Signatures and Rough Paths

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Aspects of Higher Connections and Parallel Transport in Geometry

(a high-level introduction)

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Outline

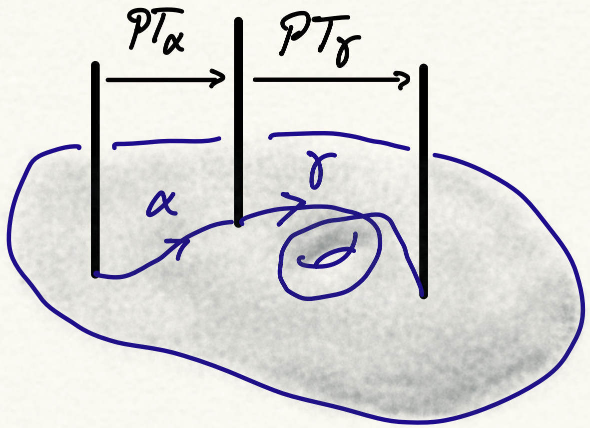
- Parallel transport holonomy and curvature — Review
- Bundles with fibre Vec — Intuition building for higher-dim case
- Formalising (higher) parallel transports systematically — categories + functors, outlook / results

1 Parallel Transport

- Let G be a Lie group, $P \xrightarrow{G} M$ a principal G-bundle.

My level: piecewise smooth

↳ A parallel transport assigns to each path $\gamma : [0,1] \longrightarrow M$ a G-equivar. diffeomorphism $PT_\gamma : P|_{\gamma(0)} \xrightarrow{\cong} PT|_{\gamma(1)}$, such that



(1) (Units) $PT_{const(x)} = id_{P|x}$,

(2) (Composition) $PT_\gamma \circ PT_\alpha = PT_{\gamma * \alpha}$

(3) (Smoothness) PT_γ depends smoothly on γ

(4) (Thin homotopy invariance) If $h : [0,1] \times [0,1] \longrightarrow M$ has $rk(h_{*|(s,t)}) < 2$ $\forall s,t \in [0,1]$, then $PT_{h(-,0)} = PT_{h(-,1)}$.

Remarks

- Equivalent formulation via connections:

(Globally) $A \in \Omega^1(P; \mathfrak{g})^{G, Ad}$ ← G-action on P translates to adjoint action on $\mathfrak{g}$

↳ Defines horizontal subspaces

$$T^h P := \ker(A) \subset TP$$

↳ Allows us to lift each γ uniquely to a horizontal path $\hat{\gamma}$:

$$\begin{tikzcd}
\{0\} \arrow[r, "\hat{\gamma}(0)"] \arrow[d, hook] & P \arrow[d, "\pi"] \\
{[0,1]} \arrow[r, "\gamma"'] \arrow[ur, "\exists! \hat{\gamma}" near start, "\text{hor.}"' near start] & M
\end{tikzcd}$$

↳ $PT_\gamma(y) := \hat{\gamma}(1)$.

⇒ Converse direction: $A\left(\frac{d}{dt}\big|_0 \hat{\gamma}\right) = \frac{d}{dt}\big|_0 PT_\gamma(\hat{\gamma}(0))$

↳ Equivalence: From ODE theory (Francis' mini course)

Vector bundles

- Admit analogous notion of connection and parallel transport:
  ↳ for $E \longrightarrow M$ a vector bundle on M,

$$PT_\gamma : E|_{\gamma(0)} \longrightarrow E|_{\gamma(1)}$$

is an isomorphism of vector spaces.

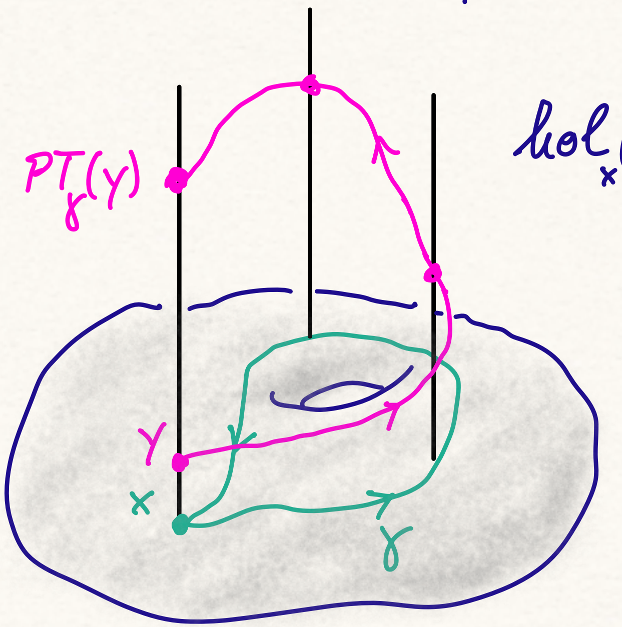
- Example: Every Riemannian metric on M determines a Levi-Civita connection on the tangent bundle TM.
  ↳ Force-free trajectories in General Relativity (geodesics).

Curvature and holonomy

- The curvature of a connection / PT is the obstruction for PT_γ to be independent of all homotopies of paths.

  ↳ Geometrically measured by a 2-form

- The holonomy of a PT on $P \longrightarrow M$ (or vbdl $E \longrightarrow M$) at a point $x \in M$ is the map



$\cong G$, but non-canonically

$$hol_x(P,-) = PT_{|\Omega_x^{\le 1}M} : \Omega_x^{\le 1}M \longrightarrow Aut(P_{|x})$$

$(\text{resp } Aut(E_{|x}))$

smooth group of loops in M based at x, modulo thin homotopy.

Examples

- The electromagnetic potential is a connection on a $U(1)$-bundle.
  - ↳ Its holonomies are numbers in $U(1)$.
    - ↳ Famously measured in the Aharonov-Bohm effect as a change to an electrons QM wave function.
- The path signature can be viewed as a PT on a bdl on $\mathbb{R}^n$ with structure group built from the tensor algebra $T^{\otimes}\mathbb{R}^n$.
  - ↳ Distinguishes (rough) paths up to thin homotopy [Lyons, McLeod].

(Nicolas' mini course)

Towards higher-dimensional parallel transport

2 Bundles with fibre Vec

- Let Vec denote the collection of all fin-dim complex vector spaces and their $\mathbb{C}$-linear maps.

Numbers	Vector spaces
$\mathbb{C}$	Vec
$(+, 0)$	$(\oplus, 0)$
$(\cdot, 1)$	$(\otimes, \mathbb{C})$
invertibles $\mathbb{C}^\times$	$\mathrm{Vec}^\times = $ 1-d vsp. L
no maps between numbers (a set)	additional layer of structure: maps in Vec! (a category — more later)

(what are "negatives"?)

$\left(L^{-1} \longleftrightarrow L^* : \; L \otimes L^* \underset{\text{can}}{\cong} \mathbb{C} \cong L^* \otimes L \right)$

↳ Understand $(\mathrm{Vec}, \oplus, 0, \otimes \mathbb{C})$ as an analogue of $\mathbb{C}$ — a 1d vector space.

- Consider a trivial bundle-like object $\mathcal{G}$ whose fibre at each $x \in M$ is $(\mathrm{Vec}, \oplus, 0)$ (like the vector space $\mathbb{C}$!)

↳ What would a parallel transport on $\mathcal{G}$ look like ?

$$
\begin{tikzcd}
PT_\gamma : \mathcal{G}|_{\gamma(0)} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{G}|_{\gamma(1)} \arrow[d, phantom, "\cong" rotate=90] \\
(\mathrm{Vec}, \oplus, 0) \arrow[r, "L_\gamma \otimes (-)"'] & (\mathrm{Vec}, \oplus, 0) \\
V \arrow[r, mapsto] & L_\gamma \otimes V
\end{tikzcd}
$$

PT is a "linear isomorphism"

tensor by a vsp. L_γ

L_γ needs to be $\otimes$-invertible, i.e. $\dim(L_\gamma) = 1$.

$\Rightarrow$ PT_γ assigns a 1d vsp to each path γ.

Properties vs data

- In Vec, isomorphisms allow us to identify vector spaces.

Incomplete list, to illustrate concepts

↳ In $\mathbb{C}$, $z_1 = z_2$ is a property, in Vec $V_1 \xrightarrow[\varphi]{\cong} V_2$ is data!

(1) (Composition) $PT_\alpha \circ PT_\gamma = L_\alpha \otimes L_\gamma \xrightarrow[\cong]{c_{\alpha,\gamma}} L_{\alpha * \gamma} = PT_{\alpha * \gamma}$

new data

coherence (for consistency — attempt):

$$
\begin{tikzcd}
L_\alpha \otimes (L_\beta \otimes L_\gamma) \arrow[r, "id \otimes c_{\beta,\gamma}"] \arrow[d, "can"', "\cong"] & L_\alpha \otimes L_{\beta * \gamma} \arrow[r, "c_{\alpha, \beta * \gamma}"] & L_{\alpha * (\beta * \gamma)} \arrow[d, dashed, "?"] \\
(L_\alpha \otimes L_\beta) \otimes L_\gamma \arrow[r, "c_{\alpha,\beta} \otimes id"'] & L_{\alpha * \beta} \otimes L_\gamma \arrow[r, "c_{\alpha * \beta, \gamma}"'] & L_{(\alpha * \beta) * \gamma}
\end{tikzcd}
$$

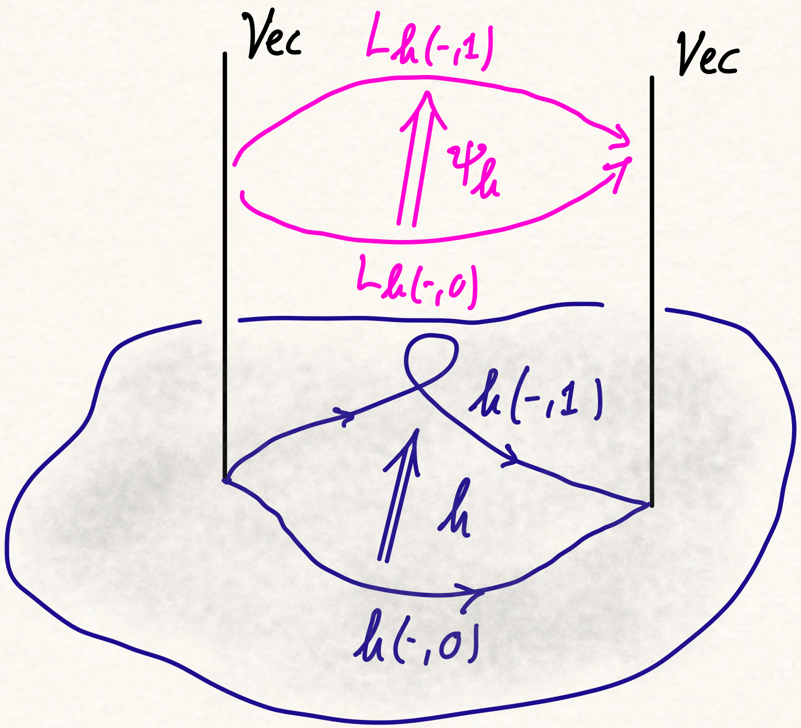
need a specified linear iso. here!

(2) (Homotopies of paths) PT_α, PT_β are vector spaces

$\Rightarrow$ They admit maps between them!

To each homotopy with fixed endpoints $h : [0,1]^2 \longrightarrow M$, assign a vsp. isomorphism $PT_h : PT_{h(-,0)} \xrightarrow{\cong} PT_{h(-,1)}$

new data, associated to 2d shapes in M!



Properties:

- For every thinly homotopic maps h, h',

$$PT_h = PT_{h'}.$$

- For every thin homotopies h, h' between the same paths, $PT_h = PT_{h'}$.

↳ In prt. $\exists$ canonical iso.

$$PT_{\alpha*(\beta*\gamma)} \longrightarrow PT_{(\alpha*\beta)*\gamma}$$

Coherence

$$\begin{tikzcd}
L_\alpha \otimes (L_\beta \otimes L_\gamma) \arrow[r, "id \otimes c_{\beta,\gamma}"] \arrow[d, "\mathrm{can}"', "\cong"] & L_\alpha \otimes L_{\beta * \gamma} \arrow[r, "c_{\alpha, \beta*\gamma}"] \arrow[d, phantom, "/\!/\!/\ (\text{commutes})"] & L_{\alpha * (\beta * \gamma)} \arrow[d] \\
(L_\alpha \otimes L_\beta) \otimes L_\gamma \arrow[r, "c_{\alpha,\beta} \otimes id"'] & L_{\alpha * \beta} \otimes L_\gamma \arrow[r, "c_{\alpha*\beta,\gamma}"'] & L_{(\alpha * \beta) * \gamma}
\end{tikzcd}$$

✓

• **Rmk:** • More data: $PT_{\mathrm{const}(x)} \xrightarrow{\cong} id_{\mathrm{Vec}}$,

+ more coherence.

• Implement smoothness : here, $\gamma \longmapsto L_\gamma$ is smooth line bundle over the path space PM.

Formulating (higher) parallel transports systematically

- Appropriate mathematical tool: (higher) categories, homotopy theory

- **Def.:** A category $\mathcal{C}$ consists of sets C_0 (objects), C_1 (morphisms), and maps $C_0 \rightleftarrows C_1$ (source/target of each mp, identity mp on each object), $C_1 \underset{C_0}{\times} C_1 \xrightarrow{(-)\circ(-)} C_1$, $(\alpha, \beta) \longmapsto \alpha \circ \beta$ (composition of composable pairs of morphisms), s.t.

  $(-)\circ(-)$ is associative and unital.

- **E.g.:** (1) $\mathcal{V}ec = \big(\{\text{fin-dim vsp. } V\} \rightleftarrows \{\text{linear maps } \psi : V_0 \longrightarrow V_1\}\big)$

  (2) G a group $\rightsquigarrow$ $BG = \big(\{*\} \rightleftarrows G\big)$ comp. = mult. in G.

  (3) $\pi_{\leq 1}^{th} M = \big(\{\text{points } x \in M\} \rightleftarrows \{\text{thin-homotopy classes of paths in } M\}\big)$

  piecewise smooth/...

Maps between categories: functors

· **Def.:** Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a pair of maps $F_0 : \mathcal{C}_0 \longrightarrow \mathcal{D}_0$, $F_1 : \mathcal{C}_1 \longrightarrow \mathcal{D}_1$, s.t.

F preserves source, target and unit maps

(e.g. $F\left(c_0 \xrightarrow{\psi} c_1\right) = \left(F_0 c_0 \xrightarrow{F_1 \psi} F_0 c_1\right)$ in $\mathcal{D}$)

and respects composition of maps, $F(\psi \circ \varphi) = F\psi \circ F\varphi$ $\forall$ maps φ, ψ in $\mathcal{C}$.

· **E.g.:** (1) A vector bundle with parallel transport is a (smooth) functor

$$\pi_{\leq 1}^{th} M \longrightarrow Vec, \quad \begin{array}{c} x_1 \\ \uparrow \gamma \\ x_0 \end{array} \longmapsto \begin{array}{c} E|_{x_1} \\ \uparrow PT_\gamma \\ E|_{x_0} \end{array}$$

(2) A parallel transport on the trivial G-bundle is a smooth fctr.

$$\pi_{\leq 1}^{th} M \longrightarrow BG = \left(\begin{array}{c} G \\ \downarrow\uparrow\downarrow \\ * \end{array}\right), \quad \begin{array}{c} x_1 \\ \uparrow \gamma \\ x_0 \end{array} \longmapsto \begin{array}{c} * \\ \uparrow PT_\gamma \in G \\ * \end{array}.$$

• 2-categories, 2-functors, 2-dimensional parallel transport

• <u>Def. (sketch):</u> A 2-category $\mathcal{C}$ consists of sets $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ of objects, 1-maps and 2-maps. s.t.

for each $c_0, c_1 \in \mathcal{C}$, there is a category of maps $\mathcal{C}(c_0, c_1)$

$$
\begin{tikzcd}[column sep=large]
c_0 \arrow[r, bend left=60, "f_2"] \arrow[r, "f_1" description] \arrow[r, bend right=60, "f_0"'] \arrow[r, phantom, shift left=3, "\Uparrow \varphi_{12}"] \arrow[r, phantom, shift right=3, "\Uparrow \varphi_{01}"] & c_1
\end{tikzcd}
\xmapsto{\text{vert. comp}}
\begin{tikzcd}[column sep=large]
c_0 \arrow[r, bend left=50, "f_2"] \arrow[r, bend right=50, "f_0"'] \arrow[r, phantom, "\Uparrow \varphi_{12} \cdot \varphi_{01}"] & c_1
\end{tikzcd}
$$

and composition functors $\mathcal{C}(c_1, c_2) \times \mathcal{C}(c_0, c_1) \xrightarrow{\text{hor comp.}} \mathcal{C}(c_0, c_2)$, satisfying the interchange law

$$
\begin{tikzcd}[column sep=large]
c_0 \arrow[r, bend left=60, "f_2"] \arrow[r, "f_1" description] \arrow[r, bend right=60, "f_0"'] \arrow[r, phantom, shift left=3, "\Uparrow \varphi_{12}"] \arrow[r, phantom, shift right=3, "\Uparrow \varphi_{01}"] & c_1 \arrow[r, bend left=60, "g_2"] \arrow[r, "g_1" description] \arrow[r, bend right=60, "g_0"'] \arrow[r, phantom, shift left=3, "\Uparrow \psi_{12}"] \arrow[r, phantom, shift right=3, "\Uparrow \psi_{01}"] & c_2
\end{tikzcd}
\longmapsto
\begin{tikzcd}[column sep=huge]
c_0 \arrow[r, bend left=50, "g_2 \circ f_2"] \arrow[r, bend right=50, "g_0 \circ f_0"'] \arrow[r, phantom, "\Uparrow"] & c_1
\end{tikzcd}
$$

$$(\psi_{12} \circ_h \varphi_{12}) \circ_v (\psi_{01} \circ_h \varphi_{01})$$
$$= (\psi_{12} \circ_v \psi_{01}) \circ_h (\varphi_{12} \circ_v \varphi_{01})$$

$\longleftarrow$ coherence / consistency condition

• Remarks :

- Composition in a 2-cat is, in gen. only assoc. up to coherent isomorphism.
- There is a notion of 2-fctrs between 2-cats, $F : \mathcal{C} \longrightarrow \mathcal{D}$.

• Examples:

- $\pi_{\leq 2}^{th}(M)$:

$$\left\{ \begin{tikzcd} x_0 \arrow[rr, bend left, "\gamma_1"] \arrow[rr, bend right, "\gamma_2"'] \arrow[rr, phantom, "\Uparrow [h]"] & & x_1 \end{tikzcd} \;\middle|\; \begin{array}{l} x_0, x_1 \in M \quad \text{(objects)} \\ \gamma_0, \gamma_1 : [0,1] \longrightarrow M \quad \text{(1-mps)} \\ [h] \text{ thin ho. class of homotopies } \gamma_0 \rightsquigarrow \gamma_1 \\ \qquad\qquad\qquad\qquad \text{(2-mp)} \end{array} \right\}$$

- $BVec^{\times}$: obj $= \{*\}$, $BVec^{\times}(*,*) = Vec^{\times, iso}$, ← take only isomps in $Vec^{\times}$.

  composition functor : $(-) \otimes (-) : Vec^{\times, iso} \times Vec^{\times, iso} \longrightarrow Vec^{\times, iso}$.

- 2d- parallel transport on a trivial "2-vector bundle" :

  (Smooth) 2-fctr $PT : \pi_{\leq 2}^{th}(M) \longrightarrow \underbrace{BVec^{\times}}_{\text{categorified group : "2-group"}}$.

2-groups

- A monoidal structure on a category $\mathcal{C}$ consists of
  - a functor $(-)\otimes(-) : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$
  - an associator $\alpha_{a,b,c} : a\otimes(b\otimes c) \xrightarrow{\cong} (a\otimes b)\otimes c$ + coherence
  - a unit object $\mathbb{1} \in \mathcal{C}$
  - unitors $\lambda_a : \mathbb{1}\otimes a \xrightarrow{\cong} a$, $\rho_a : a\otimes \mathbb{1} \xrightarrow{\cong} a$ + coherence

- Def.: A 2-group is a category with monoidal structure $(\mathcal{C},\otimes)$ s.t.

  (1) every object has a (weak) inverse,

  $$\mathbb{1} \xrightarrow{\exists \cong} a\otimes a^{-1} \quad , \quad a^{-1}\otimes a \xrightarrow{\exists \cong} \mathbb{1}$$

  (2) every morphism has a (strict) inverse : $a \xrightarrow{\psi} b$ has

  $$\psi^{-1}\circ\psi = id_a \quad , \quad \psi\circ\psi^{-1} = id_b \, .$$

• **Example:** • $(\mathrm{Vec}^{\times, iso}, \otimes)$ is a 2-group.

• For G a Lie group, $BG = (* \leftleftarrows G)$ is a 2-group

↳ Rmk: $B\mathbb{C}^\times \longrightarrow (\mathrm{Vec}^{\times, iso}, \otimes)$ is an equivalence.

• Any crossed module and Lie 2-group is a 2-group.

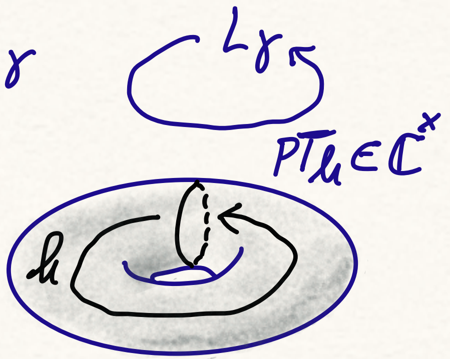
• One can define principal 2-bundles on manifolds

↳ structure group is a 2-group.

• For $(\mathrm{Vec}^{\times, iso}, \otimes) \simeq B\mathbb{C}^\times$, these are called (bundle) gerbes.

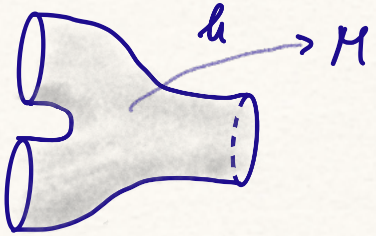
↳ Their PT is understood : $(\gamma : S^1 \longrightarrow M) \longmapsto L_\gamma$

↳ holonomy gives line bundle on $C^\infty(S^1, M)$.



• Final remarks and some results:

- 2d PTs have been constructed on various types of categorified bundles [Baez, Schreiber; Faria Martins, Picken; Schreiber, Waldorf; SB, Müller, Szabo; Bischoff, Lee; ...]
- There exist higher-degree generalisations of categories and 2-cats, up to ∞-cats [Joyal; Lurie; ...] (vast field, vast impact in maths!)
- PT and connections in generic dimension + cat. level are not yet well-understood, but a history of important developments exists [Fiorenza, Sati, Stasheff, Schreiber; Arias Abad, Schätz; Kapranov; SB, Müller, Nuiten, Szabo (to appear); ...]
- If the target is sufficiently abelian, one can also transport around more general shapes: (functorial) field theories



$$\longmapsto \left(\begin{matrix} L_{\gamma_{in}} \\ \otimes \\ L_{\gamma'_{in}} \end{matrix} \longrightarrow L_{\gamma_{out}} \right)$$

[SB, Waldorf]

Thank you for your attention!