The puzzling path integral for gravity

Feynman's path integral is the most elegant known formulation of the quantum physics of continuous systems. It is particularly suited to relativistic theories, including gauge theories and gravity. However, it is an enduring embarrassment that path integrals have never been rigorously formulated except by ``Wick rotating" to imaginary (``Euclidean") time. Unfortunately, this rotation does violence to the physics, replacing quantum interference with statistical physics. Continuing back to real (Lorentzian) time, where observations are made, is usually impossible (outside of perturbation theory). So one cannot describe dynamical processes. I will describe recent progress towards defining and evaluating path integrals, using physical insights and generalizing Picard-Lefschetz (steepest descent) methods for finite-dimensional highly oscillatory integrals. I will describe some first steps towards a satisfactory definition of Lorentzian path integrals and, if time allows, mention some exciting new work on calculating the gravitational entropy of the universe. I will explain why a Euclidean formulation of quantum field theory, even generalized to complex metrics, is unlikely to describe the real universe.