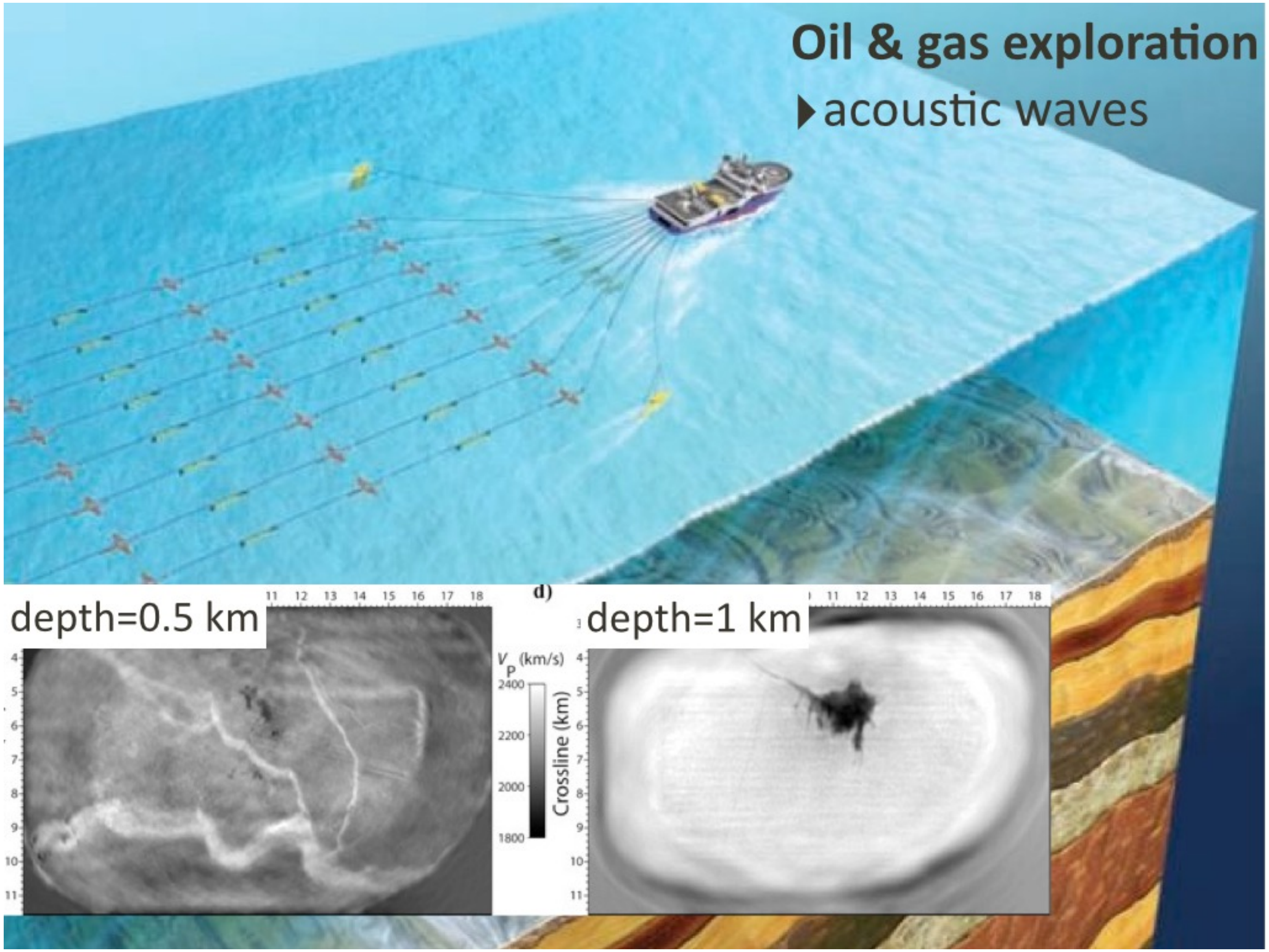
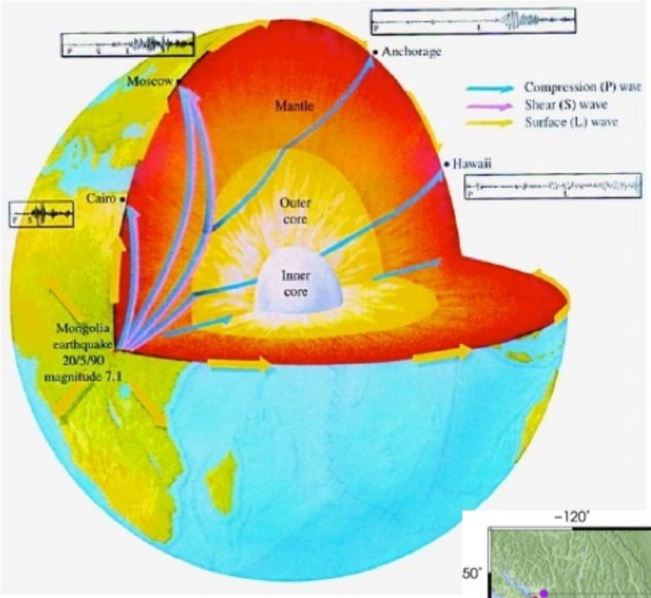
Large-scale inverse problems with time-harmonic waves

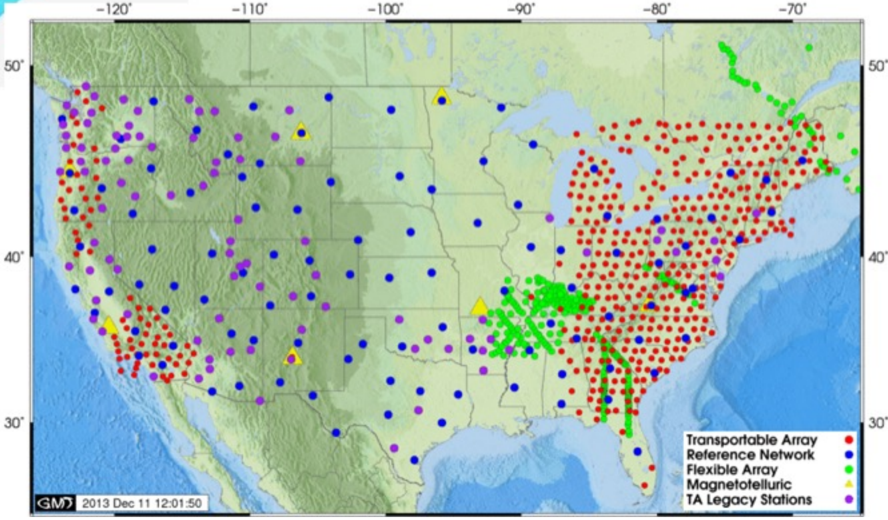
Tristan van Leeuwen


Oil & gas exploration
acoustic waves
depth=0.5 km
depth=1 km
d)
V_P (km/s)
2400
2200
2000
1800
Crossline (km)

Seismology

- elastic waves





CWI

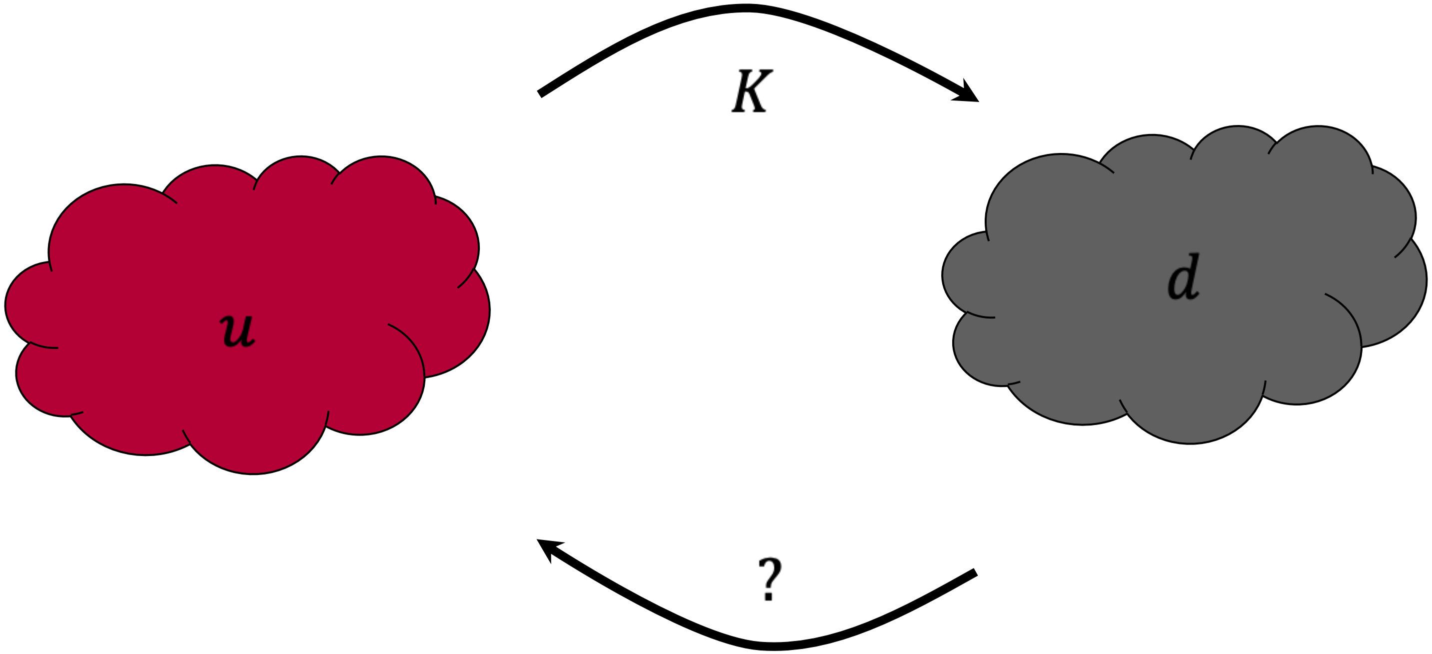
Medical imaging

- ultrasound (sound)
- MRI (EM)
- Optical tomography (light)

Outline

- The inverse problem
- PDE-constrained optimisation
- Challenges for solvers
- Wrap-up

The Inverse Problem



The Inverse Problem

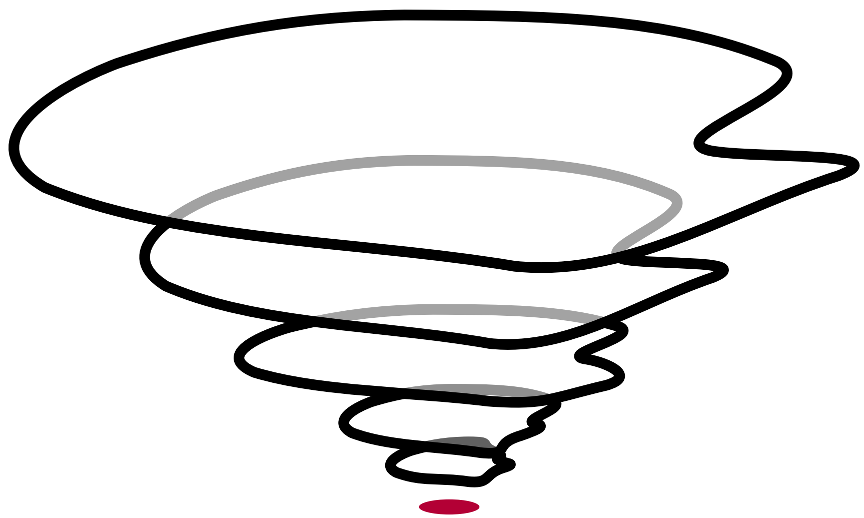
Recover coefficients $m(\mathbf{x})$ from measurements

$$d_{ijk} = u_i(\omega_j, \mathbf{x}_k)$$

where

$$\left(\omega^2 m(\mathbf{x}) + \nabla^2\right) u_i(\omega, \mathbf{x}) = q_i(\omega, \mathbf{x})$$

PDE-constrained optimisation



$$\min_{m} \boldsymbol{J}(m)$$

PDE-constrained optimisation

Discretise-then-optimise:

$$\min_{\mathbf{m},U} \sum_{i=1}^{n_s} \|P\mathbf{u}_i - \mathbf{d}_i\|_2^2 \quad \text{s.t.} \quad A(\mathbf{m})\mathbf{u}_i = \mathbf{q}_i$$

- All-at-once
- Reduced
- Penalty method

All-at-once methods

$$\min_{\mathbf{m},U}\max_{V}\sum_{i=1}^{n_s}\|P\mathbf{u}_i-\mathbf{d}_i\|_2^2+\mathbf{v}_i^T(A(\mathbf{m})\mathbf{u}_i-\mathbf{q}_i)$$

- **Set up saddle-point problem for** (m,U,V)
- **Solve using Newton-Krylov method**
- **Avoids solving PDE at each step**
- **Requires large amount of storage**

Reduced-space approach

$$\min_{\mathbf{m}} \sum_{i=1}^{n_s} \|PA(\mathbf{m})^{-1}\mathbf{q}_i - \mathbf{d}_i\|_2^2$$

- **Solve PDE explicitly for each source term**
- **Non-linear optimization in** m
- **Requires adjoint-solve to compute gradient w.r.t.** m

A penalty method

$$\min_{\mathbf{m},U} \sum_{i=1}^{n_s} \|P\mathbf{u}_i - \mathbf{d}_i\|_2^2 + \lambda \|A(\mathbf{m})\mathbf{u}_i - \mathbf{q}_i\|_2^2$$

- **Add PDE as quadratic penalty**
- **Solve for U explicitly**
- **Non-linear optimization in m**
- **No explicit adjoint-solve required**
- **Reduces the non-linearity**

Challenges for solvers

- Need to solve for many r.h.s.
- Robust and predictable convergence behaviour is desired
- May not need a very accurate solution
- Solve overdetermined problems involving Helmholtz

Challenges for solvers

- A simple preconditioner
- Harnessing inexactness
- Re-ordering the computations
- Data-augmented systems

Challenges for solvers

- **A simple preconditioner**
- Harnessing inexactness
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Simple preconditioning

A symmetric Kaczmarz-sweep leads to

$$\mathbf{u}^{(k+1)} = Q\mathbf{u}^{(k)} + R\mathbf{q}$$

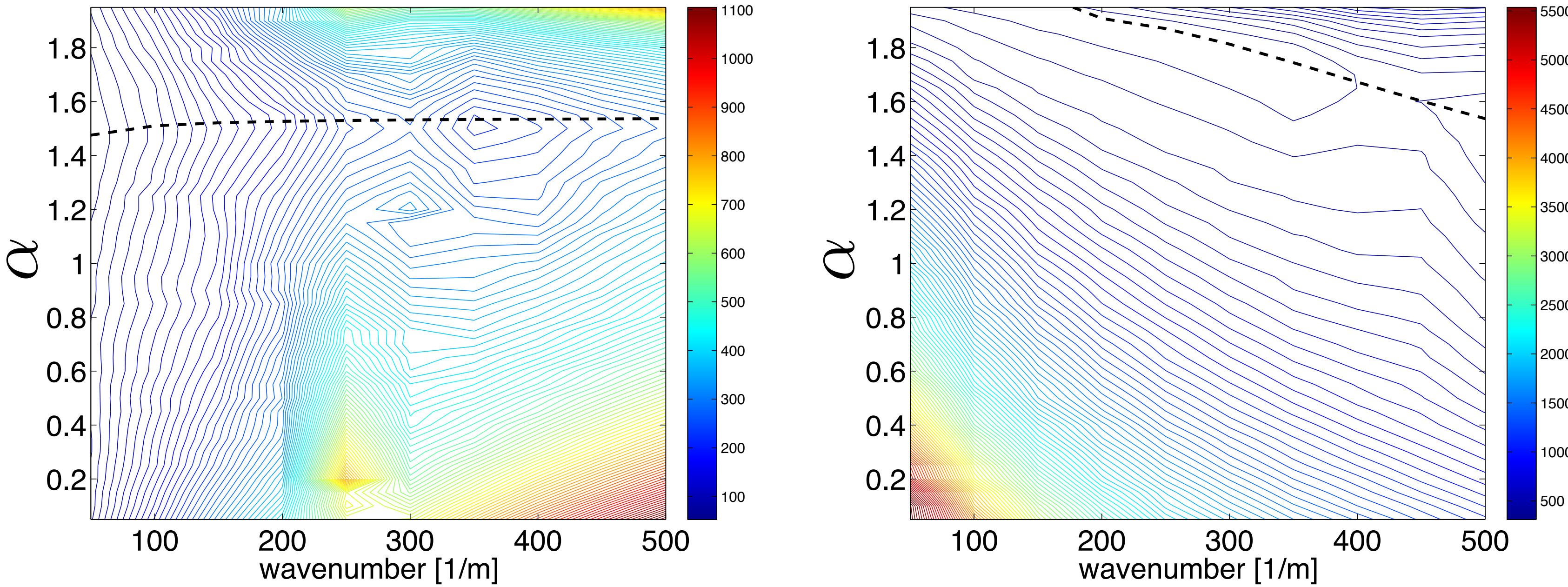
with

$$Q = I - A^* H A, \quad H = \alpha(2-\alpha)(D+\alpha U)^{-1} D (D+\alpha L)^{-1}$$

Now solve using CG.

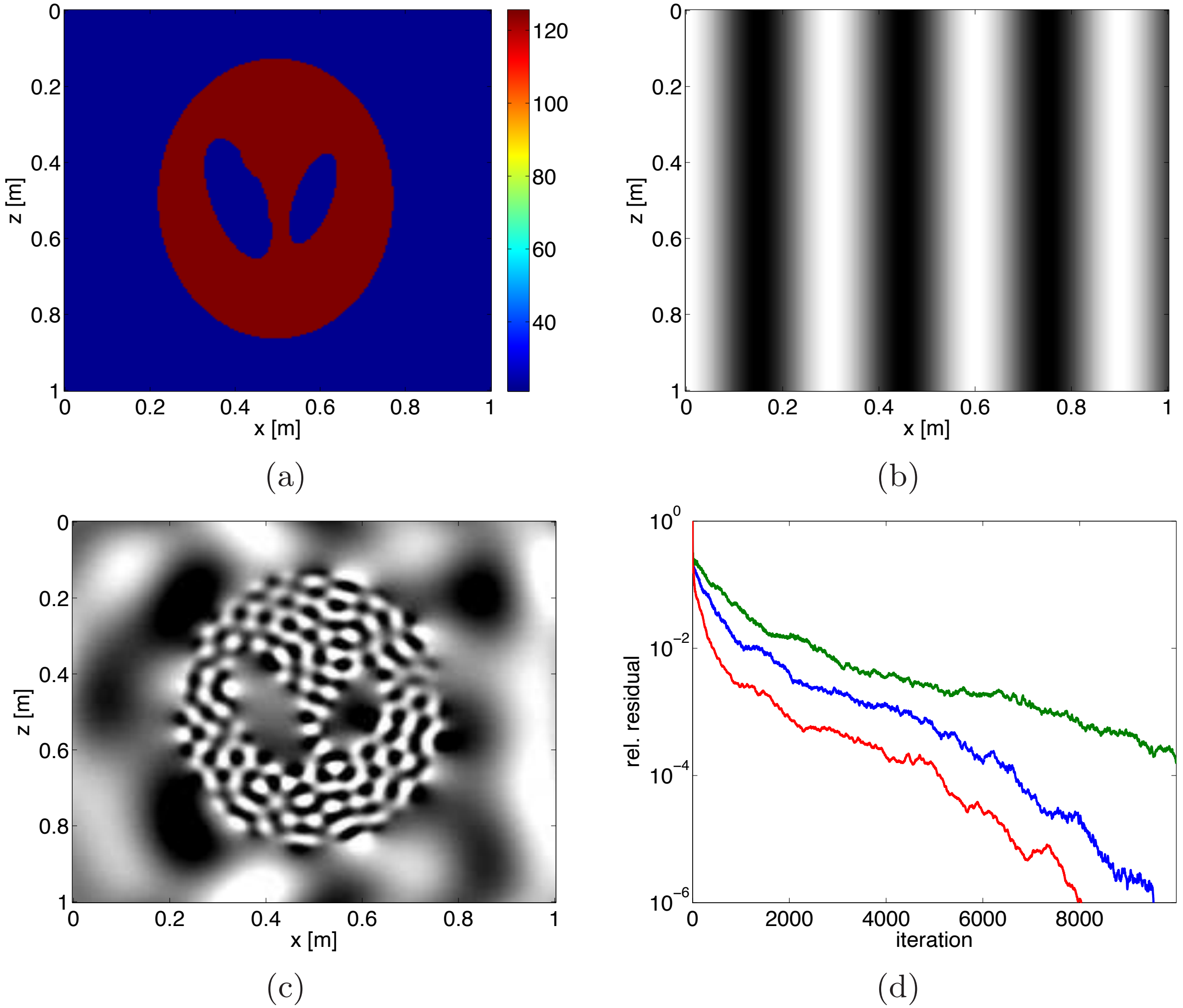
Simple preconditioning

Fourier analysis yields an optimal choice for α for each frequency



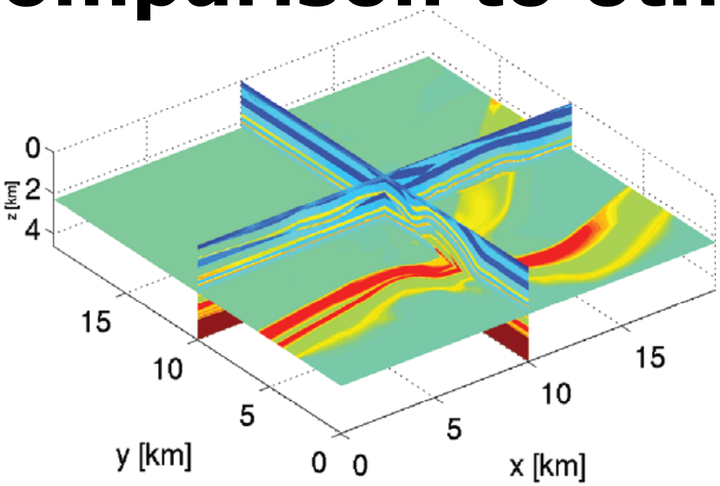
(a) (b)

Simple preconditioning

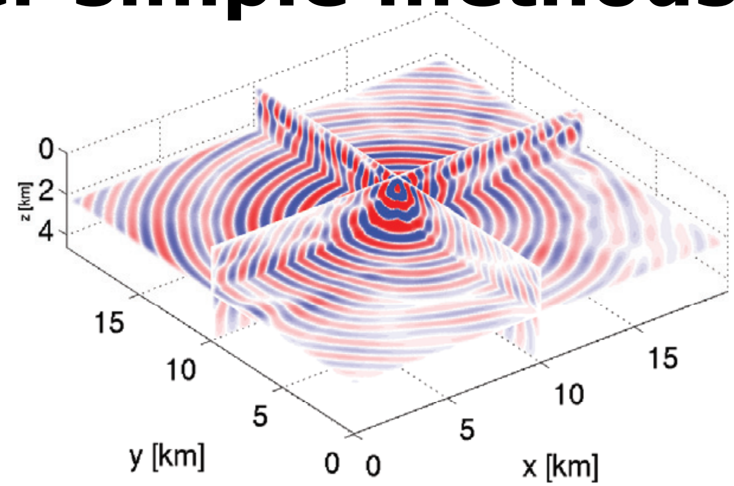


Simple preconditioning

Comparison to other simple methods



(a)



(b)

f [Hz]	N	CGMN	BiCGs	GMRES(5)
0.5	23276.0	308.0	81.0	139.0
1.0	186208.0	564.0	150.0	425.0
2.0	1455808.0	960.0	911.0	1603.0
4.0	11646464.0	2123.0	*	*

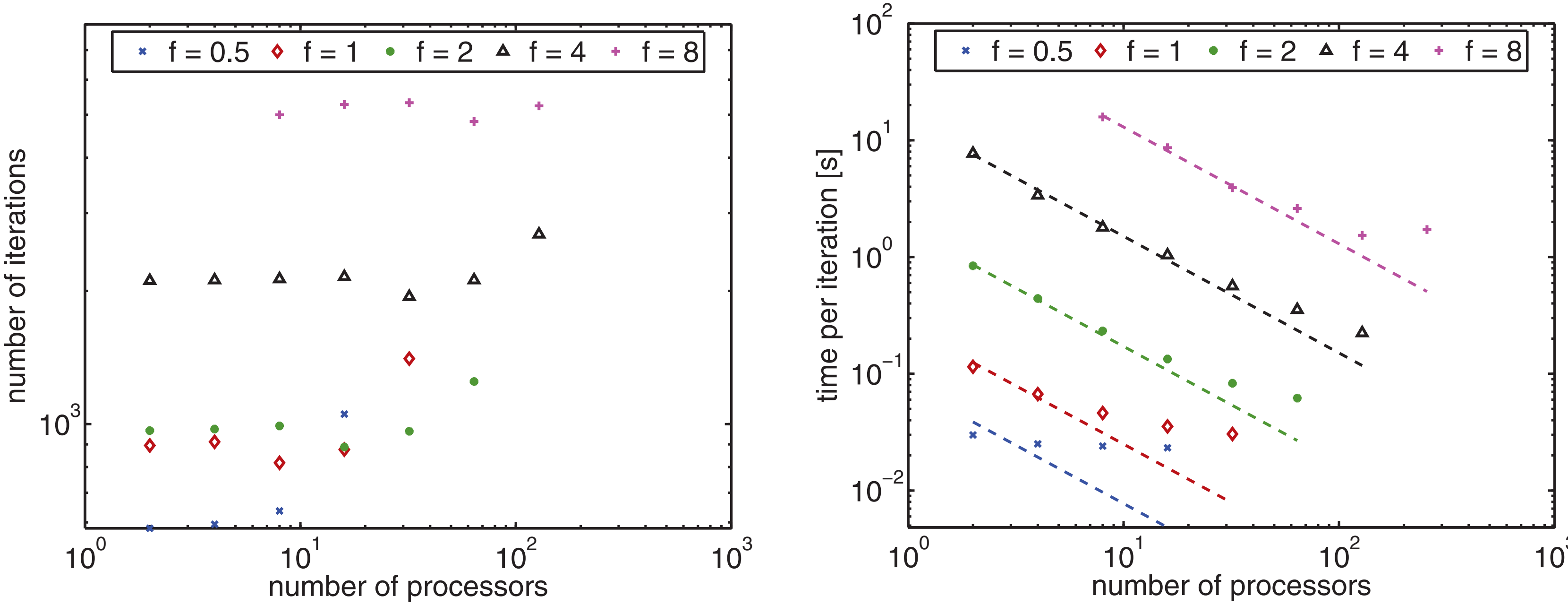
Simple preconditioning

Block-iterative methods

f [Hz]	N	block size	iter	time [s]
0.5	23276	1	291	35.9
		2	278	43.3
		5	200	29.7
		10	115	15.2
1.0	186208	1	484	2859.9
		5	477	2419.8
		10	456	2279.7
		50	220	1067.7
2.0	1455808	1	828	125358.2
		10	811	122732.7
		50	716	109424.7
		100	559	82938.2

Simple preconditioning

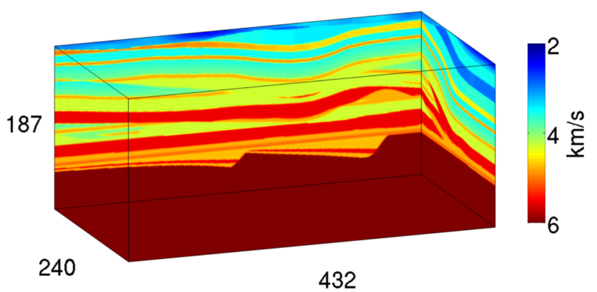
Parallelization

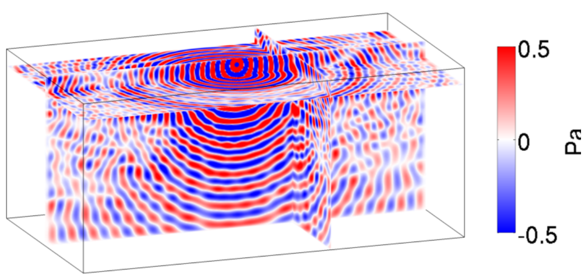


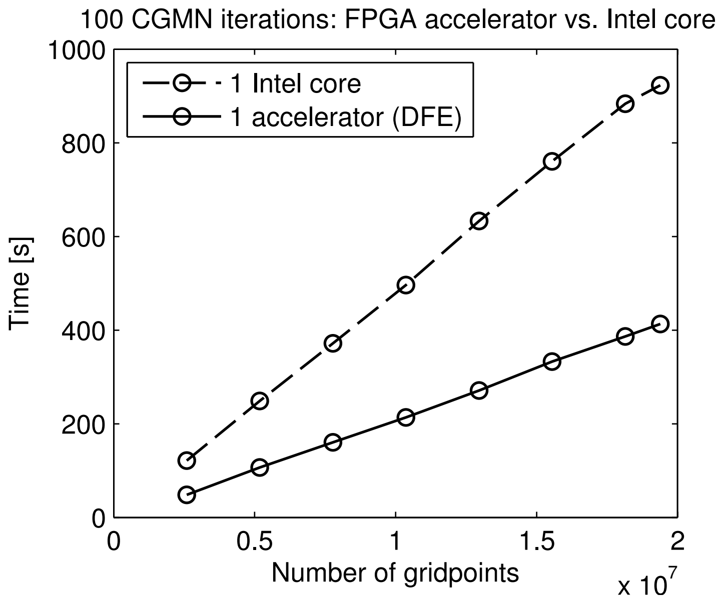
(a) (b)

Simple preconditioning

The method is amenable to FPGA accelaration







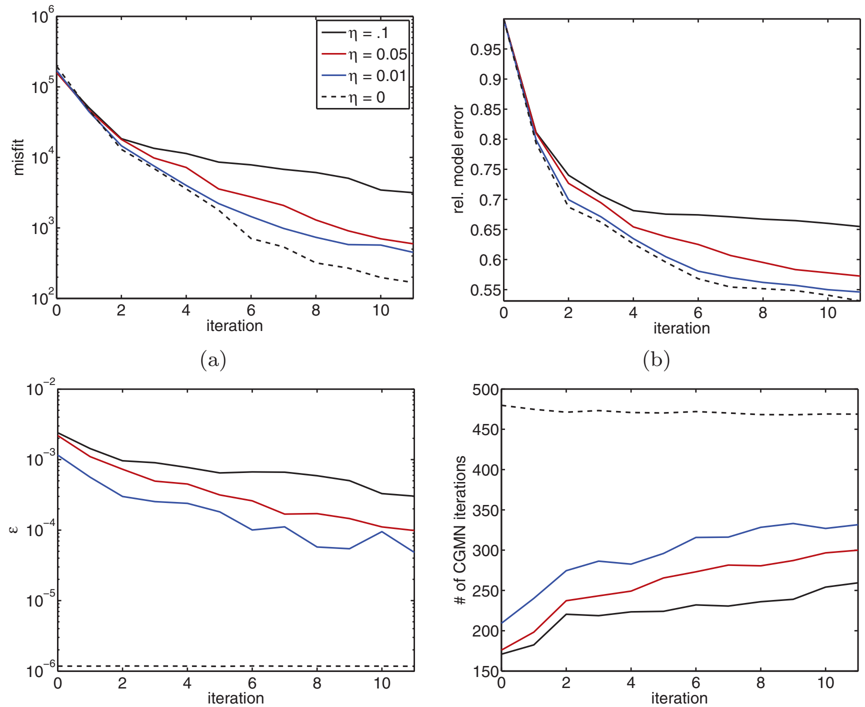
Challenges for solvers

- A simple preconditioner
- **Harnessing inexactness**
- Re-ordering the computations
- Data-augmented systems

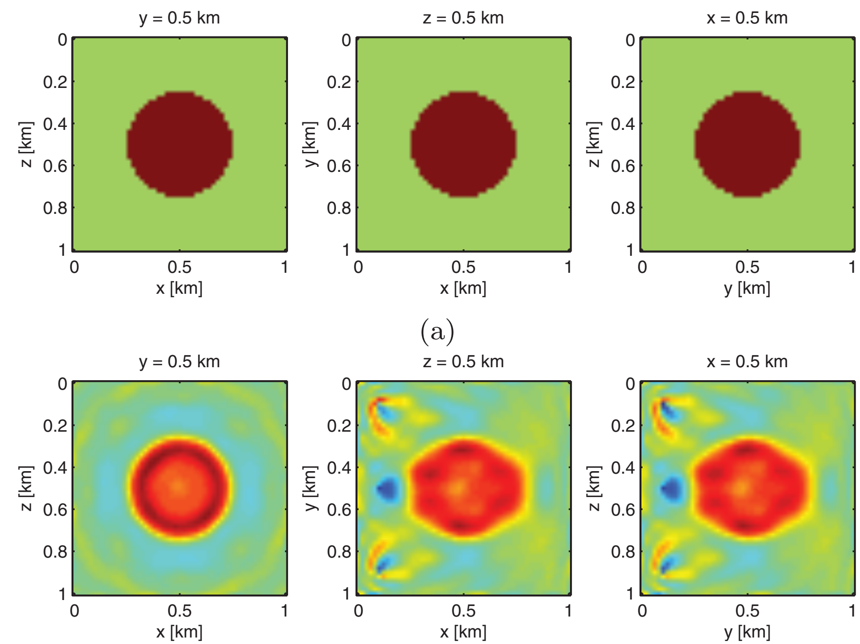
Harnessing inexactness

- **Use stochastic approximation to compute gradient**
- **PDE-solves need not be more accurate than induced error**
- **Use an adaptive tolerance based on current data-fit**

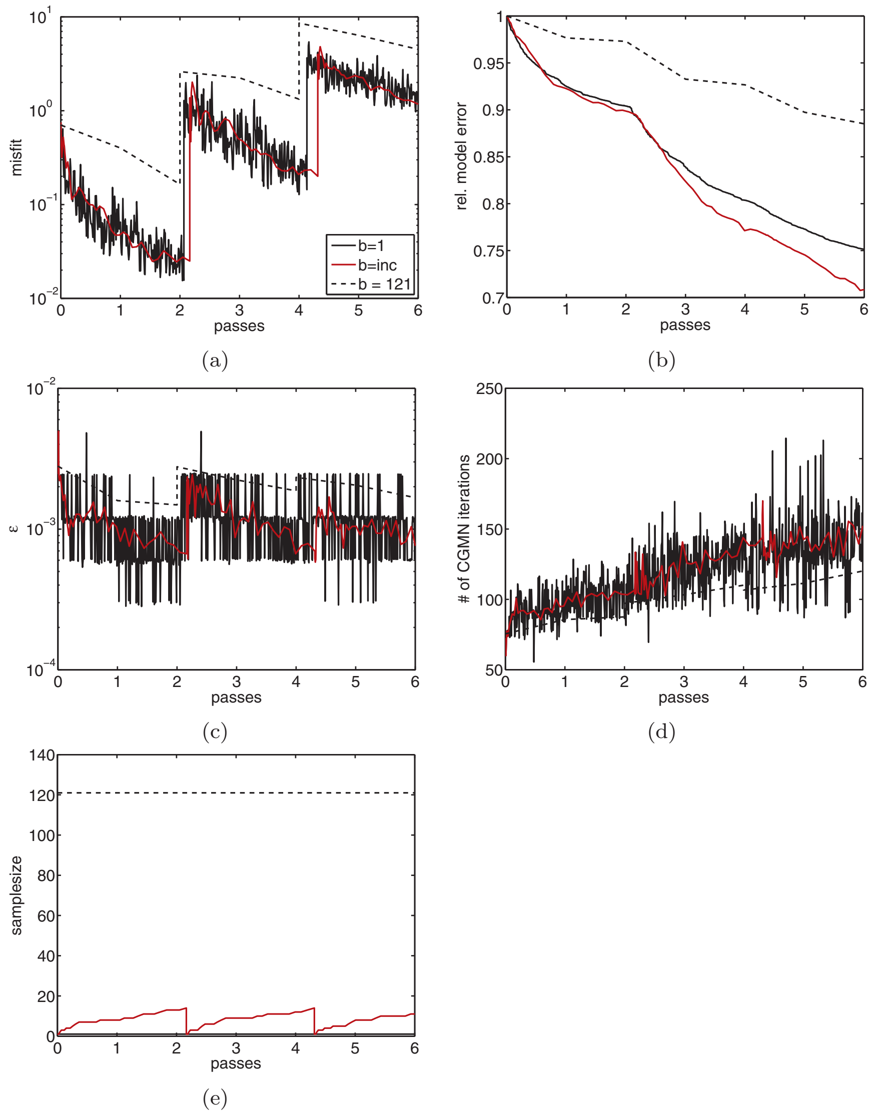
Harnessing inexactness

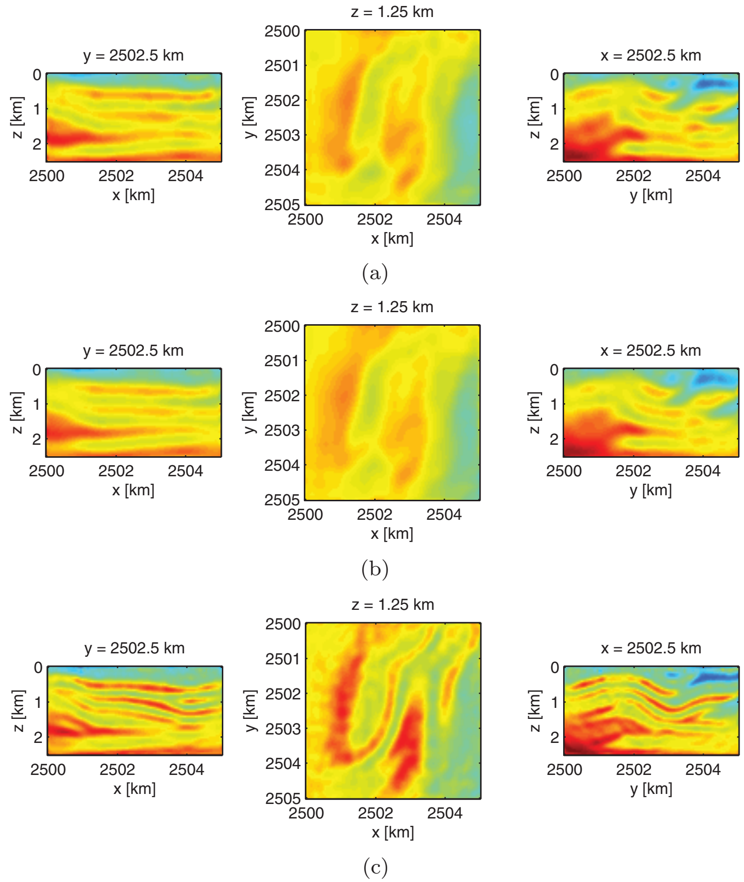


(a) (b) (c) (d)



(a) (b)


misfit
passes
b=1
b=inc
b = 121
(a)
rel. model error
passes
(b)
ε
passes
(c)
# of CGMN iterations
passes
(d)
samplesize
passes
(e)


y = 2502.5 km
z = 1.25 km
x = 2502.5 km
z [km]
y [km]
x [km]
(a)
(b)
(c)

Challenges for solvers

- A simple preconditioner
- Harnessing inexactness
- **Re-ordering the computations**
- Data-augmented systems

Re-ordering the computations

We need to solve forward and adjoint system(s)

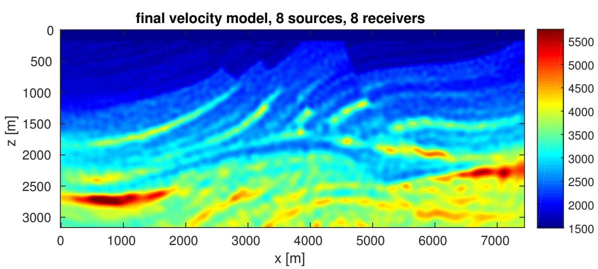
$$AU = Q$$
$$A^*V = P^T(PU - D)$$

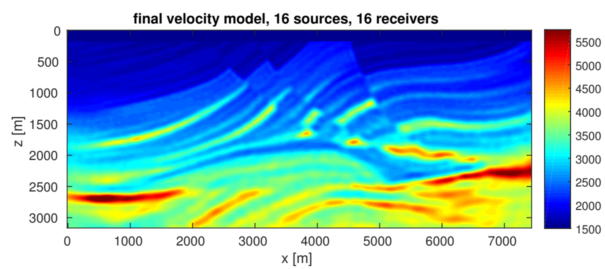
Which can be parallelized when considering

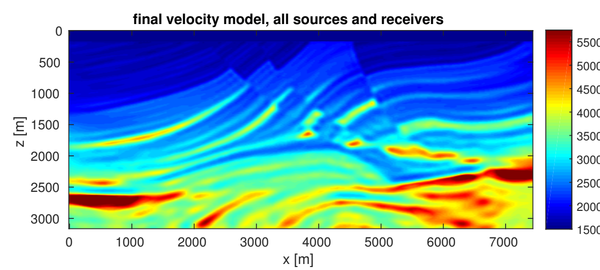
$$A^*W = P^T$$

Re-ordering the computations

Computational cost can be further reduced by stochastic subsampling







Challenges for solvers

- A simple preconditioner
- Harnessing inexactness
- Re-oredering the computations
- **Data-augmented systems**

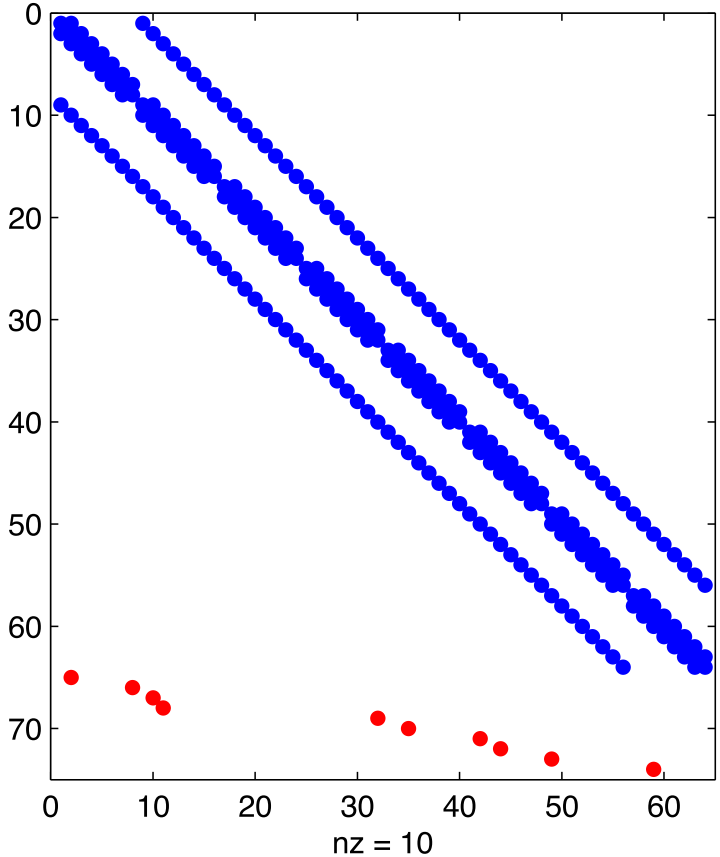
Data-augmented systems

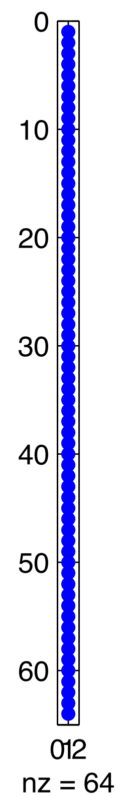
In the penalty approach we need to solve

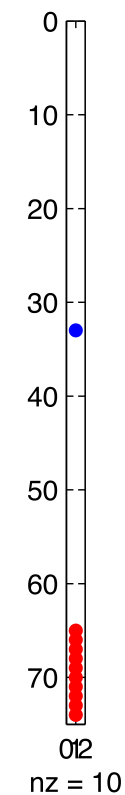
$$\begin{pmatrix} \lambda A \\ P \end{pmatrix} \mathbf{u} = \begin{pmatrix} \lambda \mathbf{q} \\ \mathbf{d} \end{pmatrix}$$

- **How are direct solvers affected?**
- **Can we re-use existing preconditioners?**

Data-augmented systems







Data-augmented systems

Start from normal equations

$$\left(A^*A + P^TP\right)\mathbf{u} = A^*\mathbf{q} + P^T\mathbf{d}$$

and re-write

$$\left(I + WW^*\right)A\mathbf{u} = \mathbf{q} + W\mathbf{d}, \quad W = A^{-*}P^T$$

so

$$\mathbf{u} = A^{-1}\left(I + WW^*\right)^{-1}\left(\mathbf{q} + W\mathbf{d}\right)$$

Data-augmented systems

$$\mathbf{u} = A^{-1}\left(I + WW^*\right)^{-1}\left(\mathbf{q} + W\mathbf{d}\right)$$

- W **is tall and skinny**
- **Replace** A^{-1} **by** M
- **Use randomized subsampled to reduce size of** W

Wrap-up

- Inverse problems pose interesting challenges and opportunities for PDE-solvers
- Robustness may be more important than accuracy
- Need all the tools in the shed to make it work

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