Variational methods for some singular stochastic elliptic PDEs

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Some classical results

- Consider the focusing cubic nonlinear Schrödinger equation on $\mathbb{R}^2$

$$i\partial_t u - \Delta u - |u|^2 u = 0.$$

This equation has conservation laws

$$M(u(t)) := \frac{1}{2}\int_{\mathbb{R}^2} |u(t,x)|^2 dx = M(u_0);$$

$$E(u(t))) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u(t,x)|^2 dx - \frac{1}{4}\int_{\mathbb{R}^2} |u(t,x)|^4 dx.$$

- Global well-posedness holds for $u_0 \in H^1(\mathbb{R}^2)$ iff $\|u_0\|_{L^2} < \|Q\|_{L^2}$: dispersive estimates + conservation of energy.
- Here Q is the unique positive radial minimal solution to

$$-\Delta Q + Q - |Q|^2 Q = 0.$$

This is also the optimizer of the Gagliardo-Nirenberg inequality

$$\|u\|_{L^4}^4 \leq C_{\text{GNS}} \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2, \text{ with } C_{\text{GNS}} = \frac{2}{\|Q\|_{L^2}^2}.$$

The Anderson NLS

What happens if we add a rough potential?

- Consider the cubic Schrödinger equation with spatial white noise potential

$$i\partial_t u - \Delta u + u\xi - |u|^2 u = 0 \text{ on } \mathbb{T}^2.$$

  ξ is a Gaussian field with covariance $\mathbb{E}\big[\xi(x)\xi(y)\big] = \delta_0(x,y)$. In particular ξ is a random distribution: $\xi \in C^{-1-}(\mathbb{T}^2)$ a.s.
- Local and global well-posedness? Is there a dichotomy for globalizing any finite energy solution? Even if $u(0) \in C^\infty(\mathbb{T}^2)$, $u\xi \in C^{-1-}(\mathbb{T}^2)$ so $u(t) \in H^{-1-}(\mathbb{T}^2)$ and $|u|^2 u$ is ill-defined.
- Take a smooth approximation $\xi_\varepsilon = \chi_\varepsilon \star \xi$ and look at the global smooth solution u_ε. Then [Hairer-Labbé '15, Debussche-Weber '18, Debussche-Martin '19, Tzvetkov-Visciglia '21&'22] define $Y_\varepsilon = \Delta^{-1}\xi_\varepsilon \in C^{1-}(\mathbb{T}^2)$ and transform $v_\varepsilon = e^{Y_\varepsilon} u_\varepsilon$. Then v_ε solves

$$i\partial_t v_\varepsilon - \Delta v_\varepsilon + 2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon + v_\varepsilon |\nabla Y_\varepsilon|^2 - |v_\varepsilon|^2 v_\varepsilon e^{-2Y_\varepsilon} = 0.$$

  Renormalization is needed!

Renormalization and global well-posedness

- Define $v_\varepsilon(t) = e^{-ic_\varepsilon t + Y_\varepsilon} u_\varepsilon$ with $c_\varepsilon = \mathbb{E}|\nabla Y_\varepsilon|^2 \sim |\log \varepsilon|$, then v_ε solves

$$i\partial_t v_\varepsilon - \Delta v_\varepsilon + 2\nabla v_\varepsilon \cdot \nabla Y_\varepsilon + v_\varepsilon :|\nabla Y_\varepsilon|^2: - |v_\varepsilon|^2 v_\varepsilon e^{-2Y_\varepsilon} = 0$$

with $:|\nabla Y_\varepsilon|^2: = |\nabla Y_\varepsilon|^2 - c_\varepsilon$ converges a.s. in $C^{0-}(\mathbb{T}^2)$.

[Debussche-Weber '18, Debussche-Martin '19, Tzvetkov-Visciglia '21&'22]

Assume $v(0) \in H^2(\mathbb{T}^2)$, i.e. $e^Y u_0 \in H^2(\mathbb{T}^2)$. Then v_ε converges a.s. to the unique global solution $v \in C_t H^{2-}(\mathbb{T}^2)$ to
$i\partial_t v - \Delta v + 2\nabla v \cdot \nabla Y + v :|\nabla Y|^2: - |v|^2 v e^{-2Y} = 0$.

- Proof relies on Brezis-Gallouet inequality since there is no conservation law at this level of regularity regularity.
- What about lower regularity?

The Anderson Hamiltonian

- Let us go back to the renormalized equation $i\partial_t u_\varepsilon - \Delta u_\varepsilon + u_\varepsilon \xi_\varepsilon - u_\varepsilon c_\varepsilon - |u_\varepsilon|^2 u_\varepsilon = 0$.
- Start with the linear equation (eigenvalue problem) $-\Delta u + u(\xi_\varepsilon - c_\varepsilon) - \lambda u = 0$, $\lambda \in \mathbb{C}$. We need to impose a structure on u.

The family of operators $H_\varepsilon = -\Delta + \xi_\varepsilon - c_\varepsilon : H^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ converges almost surely in resolvent norm to an unbounded operator H on $L^2(\mathbb{T}^2)$.

- There is an enhanced noise $\Xi(\xi) \in C^{1-}$ such that $\Xi(\xi_\varepsilon) \to \Xi(\xi)$ and the domain of H is random and consists of functions behaving locally as Ξ modulo smooth (H^2) remainder;
- It holds $D(H) \subset C^{1-}$ but $C^\infty \notin D(H)$;
- The operator $H : D(H) \to L^2(\mathbb{T}^2)$ is closed, self-adjoint, and with compact resolvent and $\inf \sigma(H) > -\infty$ a.s.
- There exists a random $\lambda > 0$ such that $H + \lambda > 0$. Then $D\big((H+\lambda)^{\frac{s}{2}}\big) = H^s(\mathbb{T}^2)$ for $s \in [0;1)$ but **not** $s \geq 1$.

Further remarks

- [Allez-Chouk '15] on $\mathbb{T}^2$, [Gubinelli-Ugurcan-Zachhuber '19] on $\mathbb{T}^3$: using para-controlled calculus.
- [Labbé '19] using regularity structures.
- [Mouzard '21], [Bailleul-Dang-Mouzard '22] on $\mathcal{M}$. In particular there is the Weyl law $\lambda_n \sim \frac{4\pi}{|\mathcal{M}|} n$. On $\mathcal{M} = \mathbb{T}^2$, we should have better localization of the spectrum.
- As for the dynamical problem, $e^Y u_0 \in H^2 \iff u_0$ is para-controlled by Y. Here $\Xi = Y+$more regular terms to get better cancellations, i.e. $H : D(H) \to L^2(\mathbb{T}^2)$ instead of $H : D(H) \to H^{0-}(\mathbb{T}^2)$.
- [Gubinelli-Ugurcan-Zachhuber '19] Global well-posedness of Anderson-NLS in $D(H)$.
- [Mouzard-Zachhuber '21] Local well-posedness in $D\big((H+\lambda)^{\frac{s}{2}}\big)$, $s > \frac{1}{2}$ using dispersive estimates.
- Propagation of higher Sobolev regularity is not known, and maybe false.

The equation

The starting point for optimal mass threshold in globalizing is to consider the nonlinear stochastic elliptic PDE (Anderson ground state equation)

$$-\Delta u + u\xi - |u|^2 u = 0$$

for $u : \mathbb{T}^2 \to \mathbb{R}$. Existence, uniqueness of solutions?

[Bailleul-Eulry-R. '22]

The (renormalized) ground state equation has infinitely many solutions.

- Again, as ξ has almost surely very low regularity, $u\xi$ is ill-defined for general u. Renormalization is needed.
- The result holds for much more general variable coefficients nonlinearities.
- In the case without potential, uniqueness modulo symmetries of the equation. Here: no (path-wise) rotational invariance, sign definiteness, translation invariance. In particular, extension from $\mathbb{T}^2$ to $\mathbb{R}^2$ unclear due to lack of compact embeddings.

Variational approach

- Define $\Phi(u) = \frac{1}{2}\langle u, (H+\lambda)u\rangle - \frac{1}{4}\int_{\mathbb{T}^2}|u|^4 dx - \frac{\lambda}{2}\int_{\mathbb{T}^2}|u|^2$ for $u \in D(\sqrt{H})$.
- $u \in D(\sqrt{H})$ is a weak solution to the ground state equation $\Leftrightarrow$ u is a critical point of Φ.

Φ satisfies the Palais-Smale condition and has a mountain pass geometry on $D(\sqrt{H})_+(\mathbb{T}^2)$.

- Other elliptic SPDES: [Otto-Weber '19], [Albeverio-De Vecchi-Gubinelli '20].
- Here, no smallness or coercivity assumption is needed.
- Variational approach for singular SPDEs only investigated in [Ignat-Otto-Ried-Tsatsoulis '20], [Duan-Zhang '21].

Symmetry $u \to -u$: infinite number of solutions.

- The spectral approach allows to deal with non-variational equations, for example the singular stochastic defocusing Choquard-Pekar equation on $-\Delta u + u\xi + \langle\nabla\rangle^{-\alpha}(|u|^p)|u|^{q-2}u = 0$ on $\mathbb{T}^2$, for any $p, q > 1$.

Thank you for your attention!