Counting low degree number fields by successive minima

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Let $\mathcal{O}$ be an order of absolute discriminant Δ in a degree n number field.
Let $\sigma_1, \ldots, \sigma_n$ be the nonzero homomorphisms $\mathcal{O} \to \mathbb{C}$.
Endow $\mathcal{O}$ with the norm

$$|x| := \sqrt{\frac{1}{n} \sum_{i=1}^{n} |\sigma_i(x)|^2}.$$

Question

What lattices arise from orders in number fields and how are they distributed?

Definition

For $0 \leq i < n$, the *i-th successive minimum* $\lambda_i(\mathcal{O})$ is the smallest $r \in \mathbb{R}_{>0}$ s.t. $\mathcal{O}$ contains at least $i+1$ linearly independent elements of length $\leq r$.

Question

How is $(\log_\Delta \lambda_1, \ldots, \log_\Delta \lambda_{n-1}) \in \mathbb{R}^{n-1}$ distributed as we range across orders in degree n number fields?

Cubic rings: setup

Fix a large $C \in \mathbb{R}_{>0}$ and let $p = (p_1, p_2) \in \mathbb{R}^2$.

Question

What is

$$\#\left\{ \begin{array}{c} \text{cubic orders } \mathcal{O} \text{ such that} \\ \mathrm{Gal}(\mathcal{O} \otimes \mathbb{Q}) \simeq S_3,\ \Delta \leq X, \\ \text{and } \max_{i=1,2}\{|\log_\Delta \lambda_i(\mathcal{O}) - p_i|\} \leq \frac{C}{\log X} \end{array} \right\}$$

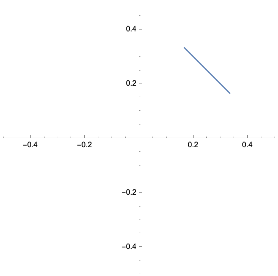
as $X \to \infty$?

This term is $\asymp X^{d_{S_3}(p)}$ for a function $d_{S_3} : \mathbb{R}^2 \to \mathbb{R}$.

Cubic rings: previous work

Theorem (Chiche-lapierre)

$$d_{S_3}(p) = \begin{cases} 1-(p_2-p_1) & \textit{if } p \in \textit{line segment} \\ 0 & \textit{else.} \end{cases}$$



Quartic rings: setup

Let $p = (p_1, p_2, p_3, q_1, q_2) \in \mathbb{R}^5$.

Question

What is

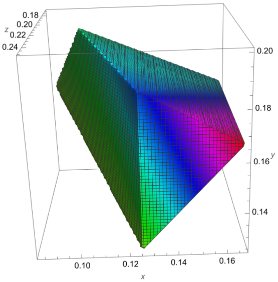
$$\# \left\{ \begin{array}{c} \text{quartic orders } \mathcal{O} \text{ equipped with a cubic resolvent} \\ \text{ring } R \text{ such that } \mathrm{Gal}(\mathcal{O} \otimes \mathbb{Q}) \simeq S_4,\ \Delta \leq X, \\ \text{and } \max_{i=1,2,3}\{|\log_\Delta \lambda_i(\mathcal{O}) - p_i|\} \leq \frac{C}{\log X} \\ \text{and } \max_{i=1,2}\{|\log_\Delta \lambda_i(R) - q_i|\} \leq \frac{C}{\log X} \end{array} \right\}$$

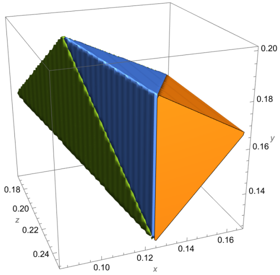
as $X \to \infty$?

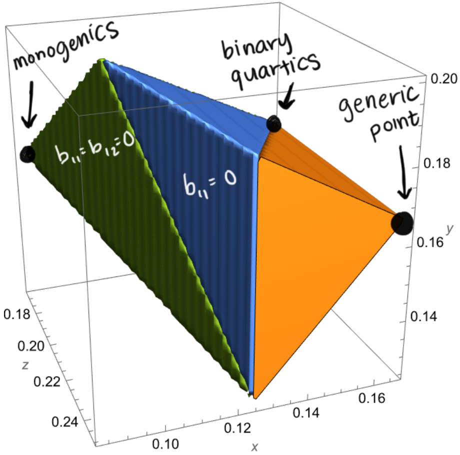
This term is $\asymp X^{d_{S_4}(p)}$ for a function $d_{S_4} : \mathbb{R}^5 \to \mathbb{R}$.

Theorem

$$d_{S_4}(p) = \begin{cases} 1 - (p_3 - p_2) - (p_3 - p_1) - (p_2 - p_1) & \textit{if } p \in \textit{ polytope} \\ \quad -(q_2 - q_1) + \sum_{i,j,k} \max\{0, q_k - p_i - p_j\} & \textit{below} \\ 0 & \textit{else.} \end{cases}$$






monogenics
binary quartics
generic point
$b_{11}=b_{12}=0$
$b_{11}=0$
0.20
0.18
0.16
y
0.14
0.18
0.20
z
0.22
0.24
0.10
0.12
x
0.14
0.16

Quintic rings: setup

Let $p = (p_1, \ldots, p_4, q_1, \ldots, q_5) \in \mathbb{R}^9$.

Question

What is

$$\#\left\{ \begin{array}{c} \text{quintic orders } \mathcal{O} \text{ equipped with a sextic resolvent} \\ \text{ring } R \text{ such that } \mathrm{Gal}(\mathcal{O} \otimes \mathbb{Q}) \simeq S_5,\ \Delta \leq X, \\ \text{and } \max_{i=1,2,3,4}\{|\log_\Delta \lambda_i(\mathcal{O}) - p_i|\} \leq \frac{C}{\log X} \\ \text{and } \max_{i=1,2,3,4,5}\{|\log_\Delta \lambda_i(R) - q_i|\} \leq \frac{C}{\log X} \end{array} \right\}$$

as $X \to \infty$?

This term is $\asymp X^{d_{S_5}(p)}$ for a function $d_{S_5} : \mathbb{R}^9 \to \mathbb{R}$.

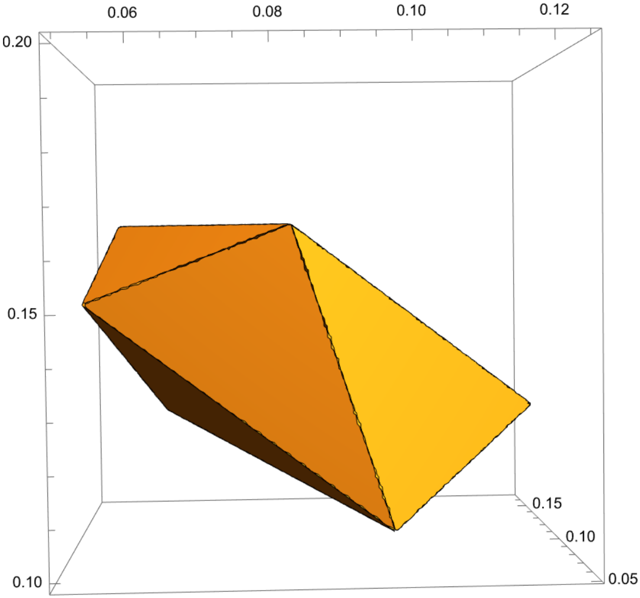
Quintic rings: theorems

Theorem

The support of d_{S_5} is a polytope; on the support, the function is given by

$$d_{S_5}(p) = 1 - \sum_{1 \le i < j \le 4} (p_j - p_i) - \sum_{1 \le i < j \le 5} (q_j - q_i) + \sum_{1 \le i < j \le 5, 1 \le k \le 4} \max\{0, q_i + q_j - p_k - 1/2\}.$$

Else, $d_{S_5}(p) = 0$.



0.06
0.08
0.10
0.12
0.20
0.15
0.10
0.15
0.10
0.05