Notation: $X : [0,T] \to V = \mathbb{R}^d$

basis $\{e_1, \dots, e_d\}$.

$$X_t = (X_t^1, \dots, X_t^d).$$

Def: Given a multi-index

$i_1 \dots i_n \in \{1, \dots, d\}^n =: W_d^n$

$$S(X)_{s,t}^{i_1 \dots i_n} := \int \dots \int_{s<u_1<\dots<u_n<t} \dot{X}_{u_1}^{i_1} \dots \dot{X}_{u_n}^{i_n} du_1 \dots du_n$$

Chen (1954)

Lyons (1998)

Signature of

$$S(X)_{s,t} = (1, S(X)_{s,t}^1, \dots, S(X)_{s,t}^d,$$
$$S(X)_{s,t}^{1,1}, \dots, S(X)_{s,t}^{d,d},$$
$$\dots)$$

Dynamical

$$dY_t = f(Y_t)\, dX_t, \quad Y_0 = \xi$$

Picard Iteration:

$$Y_t^0 = \xi$$

$$Y_t^1 = \xi + \int_0^t f(Y_s^0)\, dX_s =$$

Picard Iteration:

$$Y_t^0 = \xi$$

$$Y_t^1 = \xi + \int_0^t f(Y_s^0)\,dX_s = \xi + f(\xi)(X_t - X_0)$$

$$Y_t^2 = \xi + \int_0^t f(Y_s^1)\,dX_s \approx \xi + f(\xi)(X_t - X_0) + Df(\xi)f(\xi) \times \int_0^t (X_s - X_0)\,dX_s$$

$$\vdots$$

$$Y_t^n \approx \xi + \sum_{k=1}^{n} \sum_{i_1,\dots,i_k=1}^{d} F_{i_1} \circ \dots \circ F_{i_k}(\xi)\, S(X)_{0,t}^{i_1 \dots i_k}$$

In word notation, we may split into levels: for $n \geq 1$

$$S(X)^{(n)}_{s,t} := \left(S(X)^{w}_{s,t} : w \in \mathcal{W}_d^n \right)$$

Ex:

$$S(X)^{(1)}_{s,t} = \left(X^1_t - X^1_s, \dots, X^d_t - X^d_s \right)$$

$$S(X)^{(2)}_{s,t} = \begin{pmatrix} S(X)^{1,1}_{s,t} & \cdots & S(X)^{1d}_{s,t} \\ & \ddots & \\ & & S(X)^{d,d}_{s,t} \end{pmatrix}$$

$\vdots$

$$S(X)^{(n)}_{s,t} \in V^{\otimes n}$$

Rmk: From the def

$$S(X)^{i_1 \dots i_n}_{s,t} = \int_s^t S(X)^{i_1 \dots i_{n-1}}_{s,u} \, dX^{i_n}_u$$

In tensor notation:

$$S(X)^{(n)}_{s,t} = \int_s^t S(X)^{(n-1)}_{s,u} \otimes dX_u$$

1. $S(X)_{s,t} \in T((V)) = \prod_{n \geq 0} V^{\otimes n}$

2. $S(X)_{s,\cdot}$ solves

$$dS_{s,t} = S_{s,t} \otimes dX_t,$$

$$S_{s,s} = \mathbf{1} = (1, 0, \dots)$$

Universal parallel transport.

Pick $\alpha \in \Omega^1(V, \mathfrak{g})$, $X^*\alpha \in \Omega^1([0,T], \mathfrak{g})$

Solve: $= \Gamma(V \otimes \mathfrak{g})$

$$(L_{S_t^{-1}})_* dS_t = X^*\alpha$$

The solution S^α lives G grp of $\mathfrak{g}$:

$$S^\alpha_t = \sum_{n \geq 0} \int \cdots \int \alpha_{i_1} \cdots \alpha_{i_n} \dot{X}^{i_1}_{u_1} \cdots \dot{X}^{i_n}_{u_n} du_1 \cdots du_n$$

There is a map $A : T((V)) \to \overline{\mathcal{U}(\mathfrak{g})}$ s.t

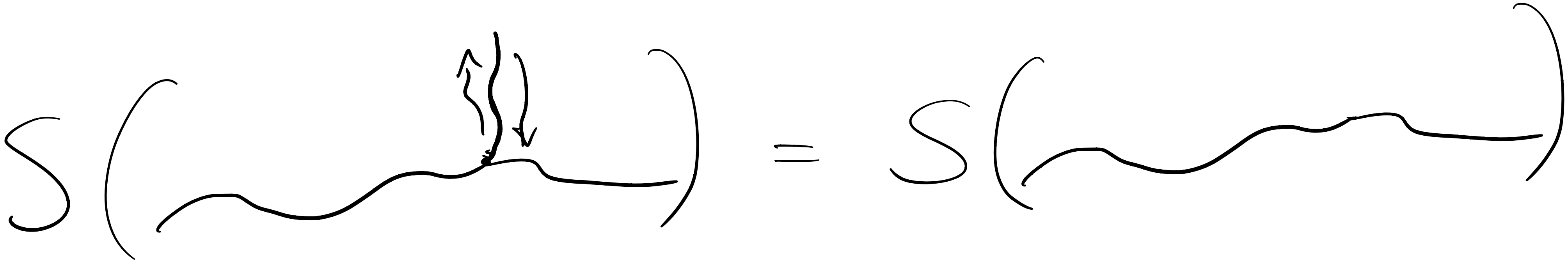
$$S^\alpha_t = A \cdot S(X)_t$$

Properties:

1. Inv. under translations: $S(X)_{s,t} = S(\overset{\text{cst. vector}}{x} + X)_{s,t}$

2. Inv. under reparametrisations: $\psi : [0,T] \to [0,T]$ and $(X^{\psi})_t := X_{\psi(t)}$. Then $S(X^{\psi})_{0,T} = S(X)_{0,T}$

3. Inv. under tree-like equiv.



4. Chen's identity: Generalization of "Charle's rule"

$$\int_a^b f = \int_a^c f + \int_c^b f \qquad c \in [a,b]$$

Thm (Chen 1954): $u \in [s,t]$

$$S(X)_{s,t}^{i_1 \dots i_n} = \sum_{k=0}^{n} S(X)_{s,u}^{i_1 \dots i_k} S(X)_{u,t}^{i_{k+1} \dots i_n}$$

Proof: $w = i_1 \dots i_n$

$$S(X)_{s,t}^{w} = \int_s^t S(X)_{s,u}^{i_1 \dots i_{n-1}} dX_u^{i_n}$$

Proof: $w = i_1 \cdots i_n$

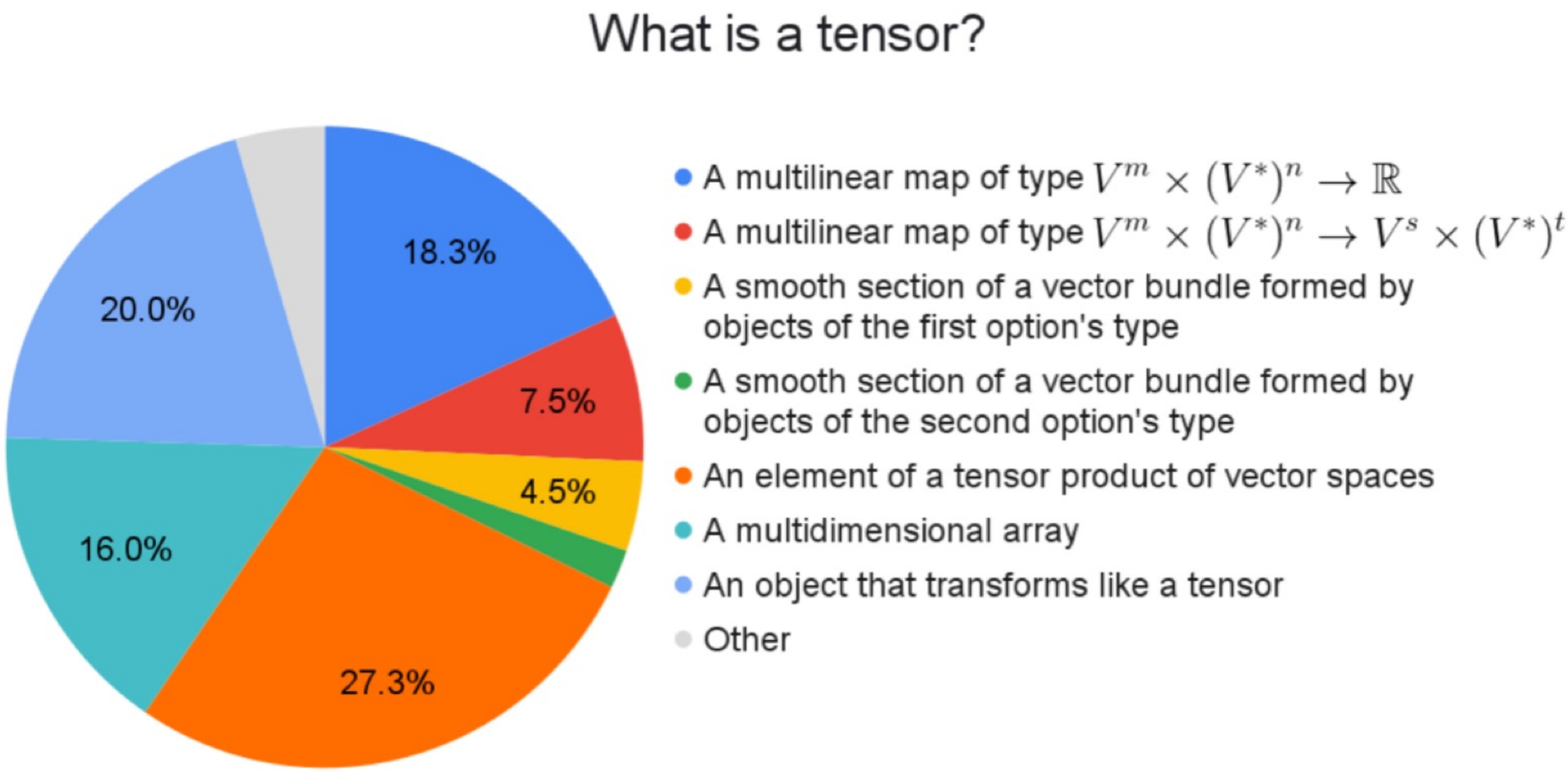
$$S(X)^{w}_{s,t} = \int_s^t S(X)^{i_1 \cdots i_{n-1}}_{s,r} \, dX^{i_n}_r$$

$$= \underbrace{\int_s^u S(X)^{i_1 \cdots i_{n-1}}_{s,r} \, dX^{i_n}_r}_{S(X)^{i_1 \cdots i_n}_{s,u}} + \int_u^t S(X)^{i_1 \cdots i_{n-1}}_{s,r} \, dX^{i_n}_r$$

$$= \cdots + \sum_{k=0}^{n-1} S(X)^{i_1 \cdots i_k}_{s,u} \int_u^t S(X)^{i_{k+1} \cdots i_{n-1}}_{u,r} \, dX^{i_n}_r$$

$$= S(X)^{i_1 \cdots i_n}_{s,u} \cdot 1 + \sum_{k=0}^{n-1} S(X)^{i_1 \cdots i_k}_{s,u} S(X)^{i_{k+1} \cdots i_n}_{u,t} \quad \blacksquare$$

Q45: What is a tensor?



"I DON'T KNOW!!!!" - Respondent #12

"dude I've been asking people this for ages and I've never gotten a straight answer" - Respondent #813

"no one knows, and anyone who says they do is bullshitting" - Respondent #864

"no-one in the history of maths has been able to give a concise answer to this question and we should just accept that (it's an array of vectors and covectors which remains the same under a change of basis even if its components themselves transform. It behaves analogous to a generalisation of a matrix (where a (1,1) tensor is a matrix, but so also are some other forms of tensors, but not all). a tensor is a tensor if it has the vibes of a tensor and it is useful to consider it as such. any object can be written as a tensor, and all of mathematics is simply different elements within a tensor, including that tensor itself. a tensor is whenever you have ten of whatever a sor is.)" - Respondent #1319

This survey has confirmed what we all knew: nobody actually knows what a tensor is.

Other responses include:

- An element of a tensor product of modules (x7)
- A bifunctor with associativity and unitality up to isomorphism
- A bifunctor in a monoidal category

I don't even remotely understand what those last two responses mean, I just copied and pasted them.

5. "Shuffle" identity. (Ree 1958)

Def: A shuffle permutation of size (n,m) is a $\sigma \in S_{n+m}$ s.t.

$$\sigma^{-1}(1) < \dots < \sigma^{-1}(n) \quad \& \quad \sigma^{-1}(n+1) < \dots < \sigma^{-1}(n+m)$$

I write $Sh(n,m)$ for the set of these.

Ex: $Sh(1,1) = S_2$

$Sh(2,1) = \{(123), (132), (312)\}$

Thm: (Ree 1958)

$$S(X)^{i_1 \dots i_n}_{s,t} \, S(X)^{i_{n+1} \dots i_{n+m}}_{s,t} = \sum_{\sigma \in Sh(n,m)} S(X)^{i_{\sigma(1)} \dots i_{\sigma(n+m)}}_{s,t}$$

Proof:

$$\int_s^t (X^i_u - X^i_s)\, dX^j_u + \int_s^t (X^j_u - X^j_s)\, dX^i_u = (X^i_t - X^i_s)(X^j_t - X^j_s) \checkmark$$

For general case it's recursive formula + integration by parts. $\square$

Algebraic structures: Shuffle algebra.

Take the spaces $T^{(n)}(\mathbb{R}^d) := \operatorname{span}_{\mathbb{R}} W_d^n \cong V^{\otimes n}$

and

$$T(\mathbb{R}^d) := \bigoplus_{n \geq 0} T^{(n)}(\mathbb{R}^d)$$

on it, define the shuffle product

$$i_1 \dots i_n \sqcup\!\sqcup\, i_{n+1} \dots i_{n+m} = \sum_{\sigma \in Sh(n,m)} i_{\sigma(1)} \dots i_{\sigma(n+m)} .$$

and the "deconcatenation" coproduct

$$\Delta\, i_1 \dots i_n = \sum_{k=0}^{n} i_1 \dots i_k \otimes i_{k+1} \dots i_n$$

We write

$$\langle S(X)_{s,t}, i_1 \cdots i_n \rangle := S(X)_{s,t}^{i_1 \cdots i_n}$$

i.e. $S(X)_{s,t} \in T(\mathbb{R}^d)^* = \prod_{n \geq 0} ((\mathbb{R}^d)^*)^{\otimes n}$

and

1. $\langle S(X)_{s,t}, i_1 \cdots i_n \rangle \langle S(X)_{s,t}, i_{n+1} \cdots i_{n+m} \rangle$

$$= \langle S(X)_{s,t}, i_1 \cdots i_n ⧢ i_{n+1} \cdots i_{n+m} \rangle$$

2. $\langle S(X)_{s,t}, i_1 \cdots i_n \rangle = \langle S(X)_{s,u} \otimes S(X)_{u,t}, \Delta(i_1 \cdots i_n) \rangle$

3. $$\left|\langle S(X)_{s,t}, i_1 \cdots i_n \rangle\right| \leq \frac{\|X\|^n_{1\text{-var}} |t-s|^n}{n!}$$

II. Hopf algebras.

Def: A Hopf algebra is 5-tuple

$(H, m, \Delta, \mathbb{1}, \varepsilon)$,

1. Coassociativity: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$
2. Counit: $(\varepsilon \otimes id)\Delta = (id \otimes \varepsilon)\Delta = id$
3. Bialgebra:

$$\Delta \circ m = (m \otimes m) \circ (id \otimes \tau \otimes id) \circ (\Delta \otimes \Delta)$$

$$
\begin{tikzcd}
H \otimes H \arrow[r, "m"] \arrow[d, "\Delta \otimes \Delta"'] & H \arrow[dd, "\Delta"] \\
H \otimes H \otimes H \otimes H \arrow[d, "(id \otimes \tau \otimes id)"'] & \\
H \otimes H \otimes H \otimes H \arrow[r, "m \otimes m"'] & H \otimes H
\end{tikzcd}
$$

This is a bialgebra.

4. Antipode: $\mathcal{A} : H \to H$ s.t. $\mathcal{A}1 = 1$

$$m \circ (id \otimes \mathcal{A}) \circ \Delta x = m \circ (\mathcal{A} \otimes id) \circ \Delta x = \varepsilon(x) 1, \ \forall x \in H$$

Graded Hopf algebra: $H = \bigoplus_{n \geq 0} H^{(n)}$ and

$$m : H^{(n)} \otimes H^{(m)} \longrightarrow H^{(n+m)}$$

$$\Delta : H^{(n)} \longrightarrow \bigoplus_{k=0}^{n} H^{(k)} \otimes H^{(n-k)}$$

Connected if $H^{(0)} \cong \mathbb{R}$.

Thm: Graded connected bialgebra is a Hopf algebra.

In the shuffle algebra

$$A(i_1 \cdots i_n) = (-1)^n i_n \cdots i_1$$

<u>Duals:</u> $H^* \cong \prod_{n\geq 0} (H^{(n)})^*$ is an algebra

via

$$f * g = (f \otimes g) \circ \Delta$$

<u>Chen:</u> $S(X)_{s,t} = S(X)_{s,u} \otimes S(X)_{u,t}$

<u>Graded dual:</u> $H^\circ := \bigoplus_{n\geq 0} (H^{(n)})^*$

We have:

$$G := \{ f \in H^* : \langle f, xy \rangle = \langle f, x \rangle \langle f, y \rangle \} \ni S(x).$$

$$\mathfrak{g} := \{ f \in H^* : \langle f, xy \rangle = \varepsilon(x) \langle f, y \rangle + \varepsilon(y) \langle f, x \rangle \}$$

One can show that G is a group under $*$.
with inverse

$$f^{*-1} = f \circ A$$

There $\exp : \mathfrak{g} \to G$ and $\log : G \to \mathfrak{g}$

$$\exp(f) = \varepsilon + \sum_{n \geq 1} \frac{1}{n!} f^{*n}.$$

One can show: $\overleftarrow{X}_t = X_{T-t}$

$$S(X)^{-1}_{s,t} = S(\overleftarrow{X})_{s,t}$$

What about $\log S(X)_{s,t}$? Can be computed by solving an ODE (Magnus 1954). $S(x) = \exp_{\otimes}(\Omega(t))$

where

$$\dot{\Omega}(x) = \frac{ad_{\Omega}}{e^{ad_{\Omega}} - 1}(\dot{x})$$

or, via Eulerian idempotent: $e = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} ⧢ \mid \circ \tilde{\Delta}^{(n-1)}$

$$\langle \log S(x)_{s,t}, w \rangle = \langle S(x)_{s,t}, e(w) \rangle$$

III Generalizations

a. Ito integration: $B^i_{st} B^j_{st} = \int_s^t B^i_{su}\, dB^j_u + \int_s^t B^j_{su}\, dB^i_u + \sigma_{ij} t$

$\Rightarrow$ quasi-shuffle Hopf algebra: Words on a new alphabet

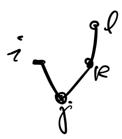
$\{1, \dots, d, [11], \dots, [1d], \dots [dd], \dots, [i_1 \cdots i_n] \cdots \}$.

$$
\begin{aligned}
a_1 \cdots a_n * a_{n+1} \cdots a_{n+m} = {} & (a_1 \cdots a_n \sqcup\!\sqcup\, a_{n+1} \cdots a_{n+m-1}) a_n \\
& + (a_1 \cdots a_{n-1} \sqcup\!\sqcup\, a_{n+1} \cdots a_{n+m}) a_n \\
& + (a_1 \cdots a_{n-1} \sqcup\!\sqcup\, a_{n+1} \cdots a_{n+m-1}) [a_n a_{n+m}]
\end{aligned}
$$

Ex

$$i \sqcup\!\sqcup\, j = ij + ji + [ij]$$

b. Branched Rough paths: Encoding by nonplanar rooted trees.


$$\longmapsto \int_s^t \left(X^i_{su} \int_s^u X^l_{\cdot}\, dX^k_{\cdot} \right) dX^j_u$$

IV.

Thm: (Universal approx.)

Take K compact set on path space. Functionals $f : K \to \mathbb{R}$ can be arbitrarily well approx. by linear combinations of signatures.

Model: (Guo, 2024).

$$y_{t+1} = S(y_{t-l:t}) \cdot \theta$$

Solve

$$\theta \in \arg\min \sum_t \left(y_{t+1} - \theta \cdot \left[S(y_{t-l:t}), x_t \right] \right)^2 + \|\theta\|_{L^1}$$