Higher Representation Theory

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Motivation

- **Representation theory**: motivated from physics, studying the action of symmetry groups on objects.
- First approach: study actions of groups via permutations of sets.
- Frobenius: study linear representations (i.e. of group algebra on vector spaces) instead.
- Geometrically, Lie algebras appear $\rightsquigarrow$ can be studied via their universal enveloping algebra.
- Study associative algebras in their own right.

Let $\mathbb{k}$ be a(n algebraically closed) field.

Algebra over $\mathbb{k}$: A $\mathbb{k}$-linear category $\mathcal{A}$ with one (or finitely many) object(s), say $\bullet$.
Representation of $\mathcal{A}$: A $\mathbb{k}$-linear functor from $\mathcal{A}$ to $\mathcal{V}ect_{\mathbb{k}}$.

Explicitly: $\bullet \mapsto V$, $\quad \mathrm{End}_{\mathcal{A}}(\bullet) \ni a \mapsto \rho(a) \in \mathrm{End}_{\mathcal{V}ect_{\mathbb{k}}}(V)$.

Motivation

What is categorification ?

Decategorification: forgetting information

Object	Decategorification
set S	number of elements in S
vector space	dimension
category	set (of isomorphism classes of objects)
additive category	(split) Grothendieck group ($[X \oplus Y] := [X] + [Y]$)
monoidal category (2-category)	algebra
n-category	$n-1$-category

Categorification: the opposite - not constructive!

Motivation

Observations:

- Translation functors on Category $\mathcal{O}$ of a Lie algebra satisfy relations of a Hecke algebra.
- Certain induction and restriction functors on affine Hecke algebras satisfy relations of a Lie algebra.

$\rightsquigarrow$ Categorification in representation theory.

Why?

More information in the higher structure
$\rightsquigarrow$ new information about the decategorified object;
now have additional information about natural transformations of these functors.

Motivation

Examples in representation theory

- categorification of Kac–Moody algebras *[Khovanov–Lauda, Rouquier]* ($\rightsquigarrow$ 4-dimensional topological quantum field theories (TQFT)?)
- categorification of Heisenberg algebras *[Khovanov]*
- categorification of Lie superalgebras *[Brundan–Stroppel]*
- categorification of Hall algebras (for cyclic quivers) *[Stroppel–Webster]*
- categorification of Hecke algebras via **Soergel bimodules** *[Soergel, Elias–Williamson]*

$\rightsquigarrow$ proof of Broué's abelian defect group conjecture for symmetric groups, proof of Kazhdan–Lusztig conjectures for all Coxeter systems, counterexample to James' conjecture for Hecke algebras, counterexamples to (and refinements of) Lusztig's conjectures

Motivation

How?

- Algebras often appear as convolution algebras of functions on certain spaces.
- Example: Hecke algebra $\mathcal{H} := \mathsf{Fun}_{B\times B}(G, \mathbb{C})$

  G conn. red. alg. group (e.g. GL_n), B Borel (e.g. $\left\{\begin{pmatrix} * & * & * \\ & \ddots & * \\ & & * \end{pmatrix}\right\}$)

Philosophy: Replace functions by sheaves, which have morphisms between them!

Issue: Difficult to work with, so find more algebraic descriptions.

2-categories

A **2-category** $\mathscr{C}$ is a category enriched over the monoidal category $\mathbf{Cat}$ of small categories, i.e. it consists of

- a class (or set) $\mathscr{C}$ of objects;
- for every $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ a small category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of morphisms from $\mathtt{i}$ to $\mathtt{j}$ (objects in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called **1-morphisms** of $\mathscr{C}$ and morphisms in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called **2-morphisms** of $\mathscr{C}$);
- functorial composition $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$;
- identity 1-morphisms $\mathbb{1}_{\mathtt{i}}$ for every $\mathtt{i} \in \mathscr{C}$;
- natural (strict) axioms.

Examples.

- the 2-category $\mathbf{Cat}$ of small categories (1-morphisms are functors and 2-morphisms are natural transformations);

2-categories

Examples.

- the 2-category $\mathbf{Cat}$ of small categories (1-morphisms are functors and 2-morphisms are natural transformations);
- the 2-category $\mathfrak{A}_{\Bbbk}^{f}$ whose
  - objects are small idempotent complete $\Bbbk$-linear additive categories with finitely many indecomposable objects up to isomorphism and finite-dimensional morphism spaces
    (that is, equivalent to the category of finitely generated projective modules over a finite-dimensional $\Bbbk$-algebra);
  - 1-morphisms are additive $\Bbbk$-linear functors;
  - 2-morphisms are natural transformations.

Remark. The objects A-proj in $\mathfrak{A}_{\Bbbk}^{f}$ are hardly ever abelian categories, since A is usually not semisimple.

2-categories

A 2-category $\mathscr{C}$ is **finitary** over $\Bbbk$ if

- $\mathscr{C}$ has finitely many objects;
- each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is in $\mathfrak{A}_{\Bbbk}^{f}$;
- composition is biadditive and $\Bbbk$-bilinear;
- identity 1-morphisms are indecomposable.

Moral: Finitary 2-categories are 2-analogues of finite dim. algebras.

Example. B finite dim. algebra
$\mathscr{C}_B$ has

- one object $\bullet$
- $\mathscr{C}_B(\bullet, \bullet) = \mathrm{add}\{B \oplus (B \otimes_{\Bbbk} B)\}$ with horizontal composition $- \otimes_B -$

Fiat 2-categories

A 2-category $\mathscr{C}$ is **(quasi-)fiat** (**f**initary - **i**nvolution - **a**djunction - **t**wo-category) if

- it is finitary;
- there is a weak involutive equivalence $(-)^*\colon \mathscr{C} \to \mathscr{C}^{\mathsf{co,op}}$ such that there exist adjunction morphisms $F \circ F^* \to \mathbb{1}_{\mathtt{i}}$ and $\mathbb{1}_{\mathtt{j}} \to F^* \circ F$.

Example. $\mathscr{C}_B$ is

- quasi-fiat iff B is Frobenius (the adjoint of $B \otimes_{\Bbbk} B$ is $B^* \otimes_{\Bbbk} B$)
- fiat iff B is quasi-symmetric.

Remark. Again, the categories $\mathscr{C}(\mathtt{i},\mathtt{j}) \simeq A_{\mathtt{i},\mathtt{j}}$-proj are typically not abelian, since $A_{\mathtt{i},\mathtt{j}}$ is not semisimple. Passing to the abelianisation loses the property of being fiat (i.e. the existence of adjunctions).

Soergel bimodules or the Hecke 2-category

(W, S) Coxeter group, $W = \langle s_i |, s_i \in S, s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle$, $m_{ij} \geq 2$

V reflection representation

$C = \mathbb{C}[V]/(\mathbb{C}[V]^W)_+$ coinvariant algebra (assume W finite)

$C_i := C \otimes_{C^{s_i}} C$ for $s_i \in S$

The 2-category $\mathscr{S} = \mathscr{S}_{W,S,V}$ of **Soergel bimodules** or **Hecke 2-category** has

- one object $\varnothing$ (identified with C-proj);
- 1-morphisms are endofunctors of $\varnothing$ isomorphic to tensoring with direct summands of direct sums of finite tensor products (over C) of the C_i;
- 2-morphisms are all natural transformations (bimodule morphisms).

Fact: $\mathscr{S}$ is fiat (for W finite) and categorifies the Hecke algebra.

Interlude - knot homology

Example $W = S_2 = \langle s | s^2 = 1 \rangle$, then $C \cong \mathbb{C}[x]/(x^2)$, $C_s \cong C \otimes_{\mathbb{C}} C$.
Let B be a braid diagram and K the associated knot (resp. link).
Associate the complex of C-C-bimodules

$$C_s \xrightarrow{a \otimes b \mapsto ab} C \qquad \text{to each crossing } L_-$$

$$C \xrightarrow{1 \mapsto \frac{x \otimes 1 + 1 \otimes x}{2}} C_s \qquad \text{to each crossing } L_+,$$

and tensor them up (over C) to obtain a complex $M^\bullet$. Then $HH_\bullet(C, M^\bullet)$ is the Homfly homology of K.

https://en.wikipedia.org/wiki/Skein_relation

2-representations

A **finitary 2-representation** $\mathbf{M}$ of a finitary 2-category $\mathscr{C}$ is a (strict) 2-functor $\mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$, i.e.

- $\mathbf{M}(\mathtt{i}) \approx B_{\mathtt{i}}$-proj for some algebra $B_{\mathtt{i}}$;
- for $\mathrm{F} \in \mathscr{C}(\mathtt{i}, \mathtt{j})$, $\mathbf{M}(\mathrm{F}) \colon \mathbf{M}(\mathtt{i}) \to \mathbf{M}(\mathtt{j})$ is an additive functor;
- for $\alpha \colon \mathrm{F} \to \mathrm{G}$, $\mathbf{M}(\alpha) \colon \mathbf{M}(\mathrm{F}) \to \mathbf{M}(\mathrm{G})$ is a natural transformation.

Examples.

- For $\mathtt{i} \in \mathscr{C}$, we have $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$.
- $\mathscr{S}$ was defined via its **defining** 2-representation.

A 2-representation $\mathbf{M}$ is **simple** if $\coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$ has no proper $\mathscr{C}$-stable ideals.

Goal. Classify simple 2-representations for interesting 2-categories.

Cell combinatorics for 2-categories

From now on, let $\mathscr{C}$ be a fiat 2-category.

On (iso-classes of) indecomposable 1-morphisms in $\mathscr{C}$, define

left preorder: $\theta_v \leq_L \theta_w$ if $\exists \theta_u$ such that θ_w is a direct summand of $\theta_u \theta_v$

left cells: equivalence classes w.r.t. $\geq_L$

Similarly:
right preorder: $\theta_v \leq_R \theta_w$ if $\exists \theta_u$ such that θ_w is a direct summand of $\theta_v \theta_u$

right cells: equivalence classes w.r.t. $\geq_R$

two-sided preorder: $\theta_v \leq_J \theta_w$ if $\exists \theta_u, \theta_{u'}$ such that θ_w is a direct summand of $\theta_u \theta_v \theta_{u'}$

two-sided cells: equivalence classes w.r.t. $\geq_J$

H-cells: intersections of left and right cells

Example

Fact: Indecomposable 1-morphisms in $\mathscr{S}$ are labelled by elements in W. In particular, indecomposable 1-morphisms descend to a cellular basis (the KL-basis) .
$\rightsquigarrow$ cell structure: left, right, two-sided, H-cells (**Kazhdan–Lusztig cells**)

Example. $W = \langle s,t | s^2 = 1 = t^2, stst = tsts \rangle$ of type $B_2 = C_2$. Cells are

1

s, sts	st
ts	t, tst

$stst$

$\mathcal{H}$-cell reduction

Let $\mathcal{H}$ be a *diagonal* H-cell in $\mathscr{C}$, contained in a two-sided cell $\mathcal{J}$.

Construct $\mathscr{C}_{\mathcal{H}}$ in several steps:

- take quotients by all two-sided cells $\mathcal{J}' \not\leq \mathcal{J}$;
- inside quotient, take additive closure of $\mathbb{1}$ and the θ_w in $\mathcal{H}$;
- factor out the maximal 2-ideal not containing id_{θ_w} for $\theta_w \in \mathcal{H}$.

In the example, take $\mathcal{H} = \{\theta_s, \theta_{sts}\}$, then $\mathscr{S}_{\mathcal{H}}$ has cell structure

$$\boxed{\mathbb{1} = \theta_1}$$

$$\boxed{\theta_s, \theta_{sts}}$$

$\mathcal{H}$-cell reduction

Theorem 1. [Mackaay–Mazorchuk–M–Zhang] There is a bijection

$$\{\text{ nontrivial simple 2-representations of } \mathscr{C}\}$$
$$\updownarrow$$
$$\{\text{ nontrivial simple 2-representations of the } \mathscr{C}_{\mathcal{H}}\}$$

where $\mathcal{H}$ runs over a choice of diagonal $\mathcal{H}$-cell in every two-sided cell.

Upshot: concentrate on $\mathscr{C}_{\mathcal{H}} \rightsquigarrow$ smaller! We call this **$\mathcal{H}$-cell reduction**.

Representations of Hecke algebras

[Lusztig]: (W, S) Coxeter group

$\mathcal{H}$ a two-sided cell or diagonal H-cell $\rightsquigarrow$ **asymptotic algebra** $\mathrm{A}_{\mathcal{H}}$ (via $q \to 0$)

Theorem. *[Lusztig]* There is a bijection

$$\{\text{simple representations of the Hecke algebra}\}$$
$$\updownarrow$$
$$\{\text{simple representations of the asymptotic algebras}\}$$

where the asymptotic algebras run over all two-sided cells or a choice of diagonal $\mathcal{H}$-cell in each two-sided cell.

Idea: Asymptotic algebras are easier to understand.

Representations of Hecke 2-categories

[Lusztig] $\mathcal{H}$ a two-sided cell or diagonal H-cell $\rightsquigarrow$ **asymptotic bicategory** $\mathscr{A}_{\mathcal{H}}$

- $\mathscr{A}_{\mathcal{H}}$ categorifies $\mathrm{A}_{\mathcal{H}}$.
- $\mathscr{A}_{\mathcal{H}}$ is a fusion category. *[Lusztig, Elias–Williamson]*
- W any finite Weyl group: $\mathscr{A}_{\mathcal{H}}$ is well-understood; simple 2-representations have been classified. *[Etingof, Ostrik et al.]*

To classify simple 2-representations of $\mathscr{S}$, want to relate 2-representations of $\mathscr{S}_{\mathcal{H}}$ to those of $\mathscr{A}_{\mathcal{H}}$.

From now on, assume (W, S) is a finite Coxeter group.

Double Centraliser Theorem

Let $\mathbf{C}$ be the so-called **cell 2-representation** of $\mathscr{S}_{\mathcal{H}}$ corresponding to $\mathcal{H}$. This is simple.

There is a canonical 2-functor

$$\mathsf{can}\colon \mathscr{S}_{\mathcal{H}} \to \mathscr{E}nd_{\mathscr{E}nd_{\mathscr{S}_{\mathcal{H}}}(\mathbf{C})}(\mathbf{C}).$$

Double Centraliser Theorem. There is an equivalence

$$\mathscr{E}nd^{\mathsf{inj}}_{\mathscr{E}nd_{\mathscr{S}_{\mathcal{H}}}(\mathbf{C})}(\mathbf{C}) \simeq \mathsf{add}(\mathcal{H}).$$

works for any fiat 2-category and any simple 2-representation

Proposition. $\mathscr{E}nd_{\mathscr{S}_{\mathcal{H}}}(\mathbf{C}) \cong \mathscr{A}_{\mathcal{H}}$.

Representations of Hecke 2-categories

Theorem 2. *[Mackaay–Mazorchuk–M.–Tubbenhauer–Zhang]* There is a bijection

$$\{\text{simple } 2\text{-representations of } \mathscr{A}_{\mathcal{H}}\}$$
$$\updownarrow$$
$$\{\text{nontrivial simple } 2\text{-representations of } \mathscr{S}_{\mathcal{H}}\}$$

even stronger: biequivalence of 2-categories of simple 2-representations

Recall:

Theorem 1. [Mackaay–Mazorchuk–M–Zhang] There is a bijection

$$\{\text{ nontrivial simple } 2\text{-representations of } \mathscr{S}\}$$
$$\updownarrow$$
$$\{\text{ nontrivial simple } 2\text{-representations of the } \mathscr{S}_{\mathcal{H}}\}$$

where $\mathcal{H}$ runs over a choice of diagonal $\mathcal{H}$-cell in every two-sided cell.

Representations of Hecke 2-categories

Combining Theorems 1 and 2, this yields

Theorem 3. *[Mackaay–Mazorchuk–M.–Tubbenhauer–Zhang]* There is a bijection

$$\{ \text{ simple } 2\text{-representations of } \mathscr{S} \}$$
$$\updownarrow$$
$$\{ \text{ simple } 2\text{-representations of the } \mathscr{A}_{\mathcal{H}} \}$$

where $\mathcal{H}$ runs over a choice of diagonal $\mathcal{H}$-cell in every two-sided cell.

Remarks

- completes classification in all finite Coxeter types apart form H_3, H_4
- for few H-cells in types H_3, H_4, $\mathscr{A}_{\mathcal{H}}$ is not well-understood

Thank you!

Thank you for your attention!