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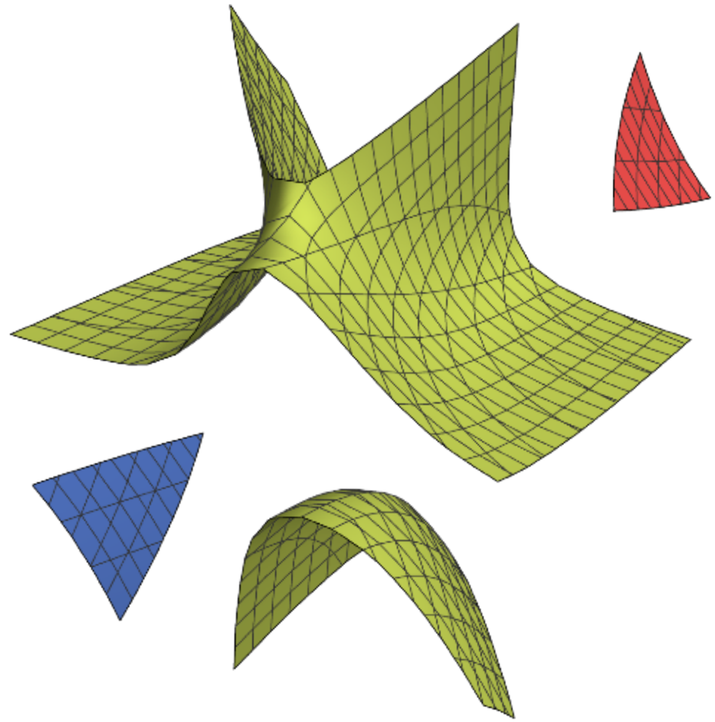
ICREA

# Segre surfaces for the Painlevé equations

with Nalini Joshi & Pieter Roffelsen

ArXiv: 2405.10541



UNIVERSITY OF BIRMINGHAM

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# 6-parameter family of Segre surfaces in $\mathbb{C}^6$

$$z_1+z_2+z_3+z_4+z_5+z_6=0$$

$$Z_q: \quad \begin{aligned} &\rho_2 z_2+\rho_3 z_3+z_4+\rho_5 z_5+\rho_6 z_6=1\\ &z_3z_4-\lambda_1 z_1 z_2=0\\ &z_5z_6-\lambda_2 z_1 z_2=0\end{aligned}$$

monodromy mfd of
qPVI

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## Our results:

- Universal: any embedded Segre surface with smooth projective completion can be put in this form via an affine transformation
- $PVI_q \uparrow PVI$ produces $\mathcal{Z}_q \uparrow \mathcal{Z}_1$ a 4 parameter family of Segre surfaces.
- $\mathcal{Z}_1$ is isomorphic as affine variety to the Jimbo-Fricke cubic
- By confluence we produce Segre surfaces for each of the Painlevé differential equations

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tools: study of lines & asymptotics

qPVI [Jimbo-Sakai '96]

$$\begin{cases} f, g : q^{\mathbb{Z}} t_0 \to \mathbb{P}^1 \\ f(t)\, f(qt) = \dfrac{\left(g(qt) - q^{\theta_0} t\right)\left(g(qt) - q^{-\theta_0} t\right)}{\left(g(qt) - q^{\theta_\infty - 1}\right)\left(g(qt) - q^{-\theta_\infty}\right)} \\ g(t)\, g(qt) = \dfrac{\left(f(t) - q^{\theta_t} t\right)\left(f(t) - q^{-\theta_t} t\right)}{q\left(f(t) - q^{\theta_1}\right)\left(f(t) - q^{-\theta_1}\right)} \end{cases}$$

6 parameters

## qPVI [Jimbo-Sakai '96]

$$
\begin{cases}
f, g : q^{\mathbb{Z}} t_0 \to \mathbb{P}^1 \\[4pt]
f(t)\, f(qt) = \dfrac{\left(g(qt) - q^{\theta_0} t\right)\left(g(qt) - q^{-\theta_0} t\right)}{\left(g(qt) - q^{\theta_\infty - 1}\right)\left(g(qt) - q^{-\theta_\infty}\right)} \\[10pt]
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\end{cases}
$$

6 parameters

## $q \uparrow 1$ gives PVI

$$f(t) = f_0(t) + f_1(t)(q-1) + f_2(t)(q-1)^2 + \cdots$$

$$g(t) = g_0(t) + g_1(t)(q-1) + \cdots$$

set $f_0 = u$, $g_0 = \dfrac{u-t}{u-1}$ $\Rightarrow$ $u(t)$ satisfies PVI

$$u_{tt} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right) u_t^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right) u_t + \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \beta \frac{t}{u^2} + \gamma \frac{t-1}{(u-1)^2} + \delta \frac{t(t-1)}{(u-t)^2}\right)$$

## Lax pair for qPVI (Jimbo-Sakai)

$$qPVI \longleftrightarrow A(z,qt)\,B(z,t) = B(qz,t)\,A(z,t)$$

Auxiliary linear system

$$Y(qz,t) = A(z,t)\,Y(z,t) \qquad A(z,t) = A_0(t) + A_1(t)z + A_2(t)z^2$$

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two singularities: $0, \infty$ $\Longrightarrow$ two local solutions $Y_0, Y_\infty$

$\Longrightarrow$ Connection matrix

$$C(z,t) := \hat{Y}_0(z,t)^{-1}\,\hat{Y}_\infty(z,t)$$

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$$C(z,t) := \hat{Y}_0(z,t)^{-1}\hat{Y}_\infty(z,t)$$

connection preserving deformation

$$Y(z,qt) = B(z,t)Y(z,t)$$

$$\Rightarrow \quad A(z,qt)B(z,t)Y(z,t) = B(qz,t)A(z,t)Y(z,t).$$

# Monodromy manifold for qPVI (Joshi - Roffelsen)

$$\mathcal{M}(\theta_0, \theta_t, \theta_1, \theta_\infty, q, t_0) = \{\text{connection matrices}\} \Big/ \begin{array}{l}\text{multiplication}\\ \text{on left \& right}\\ \text{by a diagonal}\\ \text{matrix}\end{array}$$

$$\Big\downarrow \ \text{bijection}$$

Affine Segre surface in $\mathbb{C}^6$

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 0$$

$$\rho_2 z_2 + \rho_3 z_3 + z_4 + \rho_5 z_5 + \rho_6 z_6 = 1$$

$$z_3 z_4 - \lambda_1 z_1 z_2 = 0$$

$$z_5 z_6 - \lambda_2 z_1 z_2 = 0$$

$$\mathcal{Z}_q$$

# Monodromy manifold for qPVI

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$\big\downarrow$ bijection

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$$\mathcal{Z}_q$$

[Joshi - M.M. - Roffelsen]

- smooth projective completion
- Divisor at $\infty$ is a genus 1 curve given by the intersection of two quadrics in $\mathbb{P}^3$
- Universal: generic embedded Segre with smooth projective completion can be put in this form via an affine transformation (question by Ramis)

# Limit $q \uparrow 1$

$$\mathcal{Z}_1$$

$$z_1+z_2+z_3+z_4+z_5+z_6=0$$

$$\rho_3 z_3+z_4+\rho_5 z_5+\rho_6 z_6=1$$

$$z_3 z_4-\lambda_1 z_1 z_2=0$$

$$z_5 z_6-\frac{\rho_3\lambda_1}{\rho_5\rho_6} z_1 z_2=0$$

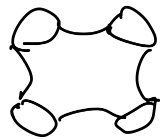
- smooth projective completion
- Divisor at $\infty$ is reducible, given by two conics intersecting at 2 points.

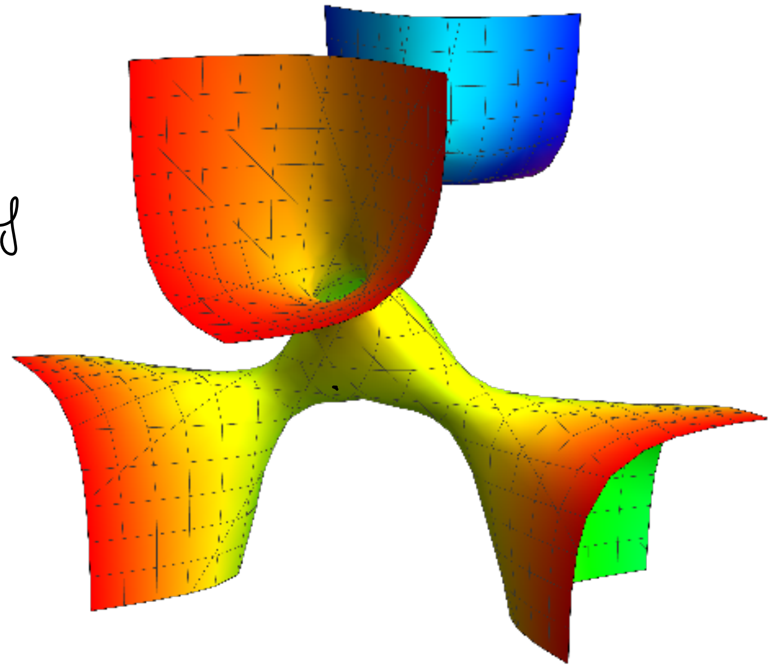
Question: how is this family related to the cubic one?

$$x_1x_2x_3-x_1^2-x_2^2-x_3^2+\omega_1 x_1+\omega_2 x_2+\omega_3 x_3+\omega_4=0$$

# the PVI monodromy manifold

$$x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0$$

- $SL_2(\mathbb{C})$ character variety of 
- Versal deformation of $D_4$ singularity
- self mirror
- For $\omega_i = 0$, Markov cubic
- Cluster algebra



- Blow up of 6 pts in $\mathbb{P}^2$ (Clebsch 1866) $\Rightarrow$ Del Pezzo of deg $=3$
- 27 lines (Cayley & Salmon 1849)

each line corresponds to a specific asymptotic behaviour of the corresponding PVI solution (Guzzetti / Klimes)

# Constructing the affine transformation

$$x_1 x_2 x_3 - x_1^2 - x_2^2 - x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4 = 0$$

$$\Big\downarrow \ \text{blow down}$$

$$y_4 = y_2 y_3$$

$$y_1 y_4 - y_1^2 - y_2^2 - y_3^2 + \omega_1 y_1 + \omega_2 y_2 + \omega_3 y_3 + \omega_4 = 0$$

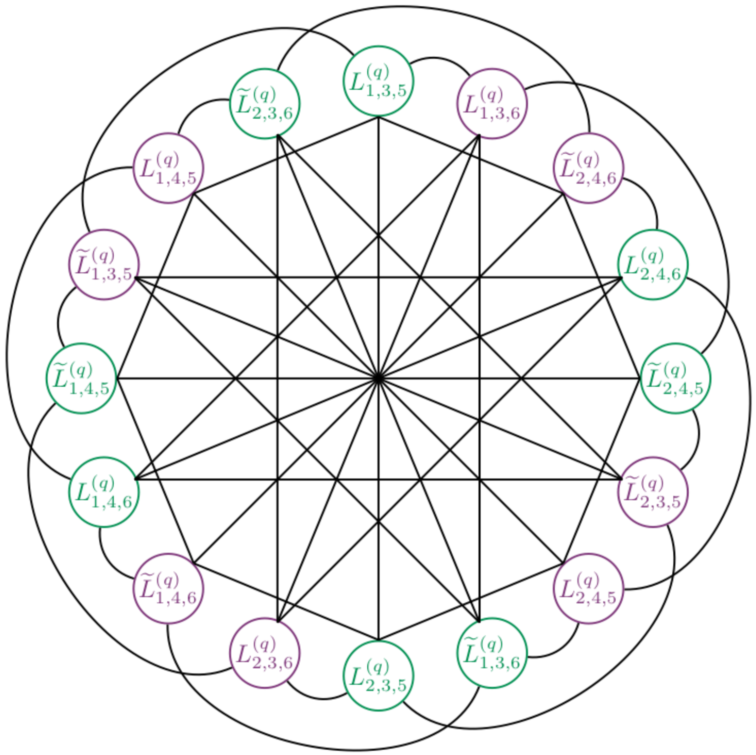
$y$- Segre surface

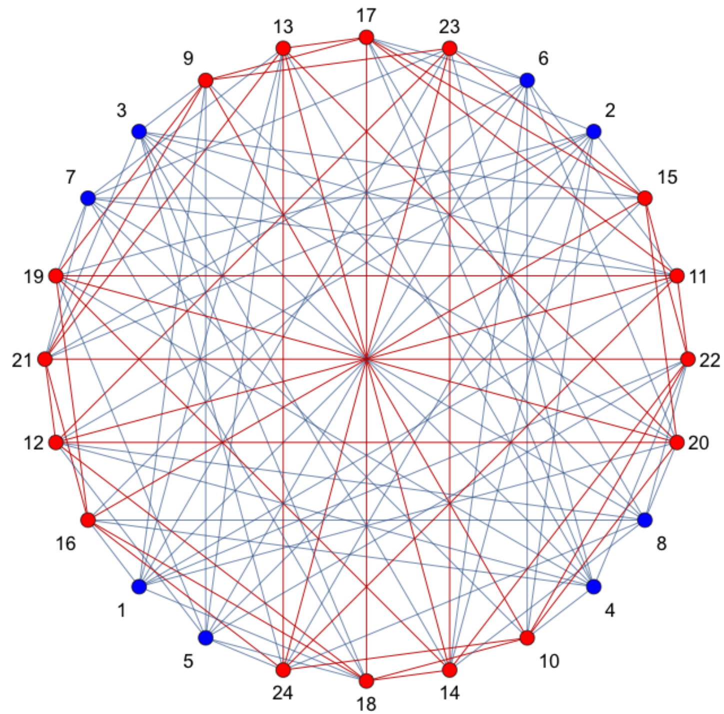
$$\Big\updownarrow \ \text{affine transformation}$$

$$\mathcal{Z}_1 : \quad \begin{aligned} & z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 0 \\ & \rho_3 z_3 + z_4 + \rho_5 z_5 + \rho_6 z_6 = 1 \\ & z_3 z_4 - \lambda_1 z_1 z_2 = 0 \\ & z_5 z_6 - \frac{\rho_3 \lambda_1}{\rho_5 \rho_6} z_1 z_2 = 0 \end{aligned}$$

# the isomorphism

Brute force




Clebsch graph for $\mathcal{Z}_1$

green lines intersect conic$_1$ at $\infty$

purple lines intersect conic 2 at $\infty$

Clebsch graph of cubic

Red subgraph for Y-segre

lines to lines

intersections to intersections

# the isomorphism

Better way : use Tyurin ratios

$$\begin{tikzcd}
\text{Tyurin ratios on } \mathcal{Z}_q \arrow[r, "q \uparrow 1"] \arrow[d, leftrightarrow] & \text{Tyurin ratios on } \mathcal{Z}_1 \\
\text{asymptotics of } qP_{VI} \arrow[r, "q \uparrow 1"] & \text{asymptotics of } P_{VI} \arrow[d, leftrightarrow, "\text{Jimbo}"] \\
& x_1, x_2, x_3 \text{ in the monodromy manifold}
\end{tikzcd}$$

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& x_1, x_2, x_3 \text{ in the monodromy manifold}
\end{tikzcd}$$

$$z_1 = +\gamma^{-1}\Big(v_\infty\Big(x_2 - v_\infty^{-1}\nu_t\Big)\Big(x_3 - v_\infty^{-1}\nu_1\Big) + (v_\infty - v_\infty^{-1})\Big(x_1 - v_\infty\nu_0\Big)\Big),$$

$$z_2 = +\gamma^{-1}\Big(v_\infty^{-1}\Big(x_2 - v_\infty\nu_t\Big)\Big(x_3 - v_\infty\nu_1\Big) - (v_\infty - v_\infty^{-1})\Big(x_1 - v_\infty^{-1}\nu_0\Big)\Big),$$

$$z_3 = -\delta^{-1}\Big(v_0 - v_t v_1 v_\infty\Big)\Big(v_0 - \frac{1}{v_t v_1 v_\infty}\Big)\Big(x_2 - \frac{v_t}{v_\infty} - \frac{v_\infty}{v_t}\Big)\Big(x_3 - \frac{v_1}{v_\infty} - \frac{v_\infty}{v_1}\Big),$$

$$z_4 = -\delta^{-1}\Big(v_0 - \frac{v_t v_1}{v_\infty}\Big)\Big(v_0 - \frac{v_\infty}{v_t v_1}\Big)\Big(x_2 - v_t v_\infty - \frac{1}{v_t v_\infty}\Big)\Big(x_3 - v_1 v_\infty - \frac{1}{v_1 v_\infty}\Big),$$

$$z_5 = +\delta^{-1}\Big(v_0 - \frac{v_t v_\infty}{v_1}\Big)\Big(v_0 - \frac{v_1}{v_t v_\infty}\Big)\Big(x_2 - \frac{v_t}{v_\infty} - \frac{v_\infty}{v_t}\Big)\Big(x_3 - v_1 v_\infty - \frac{1}{v_1 v_\infty}\Big),$$

$$z_6 = +\delta^{-1}\Big(v_0 - \frac{v_1 v_\infty}{v_t}\Big)\Big(v_0 - \frac{v_t}{v_1 v_\infty}\Big)\Big(x_2 - v_t v_\infty - \frac{1}{v_t v_\infty}\Big)\Big(x_3 - \frac{v_1}{v_\infty} - \frac{v_\infty}{v_1}\Big),$$

$$\gamma = (v_0 - 1)(v_0^{-1} - 1)(v_\infty - v_\infty^{-1})^2,$$

$$\delta = (v_0 - 1)^2(v_t - v_t^{-1})(v_1 - v_1^{-1})(v_\infty - v_\infty^{-1})^2.$$

$$v_k = e^{\pi i \theta_k}$$

Monodromy manifolds of the Painlevé differential equations

| P-eqs | Polynomials |
|---|---|
| $PVI$ | $x_1x_2x_3 - x_1^2 - x_2^2 - x_3^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$ |
| $PV$ | $x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$ |
| $PV_{deg}$ | $x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_4$ |
| $PIV$ | $x_1x_2x_3 - x_1^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \omega_4$ |
| $PIII^{D_6}$ | $x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 + \omega_2 x_2 + \omega_4$ |
| $PIII^{D_7}$ | $x_1x_2x_3 - x_1^2 - x_2^2 + \omega_1 x_1 - x_2$ |
| $PIII^{D_8}$ | $x_1x_2x_3 - x_1^2 - x_2^2 - x_2$ |
| $PII^{JM}$ | $x_1x_2x_3 - x_1 + \omega_2 x_2 - x_3 + \omega_4$ |
| $PII^{FN}$ | $x_1x_2x_3 - x_1^2 + \omega_1 x_1 - x_2 - 1$ |
| $PI$ | $x_1x_2x_3 - x_1 - x_2 + 1$ |

Saito van der Put

# the other Painlevé differential equations

$$\begin{tikzcd}
 & & P_{III} \arrow[r] \arrow[dr] & P_{III}^{D_7} \arrow[r] \arrow[dr] & P_{III}^{D_8} \\
P_{VI} \arrow[r] & P_V \arrow[ur] \arrow[r] \arrow[dr] & P_V^{deg} \arrow[ur] \arrow[dr] & \mathrm{P}_{\mathrm{II}}^{\mathrm{JM}} \arrow[r] & P_I \\
 & & P_{IV} \arrow[r] \arrow[ur] & P_{II}^{FN} \arrow[ur] &
\end{tikzcd}$$

Chekhov, M.M., Rubtsov: the confluence of the P-eqs is realised by a confluence of the monodromy manifolds $\Rightarrow$ idea: confluence the isomorphism.

Unramified cases: the isomorphism confluences

Ramified cases: confluence an isomorphism fails. But in depth study of the singularity structure of the cubic blow down allows to build an isomorphism.

# Lines

- lines clash during confluence
- they correspond to special solutions

**Example PI** $x_1x_2x_3 - x_1 - x_2 - 1 = 0$

Contains 5 affine lines and 3 at $\infty$

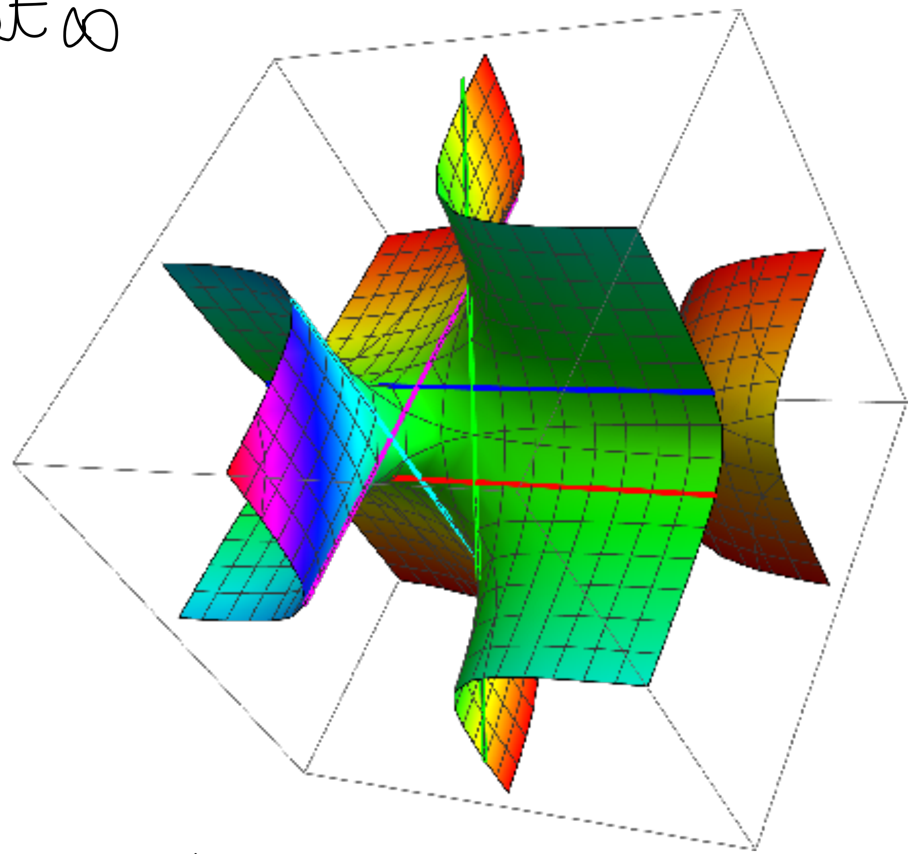
$x_1 = 0,\ x_2 = 1$

$x_1 = 1,\ x_2 = 0$

$x_3 = 0 \quad x_1 + x_2 = -1$

$x_2 = x_3 = 1$

$x_1 = x_3 = 1$



the lines correspond to tritronquée solutions (Joshi, Kitaev, van Spaendonck, Vonk)

# PI Y-Segre

$$y_4 - y_1 y_2 = 0$$

$$y_4 y_3 - y_1 - y_2 - 1 = 0$$

$L_1: y_1 = 0,\ y_4 = 0,\ y_2 = 1$

$L_2: y_1 = 0,\ y_2 = 0,\ y_4 = 0$

$L_4: y_2 = y_3 = 1$

$L_5: y_1 = y_3 = 1$

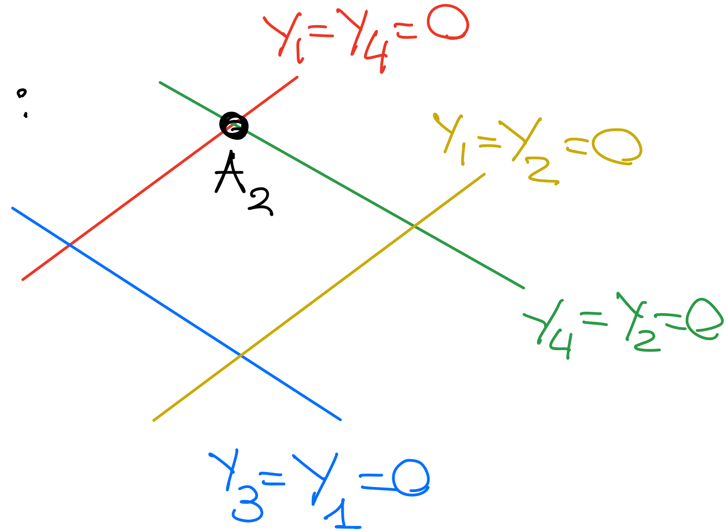
$L_3 \mapsto y_4 = y_2(1 - y_2),\ y_3 = 0$ conic

Projective completion

$$Y_0 Y_4 - Y_1 Y_2 = 0$$

$$Y_4 Y_3 - Y_0 Y_1 - Y_0 Y_2 - Y_0^2 = 0$$

For $Y_0 = 0$:




Z-Segre:

$z_1 + z_3 + z_4 + z_5 + z_6 = 0$

$z_4 = 1$

$z_3 z_4 - z_1 z_2 = 0$

$z_3 z_4 - z_5 z_6 = 0$

$y_1 = z_5 + 1,\ y_2 = z_6 + 1,\ y_3 = z_2 + 1,\ y_4 = -z_1$

and viceversa

$z_1 = -y_4,\ z_2 = y_3 - 1,\ z_3 = 1 - y_1 - y_2 + y_4$

$z_4 = 1,\ z_5 = y_1 - 1,\ z_6 = y_2 - 1$

| | |
|---|---|
| $\mathrm{P_V}$ | $z_1+z_2+z_3+z_4+z_5+z_6=0,$ <br> $z_4+\rho_5 z_5-1=0,$ <br> $z_3z_4-\lambda_1 z_1z_2=0, \quad z_5z_6-\lambda_2 z_1z_2=0.$ |
| $\mathrm{P_V^{deg}}$ | $z_1+z_3+z_4+z_5+z_6=0,$ <br> $\rho_3 z_3+z_4+\rho_5 z_5+\frac{\rho_3}{\rho_5}z_6-1=0,$ <br> $z_3z_4-z_1z_2=0, \quad z_5z_6-z_1z_2=0.$ |
| $\mathrm{P_{IV}}$ | $z_1+z_2+z_3+z_4+z_5+z_6=0,$ <br> $z_4-1=0,$ <br> $z_3z_4-\lambda_1 z_1z_2=0, \quad z_5z_6-\lambda_2 z_1z_2=0.$ |
| $\mathrm{P_{III}^{\mathit{D}_6}}$ | $z_1+z_2+z_3+z_4+z_5=0,$ <br> $z_4+\rho_5 z_5-1=0,$ <br> $z_3z_4-\lambda_1 z_1z_2=0, \quad z_5z_6-z_1z_2=0.$ |
| $\mathrm{P_{III}^{\mathit{D}_7}}$ | $z_1+z_2+z_3+z_4+z_5=0,$ <br> $z_4+\rho_5 z_5-1=0,$ <br> $z_3z_4-z_1z_2=0, \quad z_5z_6-z_1z_2=0.$ |
| $\mathrm{P_{II}^{JM}}, \mathrm{P_{II}^{FN}}$ | $z_1+z_2+z_3+z_4+z_5+z_6=0,$ <br> $z_4-1=0,$ <br> $z_3z_4-z_1z_2=0, \quad z_5z_6-\lambda_2 z_1z_2=0.$ |
| $\mathrm{P_I}$ | $z_3+z_4+z_5+z_6=0,$ <br> $z_4-1=0,$ <br> $z_3z_4-z_1z_2=0, \quad z_3z_4-z_5z_6=0.$ |

# Poisson aspects

Nambu-Takhtajan Poisson bracket:

Let $h_1, \ldots, h_{n-2} \in \mathbb{C}[t_1, \ldots, t_n]$, then

$$\{f, g\} = \frac{df \wedge dg \wedge dh_1 \wedge \cdots \wedge dh_{n-2}}{dt_1 \wedge dt_2 \wedge \cdots \wedge dt_n}$$

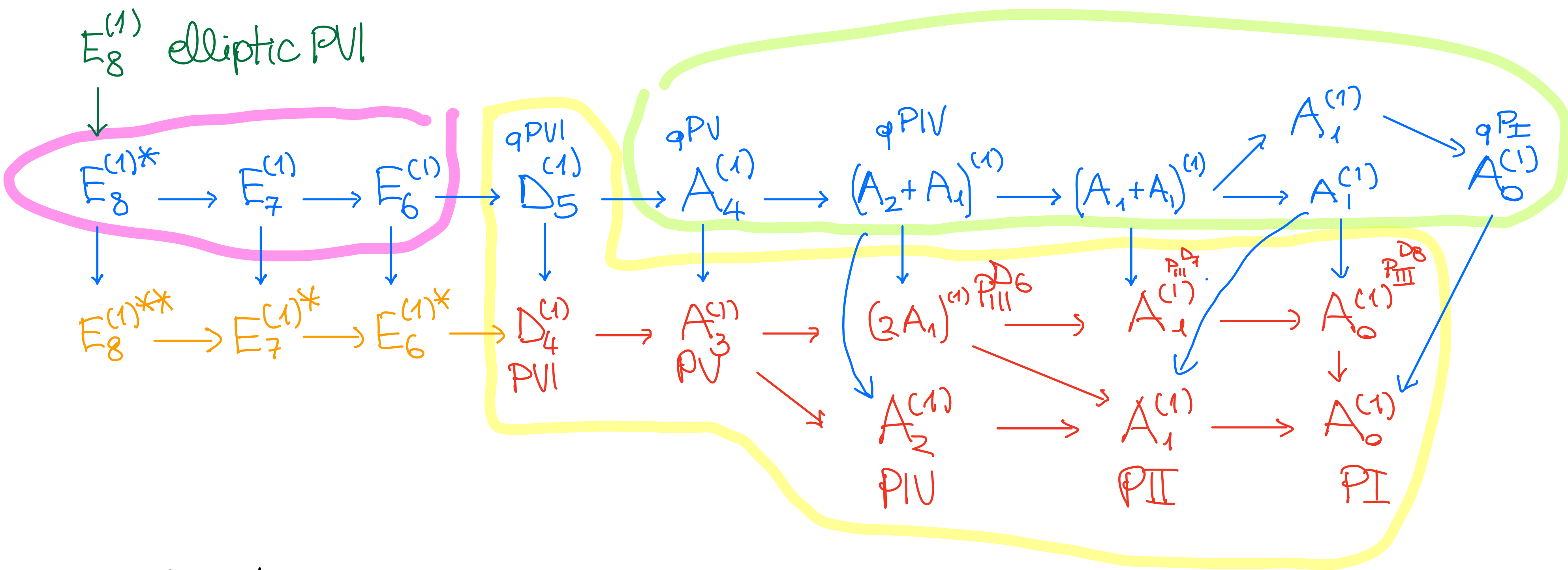
is a Poisson bracket on the algebraic variety given by the zero locus of $h_1, \ldots, h_{n-2}$.

For each Painlevé differential equation:

- cubic in $\mathbb{C}[x_1, x_2, x_3]$
- Its blow down: Segre in $\mathbb{C}[y_1, y_2, y_3, y_4]$
- Isomorphic $\mathbb{Z}$-Segre in $\mathbb{C}[z_1, z_2, \ldots, z_6]$

All these maps are Poisson.

# Outlook




- unified
- Expect $Z_q$ with different parameter specializations
- Expect higher order del Pezzo

Question: mirror construction for Segre?

Happy birthday!