longmapsto \begin{pmatrix} \Delta_1(u) & & & \\ & \Delta_2/\Delta_1(u) & & \\ & & \ddots & \\ & & & \Delta_{n+1}/\Delta_n(u) \end{pmatrix}$$

for $u = \phi^{\underline{i}}_{>0}(\underline{m})$:

$$\boxed{\mathrm{val}(\Delta(u)) = \sum_{j=1}^{N} M_j \, \alpha^\vee_{(j)}}$$

totally positive Toeplitz Matrices

Let R be a ring with a 'positive semifield $\mathcal{R}_{>0}$',

eg: $\mathcal{R}_{>0} = \mathbb{R}_{>0} \subset \mathbb{C}$

$\mathcal{R}_{>0} = \mathbb{R}_{>0}((t)) \subset \mathbb{C}((t))$

standard choice for tropicalisation

Let $\mathbb{K} = \left\{ \sum c_i t^{A_i} \;\middle|\; \begin{array}{l} A_i \in \mathbb{R}\,,\ A_i \nearrow \infty \\ \text{or } \{A_i\} \text{ finite} \\ c_i \in \mathbb{C} \end{array} \right\}$ field of "generalised Puiseaux series"

$\mathcal{R}_{>0} = \mathbb{K}_{>0} = \{ k \in \mathbb{K} \mid \text{leading coefficient in } \mathbb{R}_{>0} \}$, $\text{val}: \mathbb{K}_{>0} \longrightarrow \mathbb{R}_{\min}$

$$T_{n+1}(\mathcal{R}_{>0}) = \left\{ \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & a_2 \\ & & & \ddots & a_1 \\ & & & & 1 \end{pmatrix} \;\middle|\; \begin{array}{l} \text{all minors are} \\ \text{in } \mathcal{R}_{>0} \end{array} \right\}$$

totally positive Toeplitz matrices over R

Positive critical point theorems

Let $\mathbb{K} = \left\{ \sum c_i t^{A_i} \;\middle|\; \begin{array}{l} A_i \in \mathbb{R}, \; A_i \nearrow \infty \\ \text{or } \{A_i\} \text{ finite} \\ c_i \in \mathbb{C} \end{array} \right\}$ field of generalised Puiseaux series

$\mathbb{K}_{>0} = \{ k \in \mathbb{K} \mid \text{leading coefficient in } \mathbb{R}_{>0} \}$, $\mathrm{val}: \mathbb{K}_{>0} \longrightarrow \mathbb{R}_{\min}$

The following theorem strengthens a result of [FOOO '10]

Thm (Judd-R,'24) Let $x^{w_i} = x_1^{w_{i1}} x_2^{w_{i2}} \dots x_r^{w_{ir}}$ and $\gamma_i \in \mathbb{K}_{>0}$,

$$f(x_1, \dots, x_r) = \sum_{i=1}^{m} \gamma_i x^{w_i} \qquad \text{(Laurent polynomial in } r \text{ variables)}$$

Then f has a unique critical point in $\mathbb{K}_{>0}^r$ $\iff$ the Newton polytope of W is full-dimensional with 0 in the interior.

Suppose $p_{crit} = (p_1, \dots, p_r)$ is the positive critical point of f, then we call $\mathrm{val}(p_{crit}) = (\mathrm{val}(p_1), \dots, \mathrm{val}(p_r)) \in \mathbb{R}^r$ the tropical critical point.

Thm p_{crit} depends continuously on the coefficients $\gamma_i \in \mathbb{K}_{>0}$ (for the t-adic topology).

Thm (R '03 '06)

Let $\Delta_i\left(\begin{pmatrix} 1 & c_1 & c_2 & \cdots & c_n \\ & 1 & c_1 & \ddots & \vdots \\ & & 1 & \ddots & c_2 \\ & & & \ddots & c_1 \\ & & & & 1 \end{pmatrix}\right) := \det \begin{pmatrix} \overbrace{c_{n-i+1} \cdots c_n}^{i} \\ c_{n-i} \ddots \vdots \\ \vdots \cdots c_{n-i} \; c_{n-i+1} \end{pmatrix}$. Then

$$\mathcal{T}_{n+1}(\mathbb{R}_{>0}) = \left\{ \begin{matrix} \text{TP Toeplitz matrices} \\ \begin{pmatrix} 1 & c_1 & \cdots & c_n \\ & 1 & \ddots & \vdots \\ & & \ddots & c_1 \\ & & & 1 \end{pmatrix} \in U_+ \end{matrix} \right\} \xrightarrow{\Delta = \left(\frac{\Delta_1}{\Delta_0}, \frac{\Delta_2}{\Delta_1}, \cdots, \frac{\Delta_{n+1}}{\Delta_n}\right)} T_{SL_{n+1}}(\mathbb{R}_{>0})$$

is a homeomorphism

$\mathbb{R}_{>0}$ version

Thm (Judd-R) The map Δ over $\mathbb{K}_{>0}$ is a homeomorphism for the t-adic topology

$$\mathcal{T}_{n+1}(\mathbb{K}_{>0}) = \left\{ \begin{matrix} \text{totally positive Toeplitz} \\ \text{matrices } \begin{pmatrix} 1 & c_1 & \cdots & c_n \\ & 1 & \ddots & \vdots \\ & & \ddots & c_1 \\ & & & 1 \end{pmatrix} \text{ over } \mathbb{K} \end{matrix} \right\} \xrightarrow{\Delta = \left(\frac{\Delta_1}{\Delta_0}, \frac{\Delta_2}{\Delta_1}, \cdots, \frac{\Delta_{n+1}}{\Delta_n}\right)} T_{SL_{n+1}}(\mathbb{K}_{>0})$$

$\mathbb{K}_{>0}$ version

tropicalise

1st goal: $\boxed{\mathbb{R}_{\min} \text{ - version ?}}$

$\boxed{\mathbb{R}_{min}\text{ - version}}$

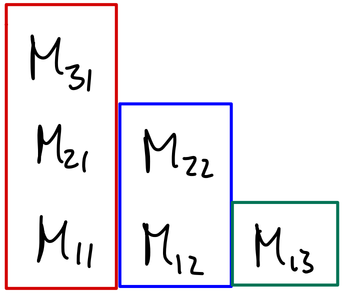
$$\underbrace{\mathcal{T}_n(\overset{\mathbb{R}_{min}}{\cancel{\mathbb{K}_{>0}}}) = \left\{ \begin{array}{l}\text{totally positive Toeplitz} \\ \text{matrices } \begin{pmatrix}1 & c_1 & \cdots & c_n\\ & 1 & \ddots & \vdots \\ & & \ddots & c_1 \\ & & & 1\end{pmatrix} \text{ over } \mathbb{K}\end{array}\right\}}_{\text{subset of } U_+(\mathbb{R}_{min}) \text{ to be described}} \cancel{\sim} \quad \Delta = \cancel{\left(\frac{\Delta_1}{\Delta_0}, \frac{\Delta_2}{\Delta_1}, \ldots, \frac{\Delta_{n+1}}{\Delta_n}\right)}$$

$$\xrightarrow[\text{Lusztig's weight map}]{} \quad T^{SL_{n+1}}(\mathbb{K}_{>0}) \cancel{\sim} \quad \hat{=} \quad \mathcal{Y}_{SL_{n+1}}$$

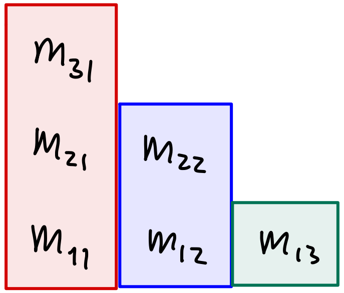
- Choose coordinate chart for $U_+(\mathbb{K}_{>0})$ corr. to $\underline{i}_0 = (n, n-1, .., 1, n, n-1, .., 2, .. n)$

$$x_{\underline{i}}(\underline{m}) = x_3(\underline{m_{31}})\, x_2(\underline{m_{21}})\, x_1(\underline{m_{11}})\, x_3(\underline{m_{22}})\, x_2(\underline{m_{12}})\, x_3(\underline{m_{13}})$$

$M_{ij} := val(m_{ij}) \rightsquigarrow$



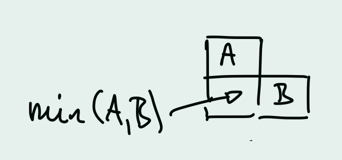
tropical coordinates



$\leftarrow$ standard coordinates

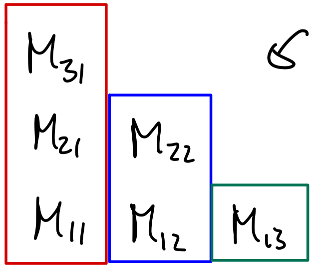
Tropical totally positive Toeplitz matrices

Theorem (transformed combination of Judd'18, Lüdenbach'23)



$$T_{n+1}(\mathbb{R}_{\min}) = \underbrace{\left\{ (M_{ij})_{i+j\leq n+1} \;\middle|\; M_{ij} = \min(M_{i+1,j}, M_{j,i+1}) \right\}}_{\text{"min-ideal fillings"}}$$

eg:



M_{31}		
M_{21}	M_{22}	
M_{11}	M_{12}	M_{13}

arises from some Toeplitz $x_{\underline{i}_0}(\underline{m})$

$$\Leftrightarrow \quad M_{21} = \min(M_{31}, M_{22})$$

$$M_{11} = \min(M_{21}, M_{12}), \quad M_{12} = \min(M_{22}, M_{13})$$

Thm (combination of above with Judd'18) Lusztig's weight map restricts to $\lambda : T_{n+1}(\mathbb{R}_{\min}) \longrightarrow \mathfrak{g}^*_{PSL_{n+1}}$ as a (piecewise linear) isomorphism

$\mathbb{R}_{\min}$ version of previous param. thms

Rmk: This gives a canonical way of writing any weight as lin comb of pos roots!

Thm (R '03 '06) Let
$$\Delta_i\left(\begin{pmatrix} 1 & c_1 & c_2 & \cdots & c_n \\ & 1 & c_1 & \ddots & \vdots \\ & & 1 & \ddots & c_2 \\ & & & \ddots & c_1 \\ & & & & 1 \end{pmatrix}\right) := \det \overbrace{\begin{pmatrix} c_{n-i+1} & \cdots & c_n \\ c_{n-i} & \ddots & \vdots \\ \vdots & \cdots\, c_{n-i} & c_{n-i+1} \end{pmatrix}}^{i}.$$
Then

$$\mathcal{T}_{n+1}(\mathbb{R}_{>0}) = \left\{ \begin{matrix}\text{TP Toeplitz matrices} \\ \begin{pmatrix} 1 & c_1 & \cdots & c_n \\ & 1 & \ddots & \vdots \\ & & \ddots & c_1 \\ & & & 1 \end{pmatrix} \in U_+ \end{matrix} \right\} \xrightarrow{\Delta = \left(\frac{\Delta_1}{\Delta_0}, \frac{\Delta_2}{\Delta_1}, \ldots, \frac{\Delta_{n+1}}{\Delta_n}\right)} T_{SL_{n+1}}(\mathbb{R}_{>0})$$
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$\mathbb{R}_{>0}$ version

Thm (Judd-R) The map Δ over $\mathbb{K}_{>0}$ is a homeomorphism for the t-adic topology

$$\mathcal{T}_{n+1}(\mathbb{K}_{>0}) = \left\{ \begin{matrix}\text{totally positive Toeplitz} \\ \text{matrices} \begin{pmatrix} 1 & c_1 & \cdots & c_n \\ & 1 & \ddots & \vdots \\ & & \ddots & c_1 \\ & & & 1 \end{pmatrix} \text{over } \mathbb{K} \end{matrix} \right\} \xrightarrow{\Delta = \left(\frac{\Delta_1}{\Delta_0}, \frac{\Delta_2}{\Delta_1}, \ldots, \frac{\Delta_{n+1}}{\Delta_n}\right)} T^{SL_{n+1}}(\mathbb{K}_{>0})$$

$\mathbb{K}_{>0}$ version

tropicalise

1st goal: $\boxed{\mathbb{R}_{min} \text{ - version ?}}$

Thm (R '03 '06) Let $\Delta_i\left(\begin{pmatrix} 1 & c_1 & c_2 & \cdots & c_n \\ & 1 & c_1 & \ddots & \vdots \\ & & 1 & \ddots & c_2 \\ & & & \ddots & c_1 \\ & & & & 1 \end{pmatrix}\right) := \det \overbrace{\begin{pmatrix} c_{n-i+1} & \cdots & c_n \\ c_{n-i} & \ddots & \vdots \\ \vdots & \cdots\ c_{n-i} & c_{n-i+1} \end{pmatrix}}^{i}$. Then

$$\mathcal{T}_{n+1}(\mathbb{R}_{>0}) = \left\{ \begin{array}{l} \text{TP Toeplitz matrices} \\ \begin{pmatrix} 1 & c_1 & \cdots & c_n \\ & 1 & \ddots & \vdots \\ & & \ddots & c_1 \\ & & & 1 \end{pmatrix} \in U_+ \end{array} \right\} \xrightarrow{\Delta = \left(\frac{\Delta_1}{\Delta_0}, \frac{\Delta_2}{\Delta_1}, \dots, \frac{\Delta_{n+1}}{\Delta_n}\right)} \mathcal{T}_{SL_{n+1}}(\mathbb{R}_{>0})$$

is a homeomorphism

$\mathbb{R}_{>0}$ version

Thm (Judd-R) The map Δ over $\mathbb{K}_{>0}$ is a homeomorphism for the t-adic topology

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$\mathbb{K}_{>0}$ version

1st goal:

Thm (Judd, Lüdenbach, R) Lusztig's weight map restricts to

$$\lambda : \mathcal{T}_{n+1}(\mathbb{R}_{\min}) \longrightarrow \mathfrak{g}^*_{PSL_{n+1}}(\mathbb{R})$$

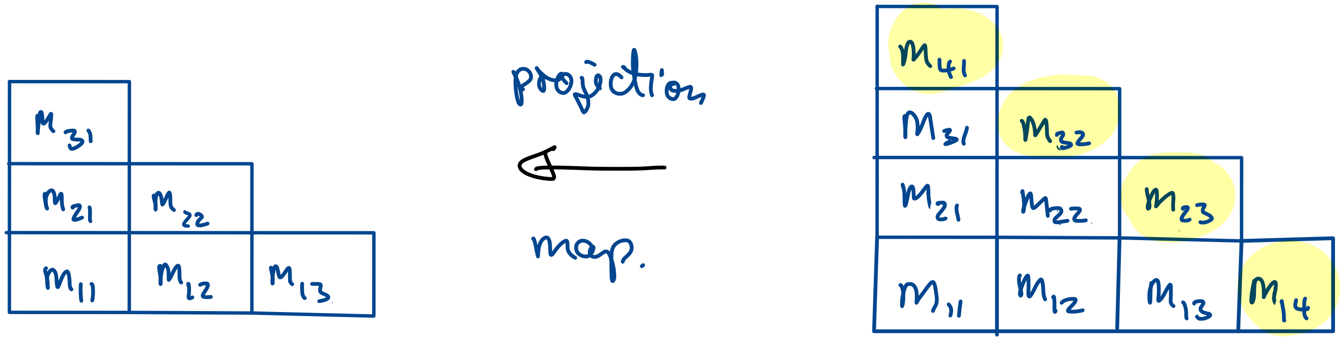
as a (piecewise linear) isomorphism

$\mathbb{R}_{\min}$ version

$\exists$ unique min-ideal filling in $U_+(\mathbb{R}_{\min})$ for every 'Lusztig weight'

Infinite totally positive Toeplitz matrices as limits of finite ones

standard coordinates



$$\begin{tikzcd} x_{321323}(m_\cdot) & x_{\underline{4}321\underline{4}32\underline{4}3\underline{4}}(m_\cdot) \arrow[l, mapsto] \\ U_+^{SL_4}(\mathbb{K}_{>0}) & U_+^{SL_5}(\mathbb{K}_{>0}) \arrow[l] \end{tikzcd}$$

- have standard coordinates $(m_{ij})_{i,j\in\mathbb{N}}$ for the projective limit $U_\infty(\mathbb{K}_{>0})$ (∞ upper-triangular matrices)
- also $(M_{ij})_{i,j\in\mathbb{N}}$ for $U_\infty(\mathbb{R}_{min})$
- Candidate for $T_\infty(\mathbb{R}_{min})$: "infinite min-ideal fillings" $\mathcal{IL}_\infty$

Parametrization Theorems for infinite min-ideal fillings

$$\mathcal{IM}_\infty = \left\{ \begin{array}{|c|c|c|c|c} \vdots & & & & \\ \hline M_{41} & \cdots & & & \\ \hline M_{31} & M_{32} & \ddots & & \\ \hline M_{21} & M_{22} & M_{23} & \ddots & \\ \hline M_{11} & M_{12} & M_{13} & M_{14} & \cdots \\ \hline \end{array} \;\middle|\; \begin{array}{l} M_{ij} \in \mathbb{R} \\ M_{ij} = \min\left(M_{i+1,j}, M_{i,j+1}\right) \end{array} \right\}$$

Theorem Let $\Omega_*(\mathbb{R}_{min}) = \left\{ (\underline{A}, \underline{B}) \in (\mathbb{R}\cup\{\infty\})^{\mathbb{N}} \times (\mathbb{R}\cup\{\infty\})^{\mathbb{N}} \;\middle|\; \begin{array}{l} A_1 \le A_2 \le \dots \\ B_1 \le B_2 \le \dots \\ \sup(\{A_i\}) = \sup(\{B_i\}) \\ \min(A_i, B_j) \in \mathbb{R} \end{array} \right\}$

Then $\mathbb{E}_* : \Omega_*(\mathbb{R}_{min}) \xrightarrow{\sim} \mathcal{IM}_\infty$

$$(\underline{A}, \underline{B}) \longmapsto (M_{ij})_{i,j\in\mathbb{N}} \quad \text{with} \quad M_{ij} = \min(A_i, B_j)$$

Example

if all $B_i = \infty$ $\quad \mathbb{E}_*(\underline{A}, \underline{B}) = \begin{array}{|c|c|c|c|c} \vdots & \vdots & & & \\ \hline A_3 & \vdots & & & \\ \hline A_2 & A_2 & \vdots & & \\ \hline A_1 & A_1 & A_1 & A_1 & \cdots \\ \hline \end{array}$

if all $A_i = \infty$ $\quad \mathbb{E}_*(\underline{A}, \underline{B}) = \begin{array}{|c|c|c|c} \vdots & \vdots & & \\ \hline B_1 & B_2 & \cdots & \\ \hline B_1 & B_2 & B_3 & \cdots \\ \hline B_1 & B_2 & B_3 & \cdots \\ \hline \end{array}$

Parametrization Theorems for infinite min-ideal fillings

Theorem Let

$$\Omega_*(\mathbb{R}_{min}) = \left\{ (\underline{A}, \underline{B}) \in (\mathbb{R}\cup\{\infty\})^{\mathbb{N}} \times (\mathbb{R}\cup\{\infty\})^{\mathbb{N}} \;\middle|\; \begin{array}{l} A_1 \le A_2 \le \dots \\ B_1 \le B_2 \le \dots \\ \sup(\{A_i\}) = \sup(\{B_i\}) \\ \min(A_i, B_j) \in \mathbb{R} \end{array} \right\}$$

"weakly interlacing"

Then

$$\mathcal{E}_* : \Omega_*(\mathbb{R}_{min}) \xrightarrow{\sim} \mathcal{IM}_\infty$$

$$(\underline{A}, \underline{B}) \longmapsto (M_{ij})_{i,j\in\mathbb{N}} \quad \text{with} \quad M_{ij} = \min(A_i, B_j)$$

Other versions

$$\mathcal{E}_*^{stable} : \Omega_*^{interlacing} \xrightarrow{\sim} \mathcal{IM}_\infty^{Stable}$$

And in between :

$$\mathcal{E}_*^{\mathbb{R}} : \Omega_*^{\mathbb{R}} \longrightarrow \mathcal{IM}_\infty^{asympt.\ real}$$

Parametrization Theorems for infinite min-ideal fillings

Theorem Let

$$\Omega_\star(\mathbb{R}_{\min}) = \left\{ (\underline{A},\underline{B}) \in (\mathbb{R}\cup\{\infty\})^{\mathbb{N}} \times (\mathbb{R}\cup\{\infty\})^{\mathbb{N}} \;\middle|\; \begin{array}{l} A_1 \leq A_2 \leq \dots \\ B_1 \leq B_2 \leq \dots \\ \sup(\{A_i\}) = \sup(\{B_i\}) \\ \min(A_i, B_j) \in \mathbb{R} \end{array} \right\}$$

"weakly interlacing"

Then

$$\mathcal{E}_\star : \Omega_\star(\mathbb{R}_{\min}) \xrightarrow{\sim} \mathcal{IM}_\infty$$

$$(\underline{A},\underline{B}) \longmapsto (M_{ij})_{i,j\in\mathbb{N}} \quad \text{with} \quad M_{ij} = \min(A_i, B_j)$$

Other versions

$$\mathcal{E}_\star^{\text{stable}} : \Omega_\star^{\text{interlacing}} \xrightarrow{\sim} \mathcal{IM}_\infty^{\text{stable}}$$

And in between :

$$\mathcal{E}_\star^{\mathbb{R}} : \Omega_\star^{\mathbb{R}} \longrightarrow \mathcal{IM}_\infty^{\text{asympt. real}}$$

Definitions

- Call $(\underline{A},\underline{B})$ interlacing if for every A_i there $\exists\, B_k \geq A_i$
  for every B_j there $\exists\, A_k \geq B_j$
- Call (M_{ij}) stable if $\sup(\{M_{ij} \mid i\}) = \max(\{M_{ij} \mid i\})$ for every j
  & $\sup(\{M_{ij} \mid j\}) = \max(\{M_{ij} \mid j\})$ for every i

Infinite TP Toeplitz matrices over $\mathbb{R}_{>0}$

Recall: Thm (Edrei 1951) Let $\Omega = \{((\alpha_i),(\beta_i)) \in \mathbb{R}^{\mathbb{N}}_{\geq 0} \times \mathbb{R}^{\mathbb{N}}_{\geq 0} \mid \begin{smallmatrix}\alpha_1 \geq \alpha_2 \geq \dots \\ \beta_1 \geq \beta_2 \geq \dots\end{smallmatrix} \quad \sum \alpha_i + \beta_i < \infty\}$

Then (c_i) is a TNN sequence $\iff \sum c_i x^i = e^{\gamma x} \dfrac{\prod(1+\beta_i x)}{\prod(1-\alpha_i x)}$ for some $((\alpha_i),(\beta_i)) \in \Omega$, $\gamma \in \mathbb{R}_{\geq 0}$

before: $\mathbb{R}_{>0} \rightsquigarrow \mathbb{K}_{>0}$ generalized Puiseux series with $\|\ \|_t$

- $\mathbb{R}_{>0} \hookrightarrow \mathbb{K}_{>0}$ has discrete topology! ✗

need better "$\mathbb{R}_{>0}$": Replace $\mathbb{K}$ by $R = C^0((0,\delta])$ where $0<\delta<1$ fixed.

Def: (valuative semifield of positive functions)

- $\mathcal{C}_{>0} = \{ f : (0,\delta] \to \mathbb{R}_{>0} \mid \exists \lim_{t \to 0} t^{-F} f(t) > 0$ for some $F \in \mathbb{R} \}$
- $\mathcal{C}_{>0} \subset \mathcal{C} := C^0((0,\delta])$ subsemifield, $\mathcal{C}_{\geq 0} = \mathcal{C}_{>0} \cup \{0\}$
- $\mathrm{val}(f) = F$ defines semifield homomorphism $\mathrm{val} : \mathcal{C}_{>0} \to \mathbb{R}_{\min}$.
- $\mathcal{O}_{>0} = \{ f \in \mathcal{C}_{>0} \mid \mathrm{val}(f) \geq 0 \} \hookrightarrow C^0([0,\delta])$

Def: Let $T_\infty(\mathcal{C}_{>0}) = \left\{ \begin{array}{l} \infty\text{-TP Toeplitz matrices over } \mathcal{C}_{>0} \\ \text{with standard coordinates } (m_{ij}) \end{array} \right\}$

$$\downarrow$$

$$T_{\infty,\mathcal{C}_{>0}}(\mathbb{R}_{min}) = \{ (M_{ij})_{i,j\in\mathbb{N}} \mid M_{ij} = val(m_{ij}) \text{ for } (m_{ij}) \in T_\infty(\mathcal{C}_{>0}) \}$$

Theorem $T_{\infty,\mathcal{C}_{>0}}(\mathbb{R}_{min}) = \mathcal{I}\mathcal{M}_\infty$

(all infinite min-ideal fillings are represented by totally positive Toeplitz matrices over $\mathcal{C}_{>0}$)

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Theorem $T_{\infty,\mathcal{C}_{>0}}(\mathbb{R}_{\min}) = \mathcal{IM}_\infty$

(all infinite min-ideal fillings are represented by totally positive Toeplitz matrices over $\mathcal{C}_{>0}$)

Topologies on $\mathcal{C}_{\geq 0}$: "strong topology" (gen. by $\mathcal{U}(f,A,\varepsilon) = t^A \underbrace{B_\varepsilon(f)}_{\| \|_{\sup}\ \varepsilon\text{-ball}}$ $f \in \mathcal{C}_{>0}$, $A \in \mathbb{R}$, $\varepsilon > 0$ and $\{0\}$)

$\mathcal{C}^{st}_{\geq 0}$

"weak topology" (pointwise convergence and continuity of $\mathrm{val}: \mathcal{C}_{\geq 0} \to \mathbb{R} \cup \{\infty\}$)

$\mathcal{C}^{wk}_{\geq 0}$

Parameters:

$$\Omega(\mathbb{R}_{\geq 0}) := \left\{ (\alpha, \beta) \;\middle|\; \begin{array}{l} \alpha_1 \geq \alpha_2 \geq \ldots \\ \beta_1 \geq \beta_2 \geq \ldots \quad \text{coefficients of } \dfrac{\prod(1+\beta_i x)}{\prod(1-\alpha_i x)} \text{ converge in } \mathbb{R}_{\geq 0} \\ \alpha_i + \beta_i \in \mathbb{R}_{\geq 0} \text{ for all } i \end{array} \right\}$$

topological semiring.

Schoenberg/Edrei/Thoma type parametrisation

The proof of Theorem $\mathcal{T}_{\infty,\ell_{>0}}(\mathbb{R}_{min}) = \mathcal{IM}_\infty$ involves the map

$$\begin{aligned} \tau :\ \Omega(\mathcal{C}^{wk}_{>0}) &\longrightarrow \mathcal{T}_\infty(\mathcal{C}_{>0}) \\ (\underline{\alpha},\underline{\beta}) &\longmapsto \text{Toeplitz matrix for } 1+\sum_{k=1}^{\infty} c_k x^k = \frac{\prod(1+\beta_j x)}{\prod(1-\alpha_i x)} \end{aligned}$$

Stronger theorem: Let $\mathcal{T}^{res}_\infty(R_{\geq 0}) := \left\{ (m_{ij}) \in \mathcal{T}_\infty(R_{\geq 0}) \;\middle|\; \begin{array}{l} \lim_{k\to\infty} m_{ik} \in R_{\geq 0} \\ \lim_{k\to\infty} m_{kj} \in R_{\geq 0} \end{array} \right\}$

"restricted" infinite Toeplitz matrices

(if $R_{\geq 0} = \mathbb{R}_{\geq 0}$ then $\mathcal{T}^{res}_\infty(\mathbb{R}_{\geq 0}) = \mathcal{T}_\infty(\mathbb{R}_{\geq 0})$ by result of Edrei)

Theorem

$$\begin{tikzcd}
(\underline{\alpha},\underline{\beta}) \arrow[r, phantom, "\in"] \arrow[d, mapsto] & \Omega^{res}(\mathcal{C}^{wk}_{>0}) \arrow[r, "\tau"] \arrow[d, "Val"] \arrow[dr, phantom, "\#"] & \mathcal{T}^{res}_\infty(\mathcal{C}^{wk}_{>0}) \arrow[r, phantom, "\ni"] \arrow[d, "Val"] & (m_{ij})_{i,j} \arrow[d, mapsto] \\
(\underline{A},\underline{B}) \arrow[r, phantom, "\in"] & \Omega_*(\mathbb{R}_{min}) \arrow[r, "\mathcal{E}_*", "\sim"'] & \mathcal{IM}_\infty(\mathbb{R}_{min}) \arrow[r, phantom, "\ni"] & (M_{ij})_{i,j}
\end{tikzcd}$$

weakly interlacing $(\underline{A},\underline{B}) \longmapsto (\min(A_i,B_j))_{i,j}$

Vershik-Kerov type Theorems

Suppose $(M_{ij}) \in \mathcal{T}_\infty(\mathbb{R}_{\min})$

$\Rightarrow (M_{ij})_{i+j\le n+1} \in \mathcal{T}_{n+1}(\mathbb{R}_{\min})$ Write $\lambda^{(n)} = (\lambda^{(n)}_1, \dots, \lambda^{(n)}_{n+1})$ for its 'Lusztig weight'

eg $\boxed{M_{11}} \mapsto (M_{11}, -M_{11}) = \lambda^{(1)}$ $\quad \begin{pmatrix} M_{11} & \\ & -M_{11} \end{pmatrix} \in \mathfrak{h}_{SL_2}$

$\begin{array}{|c|c} \hline M_{21} & \\ \hline M_{11} & \multicolumn{1}{c|}{M_{12}} \\ \hline \end{array} \mapsto (M_{12} + M_{11}, M_{21} - M_{12}, -M_{11} - M_{21}) = \lambda^{(2)}$

Define also $-w_0\lambda^{(n)}$ as '**dual representation analogue**' $-w_0\lambda^{(n)} = (-\lambda^{(n)}_{n+1}, \dots, -\lambda^{(n)}_2, -\lambda^{(n)}_1)$

Theorem For asymptotically real $(M_{ij})_{i,j} \in \mathcal{I}\mathcal{U}_\infty$ the tropical Schoenberg parameters $(\underline{A}, \underline{B}) \in \Omega^{\mathbb{R}}_{\star}$ arise as normalised limits of the finite tropical parameters $\lambda^{(n)}_i$ of the $(M_{ij})_{i+j\le n+1}$

$$A_i = \lim_{n\to\infty} \frac{\lambda^{(n)}_i}{n} \qquad\qquad B_j = \lim \frac{-w_0\lambda^{(n)}_j}{n}$$

Lusztig's weight map and its dual, when $n \to \infty$ gives tropical Schoenberg parameters.

$\mathbb{R}_{>0}$- version

Theorem Suppose $u((c_i))$ is an infinite real totally positive Toeplitz matrix such that the generating function $\sum c_i x^i$ has ∞ many roots & poles.

So $$1+\sum_{i=1}^{\infty} c_i x^i = e^{\gamma x} \frac{\prod_i (1+\beta_i x)}{\prod_i (1-\alpha_i x)}$$ with $\alpha_1 \geq \alpha_2 \geq \dots > 0$, $\beta_1 \geq \beta_2 \geq \dots > 0$ (by Edrei's theorem).

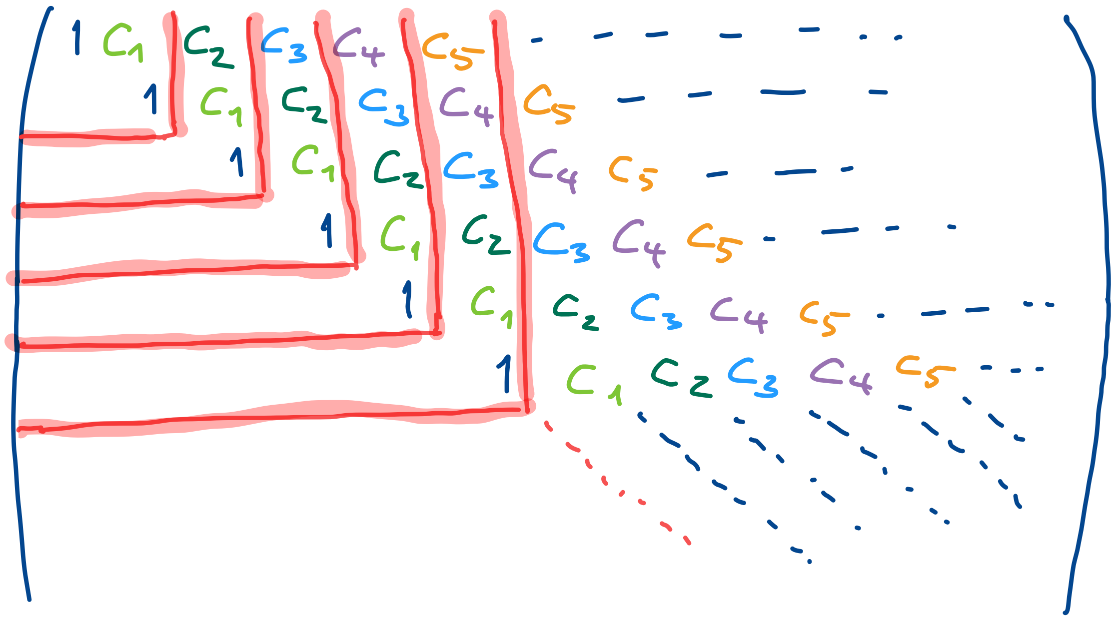
Let $$u^{(n+1)} = \begin{pmatrix} 1 & c_1 & \cdots & c_n \\ & \ddots & \ddots & \vdots \\ & & \ddots & c_1 \\ & & & 1 \end{pmatrix} \in J_{n+1}(\mathbb{R}_{>0})$$ have parameters:
$$\Delta(u^{(n+1)}) = (d_1^{(n+1)}, \dots, d_{n+1}^{(n+1)})$$

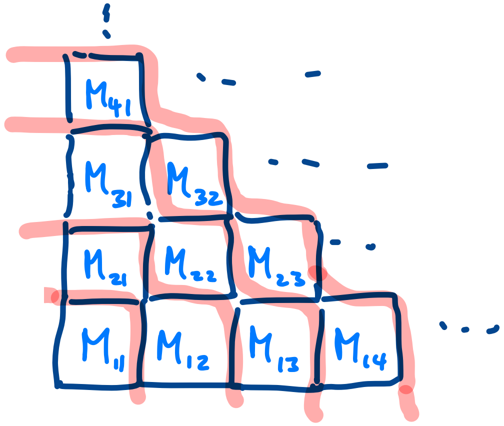
Then $$\alpha_i = \lim_{n\to\infty} \left(d_i^{(n+1)}\right)^{\frac{1}{n}}, \quad \beta_j = \lim_{n\to\infty} \left(d_{n+2-j}^{(n+1)}\right)^{-\frac{1}{n}}$$

This also adds an asymptotic interpretation to the quantum parameters $q_i^{(n)}$ of $qH^*(Gl_{n+1}/B)$ as $n \to \infty$.

$$\| \quad d_i^{(n+1)} / d_{i+1}^{(n+1)}$$

HAPPY BIRTHDAY !

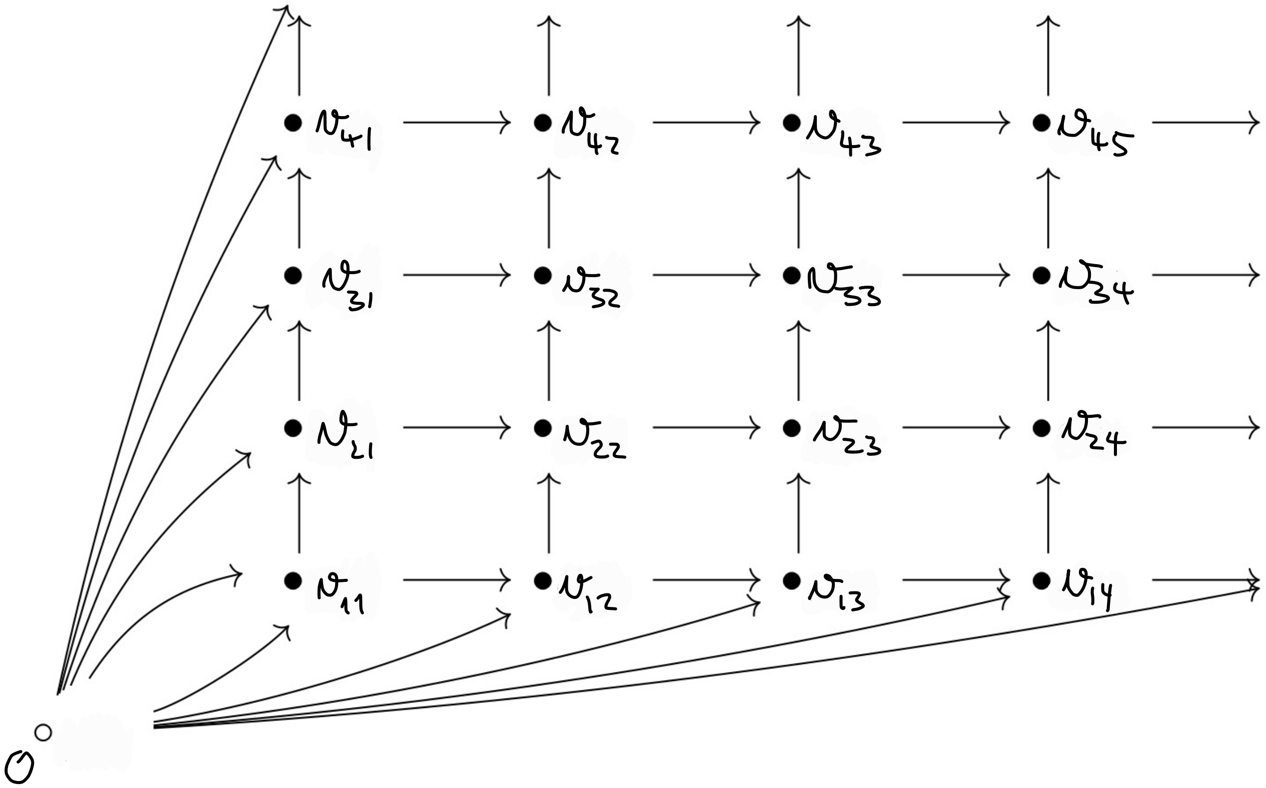




$$\frac{\prod_i (1 + \alpha_i x)}{\prod_j (1 - \beta_j x)} = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$M_{ij} = \min(A_i, B_j)$$

BONUS: Combinatorics of $T_\infty(K_{>0}) \subset U_+(K_{>0})$



$$\boxed{v_{ij} = \frac{1}{m_{ij}} + \sum_{k\geq 1} \frac{1}{m_{i-k,j-k}}}$$

$$\overset{\circ}{v} \xrightarrow{a} \overset{\circ}{v'}$$

label arrow by $v' - v$

Toeplitz relations: At every $\bullet$-vertex v : $\prod_{a\in \mathrm{in}(v)} a = \prod_{a\in \mathrm{out}(v)} a$

incoming arrows

outgoing arrows

Theorem

$$\begin{tikzcd}
\Omega^{res}(\mathcal{C}^{wk}_{>0}) \arrow[r, "\tau^{res}"] \arrow[d, "Val"] \arrow[dr, phantom, "\#"] & \mathcal{T}^{res}_{\infty}(\mathcal{C}^{wk}_{>0}) \arrow[d, "Val"] & \ni (m_{ij})_{i,j} \arrow[d, mapsto] \\
\Omega_*(\mathbb{R}_{\min}) \arrow[r, "E_*", "\sim"'] & \mathcal{IM}_{\infty}(\mathbb{R}_{\min}) & \ni (M_{ij})_{i,j} \\
\text{weakly interlacing } (\underline{A}, \underline{B}) \arrow[u, bend left] \arrow[r, mapsto] & (\min(A_i, B_j))_{i,j} &
\end{tikzcd}$$

(weak topology)

Theorem

$$\begin{tikzcd}
\Omega^{res}(\mathcal{C}^{st}_{>0}) \arrow[r, "\tau^{res}"] \arrow[d, "Val"] \arrow[dr, phantom, "\#"] & \mathcal{T}^{res}_{\infty}(\mathcal{C}^{st}_{>0}) \arrow[d, "Val"] & \ni (m_{ij})_{i,j} \arrow[d, mapsto] \\
\Omega^{interlacing}_*(\mathbb{R}_{\min}) \arrow[r, "E^{stable}_*", "\sim"'] & \mathcal{IM}^{stable}_{\infty}(\mathbb{R}_{\min}) & \ni (M_{ij})_{i,j} \\
\text{interlacing } (\underline{A}, \underline{B}) \arrow[u, bend left] \arrow[r, mapsto] & (\min(A_i, B_j))_{i,j} &
\end{tikzcd}$$

(strong topology)

These theorems interpret the parameters from the "tropical Edrei theorem" as valuations of Schoenberg-type parameters (taken over $\mathcal{C}_{>0}$), and 'detropicalise' the parametrisation maps E_*, E_*^{stable} for infinite ideal fillings.