Parallel transport along paths and surfaces

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Signatures and Rough Paths

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Linear ODEs

We're interested in differential equations of the following form

$$\frac{dy}{dt} = f(t)y,$$

and more generally systems of the form

$$\begin{aligned}
\frac{dy_1}{dt} &= f_{11}(t)y_1 + f_{12}(t)y_2 + ... + f_{1n}(t)y_n, \\
\frac{dy_2}{dt} &= f_{21}(t)y_1 + f_{22}(t)y_2 + ... + f_{2n}(t)y_n, \\
&... \\
\frac{dy_n}{dt} &= f_{n1}(t)y_1 + f_{n2}(t)y_2 + ... + f_{nn}(t)y_n.
\end{aligned}$$

We can express these systems compactly as follows

$$\frac{d\mathbf{y}}{dt} = A(t)\mathbf{y},$$

where $\mathbf{y} \in \mathbb{R}^n$ is a column vector consisting of n unknown functions, and $A(t) \in \mathrm{Mat}(n)$ is an $n \times n$ matrix of functions.

A **solution** is a function $\mathbf{y}(t)$ satisfying the above equation.

Solutions: Existence and Uniqueness.

Theorem: Given the **choice** of an initial condition $\mathbf{c} \in \mathbb{R}^n$, and an initial time $a \in \mathbb{R}$, there is a unique solution $\mathbf{y}$ of the equation

$$\frac{d\mathbf{y}}{dt} = A(t)\mathbf{y},$$

subject to the **initial condition** that $\mathbf{y}(a) = \mathbf{c}$.

Definition: Define the **parallel transport** to be the function

$$P : \mathbb{R} \times \mathbb{R} \to \mathrm{Mat}(n), \qquad (t, a) \mapsto P(t, a)$$

by letting $y(t) = P(t, a)(\mathbf{c})$ be the unique solution satisfying $y(a) = \mathbf{c}$.

Remark: If $A(t)$ is smooth, then P is smooth.

How to solve: iterated integrals

Explicitly, the parallel transport of

$$\frac{d\mathbf{y}}{dt} = A(t)\mathbf{y},$$

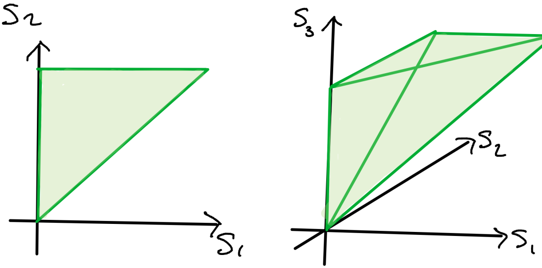
is given by an iterated integral

$$P(t,a) = id + \sum_{n\geq 1} \int_{a\leq s_1 \leq \ldots \leq s_n \leq t} A(s_n)A(s_{n-1})\ldots A(s_1)ds_1\ldots ds_n.$$

The n-symplex

The n-symplex is the set of points

$$\Delta^n = \{(s_1, s_2, ..., s_n) \in \mathbb{R}^n \mid 0 \leq s_1 \leq s_2 \leq ... \leq s_n \leq 1\}.$$



Parallel transport: algebraic properties

Using the existence and uniqueness theorem, we can deduce the basic properties of the parallel transport $P(x, y)$ of the differential equation

$$\frac{d\mathbf{y}}{dx} = A(x)\mathbf{y}.$$

1. $P(x, x) = id$, since $P(x, a)(\mathbf{c})$ solves the DE with initial condition
$$P(a, a)(\mathbf{c}) = \mathbf{c}.$$
2. $P(x, y) \circ P(y, z) = P(x, z)$ for $x, y, z \in \mathbb{R}$.
3. $P(x, y) \in \mathrm{GL}(n)$ is invertible since
$$P(x, y) \circ P(y, x) = P(x, x) = id.$$

Parallel transport

Summarizing, we conclude that the parallel transport

$$P : \mathbb{R} \times \mathbb{R} \to \mathrm{GL}(n)$$

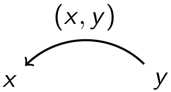
is a homomorphism.

But $\mathbb{R} \times \mathbb{R}$ is **not** a group. It is a **groupoid**.

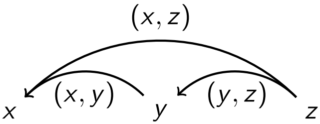
Lie groupoids

Given the space $\mathbb{R}$, the pair groupoid $\mathrm{Pair}(\mathbb{R}) \rightrightarrows \mathbb{R}$ is an example of a Lie groupoid: "a Lie group with many identities".

- It has a space $\mathbb{R}$ of 'points', and a space $\mathbb{R} \times \mathbb{R}$ of 'arrows',
- The arrows connect points:

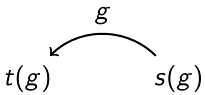


  Therefore, the arrows (x, x) are 'identities'
- Arrows with matching endpoints compose: $(x, y) * (y, z) = (x, z)$

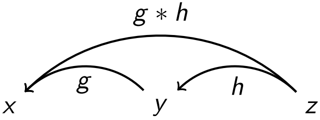


Lie groupoids

- A **Lie groupoid** $\mathcal{G} \rightrightarrows M$ has a manifold M of ‘objects/points’ and a manifold $\mathcal{G}$ of ‘arrows’.
- It is equipped with source and target maps $s, t : \mathcal{G} \to M$:

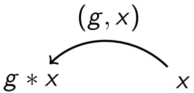


- There is a partially defined multiplication:

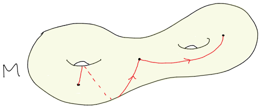


Lie groupoids: examples

- A Lie group G is a Lie groupoid with set of objects $M = \{*\}$ a point.
- A manifold M is a Lie groupoid with only identity arrows $\mathcal{G} = M$.
- Given a group action $G \circlearrowleft M$, there is an action groupoid $G \ltimes M \rightrightarrows M$ with objects M and arrows $G \times M$



- Given a manifold M, the **fundamental groupoid** $\Pi(M)$ consists of homotopy classes of paths in M.



Solving ODEs and Lie's theorem

So far, we have shown the following

Theorem: There is a one-to-one correspondence between linear ODEs and representations of the pair groupoid of $\mathbb{R}$:

$$\left\{ \begin{array}{c} \text{Linear ODEs} \\ \frac{d\mathbf{y}}{dt} = A(t)\mathbf{y} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Representations of the pair groupoid} \\ P : \mathrm{Pair}(\mathbb{R}) \to \mathrm{GL}(n) \end{array} \right\}$$

This is a special case of Lie's second theorem for Lie groupoids (Mackenzie, Xu and Moerdijk, Mrčun).

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Connections

We want to generalize the previous story in many directions. First, we increase the dimension of the space.

Definition: A **connection** on $\mathbb{R}^k$ is a differential operator

$$\nabla = d - \alpha : C^\infty(\mathbb{R}^k, \mathbb{R}^n) \to \Omega^1(\mathbb{R}^k) \otimes \mathbb{R}^n$$

where α is a matrix valued differential 1-form

$$\alpha = \sum_{i=1}^{k} \alpha_i dx_i \in \Omega^1(\mathbb{R}^k) \otimes \mathrm{Mat}(n).$$

This corresponds to the system of partial differential equations

$$\frac{\partial \mathbf{y}}{\partial x_i} = \alpha_i(x)\mathbf{y}.$$

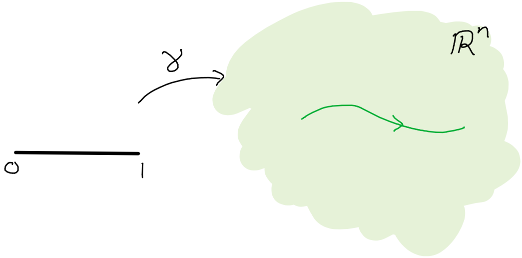
Parallel transport

The system of equations

$$\frac{\partial \mathbf{y}}{\partial x_i} = \alpha_i(x)\mathbf{y}.$$

may not admit any solutions $\mathbf{y} : \mathbb{R}^k \to \mathbb{R}^n$. But, we can define parallel transport along a path

$$\gamma : [0,1] \to \mathbb{R}^k.$$



Parallel transport

The system of equations

$$\frac{\partial \mathbf{y}}{\partial x_i} = \alpha_i(x)\mathbf{y}.$$

may not admit any solutions $\mathbf{y} : \mathbb{R}^k \to \mathbb{R}^n$. But, we can define parallel transport along a path

$$\gamma : [0,1] \to \mathbb{R}^k.$$

- Pullback the connection ∇ along the path to get a connection on the interval: $\gamma^*(\nabla) = d - \gamma^*(\alpha) = d - A(t)dt$, where

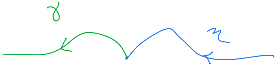
$$A(t) = \sum_{i=1}^{k} \alpha_i(\gamma(t)) \frac{d\gamma_i}{dt}.$$

- Solve the equation to get the parallel transport $P(\gamma) \in \mathrm{GL(n)}$ along the path.

Parallel transport: algebraic properties

The algebraic properties we deduced before still hold:

- Given a path γ, $P(\gamma) \in GL(n)$.
- Given two paths γ, η with matching endpoints



  we have $P(\gamma * \eta) = P(\gamma)P(\eta)$.
- Given a constant path c, $P(c) = id$.
- Given a path γ, let γ^{-1} be the path travelled in the opposite direction. Then $P(\gamma^{-1}) = P(\gamma)^{-1}$.

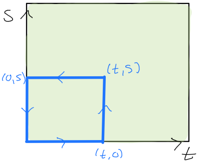
Parallel transport: dependence on paths

How does the parallel transport depend on the particular choice of path?

- Consider a connection ∇ on the square $[0,1]^2$, with

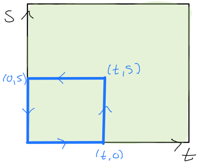
$$\alpha(t,s) = A(t,s)dt + B(t,s)ds.$$

- We consider the parallel transport along the following loop $\lambda_{(t,s)}$:



The parallel transport around the closed loop $\lambda_{(t,s)}$ has the form

$$P(\lambda_{(t,s)}) = id + (\partial_t B - \partial_s A - [A, B])|_{(0,0)} ts + \mathcal{O}((t,s)^3)$$



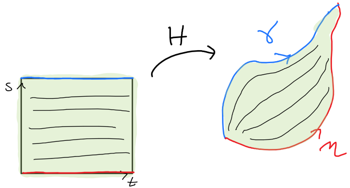
Therefore, the quantity $\partial_t B - \partial_s A - [A, B]$ measures the failure of the parallel transport being independent of the choice of path connecting two given endpoints.

Curvature

- The curvature of a connection $\nabla = d - \alpha$ is the matrix-valued 2-form
$$F(\alpha) = d\alpha - \alpha \wedge \alpha \in \Omega^2(\mathbb{R}^k) \otimes \mathrm{Mat}(n).$$
- For $\alpha(t,s) = A(t,s)dt + B(t,s)ds$
$$F(\alpha) = (\partial_t B - \partial_s A - [A,B])dt \wedge ds.$$
- A connection ∇ is **flat** if its curvature vanishes
$$F(\nabla) = 0.$$

Parallel transport

The parallel transport of a flat connection ∇ only depends on the endpoints of a path γ. In other words, it is invariant under homotopy equivalence:



Then

$$P(\gamma) = P(\eta).$$

As a result, the P depends only on the endpoints:

$$P(\gamma) = P(\gamma(1), \gamma(0))$$

Flat connections

If ∇ is flat, then the system of equations

$$\frac{\partial \mathbf{y}}{\partial x_i} = \alpha_i(x)\mathbf{y}.$$

does admit global solutions $\mathbf{y} : \mathbb{R}^k \to \mathbb{R}^n$.

Lie's second theorem

If ∇ is a flat connection, then its parallel transport defines a groupoid homomorphism

$$P : \mathrm{Pair}(\mathbb{R}^k) \to \mathrm{GL}(n).$$

Theorem: There is a one-to-one correspondence between flat connections on $\mathbb{R}^k$ and representations of the pair groupoid of $\mathbb{R}^k$:

$$\left\{ \begin{array}{c} \text{Flat connections } \nabla \\ F(\nabla) = 0 \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Representations of the pair groupoid} \\ P : \mathrm{Pair}(\mathbb{R}^k) \to \mathrm{GL}(n) \end{array} \right\}$$

What happens when ∇ is not flat?

We do not get a representation of $\mathrm{Pair}(\mathbb{R}^k)$ when our connection ∇ is not flat.

However, $P(\gamma)$ satisfies a number of properties:

- Invariant under reparametrizations $P(\gamma \circ \phi) = P(\gamma)$,
- Preserves inverses $P(\gamma^{-1}) = P(\gamma)^{-1}$
- Multiplicative $P(\gamma * \eta) = P(\gamma)P(\eta)$.
- $\implies$ Invariant under cancellation of retracings:

$$P(\gamma * \eta * \eta^{-1} * \tau) = P(\gamma * \tau).$$

Invariance under retracings

The parallel transport satisfies: $P(\gamma * \eta * \eta^{-1} * \tau) = P(\gamma * \tau)$.

These properties of the parallel transport are enough for P to define a representation from a some groupoid of paths into the group $\mathrm{GL}(n)$!

Thin homotopy

Definition: Given two paths $\gamma, \eta : [0,1] \to \mathbb{R}^k$, such that

$$\gamma(0) = \eta(0) = x, \qquad \gamma(1) = \eta(1) = y,$$

a **homotopy** from γ to η is a map

$$H : [0,1]^2 \to \mathbb{R}^k$$

such that

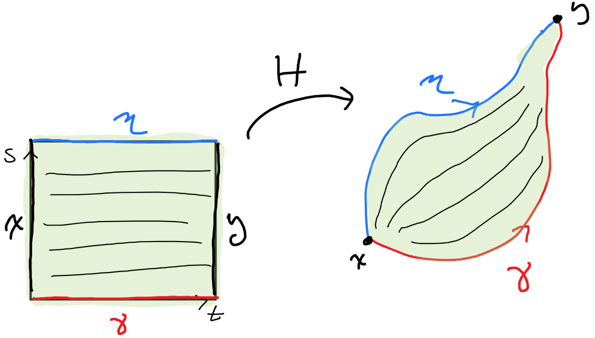
- $H(0,s) = x$ and $H(1,s) = y$ for all s,
- $H(t,0) = \gamma$ and $H(t,1) = \eta$ for all t.

The homotopy H is a **thin homotopy** if

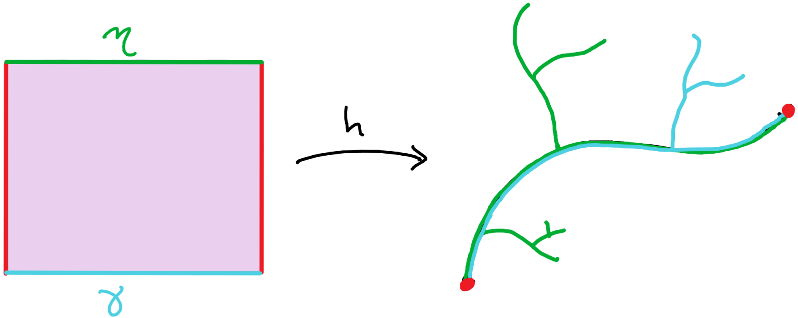
$$\mathrm{rank}(dH_{(t,s)}) \leq 1$$

for all $(t,s) \in [0,1]^2$.
Remark: Thin homotopy is equivalent to **tree-like equivalence**.

Homotopy



Thin homotopy



Thin fundamental groupoid

- Both homotopy and thin homotopy define equivalence relations on the space of paths $C^\infty([0,1],\mathbb{R}^n)$.
- Both define groupoids over $\mathbb{R}^k$: $\Pi(\mathbb{R}^k)$ and $\Pi^{\mathrm{thin}}(\mathbb{R}^k)$:
  1. The space of objects is $\mathbb{R}^k$ and the space of arrows is $C^\infty([0,1],\mathbb{R}^n)/\sim$.
  2. The maps $t, s : \Pi^{(\mathrm{thin})}(\mathbb{R}^k) \to \mathbb{R}^k$ are given by the endpoints:
  $$t(\gamma) = \gamma(1), \qquad s(\gamma) = \gamma(0).$$
  3. The multiplication is given by path concatenation.
- There is a homomorphism $\Pi^{\mathrm{thin}}(\mathbb{R}^k) \to \Pi(\mathbb{R}^k)$.
- Note that $\Pi(\mathbb{R}^k) \cong \mathrm{Pair}(\mathbb{R}^k)$ since any path in $\mathbb{R}^k$ is determined up to homotopy by its endpoints.
- The (thin) fundamental groupoids exist for any manifold M:
$$\Pi(M) \rightrightarrows M, \qquad \Pi^{\mathrm{thin}}(M) \rightrightarrows M.$$

Invariance under thin homotopy

If γ and η are thin homotopic paths, then

$$P(\gamma) = P(\eta).$$

This is because, given a thin homotopy $H : [0,1]^2 \to \mathbb{R}^k$, the curvature of $H^*(\nabla)$ vanishes.

This is because the curvature $H^*(F(\nabla))$ is a 2-form and $\mathrm{rank}(dH_{(t,s)}) \leq 1$.

Given a general connection ∇ on $\mathbb{R}^k$, its parallel transport defines a representation

$$P : \Pi^{\mathrm{thin}}(\mathbb{R}^k) \to \mathrm{GL}(n).$$

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Let G be a Lie group: a smooth manifold with a compatible group structure.

- It determines a Lie algebra $\mathfrak{g} = \mathrm{Lie}(G) = T_e G$.
- The tangent bundle of G is trivial: $TG \cong G \times \mathfrak{g}$ via right trivialization
$$(dg)g^{-1} : T_g G \to \mathfrak{g}.$$
Alternatively, we can use left trivializations.
- We can replace $GL(n)$ with any Lie group when defining differential equations and connections:
$$\frac{ds}{dt} = A(t)s(t)$$
for $A(t) \in \mathfrak{g}$.
- A solution is a map $s : \mathbb{R} \to G$ such that $(\frac{ds}{dt})s(t)^{-1} = A(t)$.

- All the previous results continue to hold in the setting of $\mathfrak{g}$-valued connections.
- We can still solve the equations using iterated integrals:

$$P(t,a) = id + \sum_{n\geq 1} \int_{a\leq s_1\leq \ldots \leq s_n \leq t} A(s_n)A(s_{n-1})\ldots A(s_1)ds_1\ldots ds_n.$$

  But this should be understood in the (completed) universal enveloping algebra $\hat{U}(\mathfrak{g})$.
- Flat $\mathfrak{g}$-connections are equivalent to representations

$$P : \mathrm{Pair}(\mathbb{R}^k) \to G,$$

- General $\mathfrak{g}$-connections are equivalent to (smooth) representations

$$P : \Pi^{\mathrm{thin}}(\mathbb{R}^k) \to G.$$

- The path signature arises as a very special example of this.

Free algebras

Let $V \cong \mathbb{R}^n$ be a real vector space. We define

- The *free associative algebra* on V:

$$T(V) = \bigoplus_{k \geq 0} V^{\otimes k} \cong \mathbb{R}\langle z_1, \dots, z_n \rangle,$$

  which is an algebra of non-commutative polynomials in n variables.

- The *free Lie algebra* on V is the Lie subalgebra $\mathcal{L}(V) \subset T(V)$ generated by V:

$$\mathcal{L}(V) = V \oplus [V, V] \oplus \dots$$

- It can be formally integrated to a group by taking exponentials

$$K_0(V) = \exp(\mathcal{L}(V)) \subset T((V)).$$

Truncations and Completions

- We will also consider truncations

$$T^{(n)}(V) = T(V)/(V^{\otimes n+1}), \qquad \mathcal{L}^{(n)}(V), \qquad K^{(n)}(V).$$

- $\mathcal{L}^{(n)}(V)$ is the *free nilpotent Lie algebra of n steps* on V. The algebraic Lie group $K^{(n)}(V)$ is its integration.
- Finally, we consider the completion

$$T((V)) = \prod_{k \geq 0} V^{\otimes k} \cong \mathbb{R}\langle\langle z_1, ..., z_n \rangle\rangle,$$

which is the algebra of non-commutative power series.
- $\mathcal{L}((V)) \subset T((V))$ is the *completed free Lie algebra* and

$$\hat{K}(V) = \exp(\mathcal{L}((V))) \subset T((V))$$

is its formal integration.

The tautological connection

The **tautological connection** on a vector space V is the following translation invariant $\mathcal{L}(V)$-connection:

$$\nabla = d - \mathrm{id}_V,$$

where $\mathrm{id}_V \in V^* \otimes V \subset \Omega^1(V) \otimes \mathcal{L}(V)$.

More concretely, let $V = \mathbb{R}^n$, with basis $z_1, ..., z_n$. Let $x_1, ..., x_n$ be the corresponding dual basis, viewed as linear functions on $\mathbb{R}^n$. Then

$$\nabla = d - \sum_{i=1}^{n} z_i dx_i.$$

It has non-zero curvature

$$F(\nabla) = \sum_{i<j} [z_i, z_j] dx_i \wedge dx_j.$$

The path signature

The parallel transport of the tautological connection ∇ gives a homomorphism

$$P : \Pi^{\mathrm{thin}}(V) \to \hat{K}(V).$$

Because ∇ is translation invariant, it satisfies $P(\gamma + v) = P(\gamma)$ for any $v \in V$. Therefore, the parallel transport factors as a homomorphism

$$S : \tau(V) \to \hat{K}(V),$$

where

$$\tau(V) = \Pi^{\mathrm{thin}}(V)/V,$$

which has the structure of a group.

Definition: Given a path $\gamma \in \tau(V)$, its **signature** is the non-commutative power series

$$S(\gamma) \in \hat{K}(V).$$

Computing the signature

Let $\gamma(t) = (x_1(t), x_2(t), ..., x_n(t)) : [0,1] \to V$ be a path and let

$$S(\gamma) = \sum_w S_w(\gamma) w$$

be its signature, where the sum is over all words w in the letters $z_1, ..., z_n$. The coefficient $S_w(\gamma)$ of the word $w = z_{i_1} ... z_{i_k}$ is computed as an iterated integral as follows:

$$S_w(\gamma) = \int_{0 \leq s_k \leq ... \leq s_1 \leq 1} x'_{i_1}(s_1) ... x'_{i_k}(s_k) ds_k ... ds_1.$$

The signature is one-to-one

Theorem (Chen) The signature

$$S_0 : \tau_1(V) \to \hat{K}(V)$$

is an injective group homomorphism.

This result was generalized to bounded variation paths by Hambly and Lyons, and to rough paths by Boedihardjo, Geng, Lyons, Yang.

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Principal bundles

Now we generalize the space on which our connections are defined from $\mathbb{R}^k$ to a general manifold M. This opens the possibility of studying connections on general **principal G-bundles**.

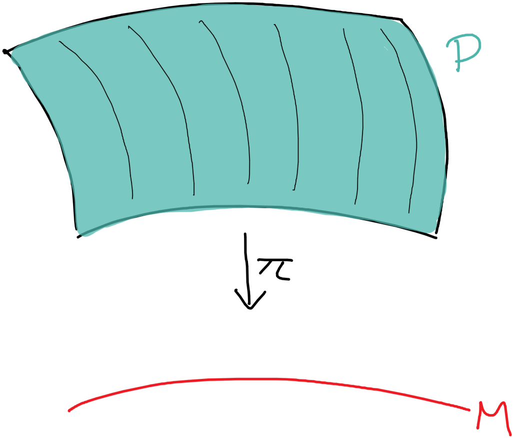
- A principal G-bundle over a manifold is a map

$$\pi : P \to M,$$

  where

  1. the space P is equipped with a free right G-action.
  2. the map π is G-invariant: $\pi(p * g) = \pi(p)$.
  3. P is locally (in M) isomorphic to a product

$$\pi^{-1}(U) \cong U \times G.$$


P
π
M

Examples

Let G be a Lie group with subgroup H, and let $M = G/H$, the coset space. Then the quotient map

$$\pi : G \to M$$

defines a principal H-bundle.

Example: Hopf bundle

Let $G = SU(2) \cong S^3$ and $H = S^1$. Then $G/H \cong S^2$. This defines the Hopf bundle

$$\pi : S^3 \to S^2.$$

Example: non-commutative power series

Recall that we defined the truncations

$$T^{(n)}(V) = T(V)/(V^{\otimes n+1}), \qquad \mathcal{L}^{(n)}(V), \qquad K^{(n)}(V).$$

Then there is a surjective group homomorphism

$$\pi : \hat{K}(V) \to K^{(n)}(V).$$

Let $\hat{H}_n(V) = \ker(\pi)$.

- Then $\hat{K}(V)$ is a principal $\hat{H}_n(V)$-bundle over $K^{(n)}(V)$.
- In particular, $\hat{K}(V)$ is a principal $\hat{H}_1(V)$-bundle over $K^{(1)}(V) = V$.

More about principal bundles

Let $\pi : P \to M$ be a principal G-bundle.

- The fibres $P_x = \pi^{-1}(x)$ of P are **non-canonically** isomorphic to G.
- This means that it is not possible to compare the different fibres without more data. In particular, there is no sense in which a path

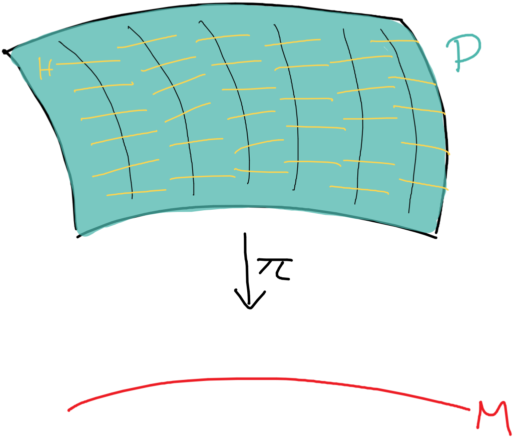
$$\gamma : [0, 1] \to M$$

  can be lifted to P in a way which is 'constant' in the fibres.
- The extra data we must choose is a **connection**: the choice of a 'horizontal' subspace

$$H_p \subset T_p P$$

  at every point $p \in P$, which is complementary to the vertical fibres.

Connections on principal bundles



Connections on principal bundles

Let $(\pi : P \to M, H)$ be a principal G-bundle equipped with connection H. Given a trivialization $P \cong M \times G$, we have

$$T_{(m,e)}P = T_m M \oplus T_e G = T_m M \oplus \mathfrak{g}.$$

The connection $H_{(m,e)} \subset T_m M \oplus \mathfrak{g}$ is the graph of a 1-form

$$\alpha \in \Omega^1(M) \otimes \mathfrak{g} \qquad H = \mathrm{graph}(\alpha : TM \to \mathfrak{g}).$$

In other words, H is locally the same thing as a connection

$$\nabla = d - \alpha.$$

Example

Let $H \subset G$ be a normal subgroup with quotient $M = G/H$ and consider the short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{m} \to 0.$$

A splitting $s : \mathfrak{m} \to \mathfrak{g}$ induces a connection on G with curvature

$$F(X, Y) = \pi_{\mathfrak{h}}([s(X), s(Y)]).$$

Example

We have the decomposition

$$\mathcal{L}(V) = \mathfrak{h}_n(V) \oplus \mathcal{L}^{(n)}(V),$$

where $\mathfrak{h}_n(V) = Lie(H_n(V))$.
Therefore, the principal bundles

$$\pi : \hat{K}(V) \to K^{(n)}(V)$$

are all naturally equipped with connections ∇_n induced from the splitting $s : \mathcal{L}^{(n)}(V) \to \mathcal{L}(V)$.

Horizontal lift

Let $(\pi : P \to M, H)$ be a principal G-bundle equipped with connection H. Given a path

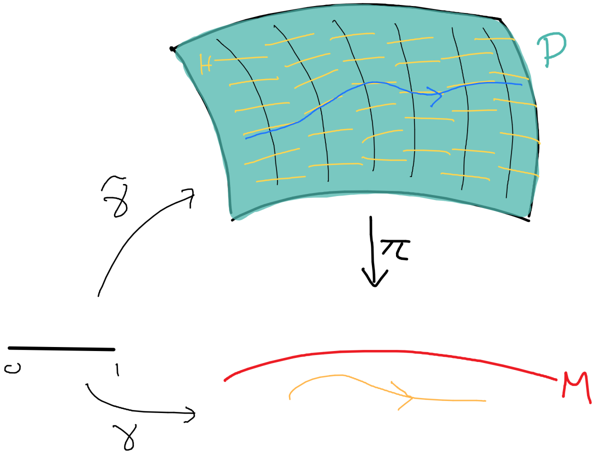
$$\gamma : [0,1] \to M,$$

a **horizontal lift** of γ is a path

$$\hat{\gamma} : [0,1] \to P$$

such that

- $\hat{\gamma}$ is a lift of γ: $\pi \circ \hat{\gamma} = \gamma$.
- $\hat{\gamma}$ is horizontal: $\frac{d\hat{\gamma}}{dt} \in H$.


P
H
π
0
1
γ
M

Horizontal lift

Choose a trivialization: $P \cong M \times G$ such that the connection H is represented by $\alpha \in \Omega^1(M) \otimes \mathfrak{g}$. The problem of finding the horizontal lift is equivalent to find

$$s : [0,1] \to G \text{ such that } (ds)s^{-1} = \alpha.$$

Therefore

- Existence and uniqueness: given $\hat{\gamma}(0) \in P_{\gamma(0)}$, there is a unique horizontal lift.
- The horizontal lift along γ defines a parallel transport isomorphism

$$P(\gamma) : P_{\gamma(0)} \to P_{\gamma(1)}.$$

The signature revisited

- Consider the principal bundle $\pi : \hat{K}(V) \to V$ equipped with the connection ∇_1 induced by the splitting $\mathcal{L}(V) = \mathfrak{h}_1(V) \oplus V$.
- Given a path $\gamma : [0,1] \to V$, the parallel transport defines a lift

$$\hat{\gamma} : [0,1] \to \hat{K}(V), \qquad P(\gamma) = \hat{\gamma}(1).$$

  This recovers the path signature.
- Bellingeri, Friz, Paycha, Preiß: a smooth rough path is a map $\gamma : [0,1] \to K^{(n)}(V)$. Using the connection ∇_n, this has horizontal lift

$$\hat{\gamma} : [0,1] \to \hat{K}(V), \qquad P(\gamma) = \hat{\gamma}(1).$$

  This defines the signature of rough paths.

$$
\begin{array}{ccccccc}
 & & \hat{H}_3(V) & & \hat{H}_2(V) & & \hat{H}_1(V) \\
 & \swarrow & & \swarrow & & \swarrow & \\
\cdots & \hat{K}(V) & \longrightarrow & \hat{K}(V) & \longrightarrow & \hat{K}(V) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\cdots & K^{(3)}(V) & \longrightarrow & K^{(2)}(V) & \longrightarrow & K^{(1)}(V) = V &
\end{array}
$$

Representations of the fundamental group

- Fundamental group: let $\pi_1(M, x)$ be the subgroup of $\Pi(M)$ consisting of loops based at x.
- Thin fundamental group: let $\pi^{\mathrm{thin}}(M, x)$ be the subgroup of $\Pi^{\mathrm{thin}}(M)$ consisting of loops based at x.
- The parallel transport of a connection defines a homomorphism

$$P : \pi^{\mathrm{thin}}(M, x) \to G.$$

- If the connection is flat, it descends to a homomorphism

$$P : \pi(M, x) \to G.$$

Riemann-Hilbert Correspondence

Let G be a Lie group and let M be a smooth manifold. The Riemann-Hilbert correspondence is an equivalence of categories:

$$\mathrm{Flat}(M, G) \cong \mathrm{Rep}(\pi_1(M), G),$$

where

- $\mathrm{Flat}(M, G)$ is the category of flat connections ∇ on principal G-bundles $P \to M$,
- $\mathrm{Rep}(\pi_1(M), G)$ is the category of G-representations of the fundamental group $\pi_1(M)$.

Riemann-Hilbert Correspondence

This equivalence of categories arises by combining two more basic equivalences

$$\mathrm{Rep}(TM, G) \xleftrightarrow{\text{Lie 2}} \mathrm{Rep}(\Pi(M), G) \xleftrightarrow{\text{Morita}} \mathrm{Rep}(\pi_1(M, x), G)$$

Thin Riemann-Hilbert Correspondence

Theorem (Barrett, Caetano-Picken)

$$\mathrm{Conn}(M, G) \cong \mathrm{Rep}_{C^\infty}(\pi_1^{thin}(M), G)$$

where

- $\mathrm{Conn}(M, G)$ denotes the category of all G-connections on M,
- $\mathrm{Rep}_{C^\infty}(\pi_1^{thin}(M), G)$ denotes the category of G-representations of $\pi_1^{thin}(M)$ which are smooth in a certain sense.

The signature is universal

- There is a sense in which the path signature $S(\gamma) \in \hat{K}(V)$ is universal.
- Chen: all possible iterated integrals along γ are contained in $S(\gamma)$. Therefore, there should be a way of extracting the parallel transport of any connection from $S(\gamma)$.
- By Chen's theorem, the group of thin paths $\tau(V)$ embeds into $\hat{K}(V)$.
- Chow's theorem: after truncating, the map $\tau(V) \to K^{(n)}(V)$ is surjective.
- Recall that the component of $S(\gamma)$ in V is the translation of a path. Therefore,

$$\pi_1^{thin}(V) \to \hat{H}_1(V).$$

  So we should think of $H_1(V)$ as the group of formal loops in V.
- Since formally $\mathrm{Lie}(H_1(V)) = [\mathcal{L}(V), \mathcal{L}(V)]$, this is the Lie algebra of 'infinitesimal loops' in V.

Theorem: "Infinitesimal Riemann-Hilbert correspondence" (Reutenauer, Kapranov)

$$\mathrm{Conn}(\mathcal{D}_n, G) \cong \mathrm{Rep}([\mathcal{L}(\mathbb{R}^n), \mathcal{L}(\mathbb{R}^n)], \mathfrak{g})$$

where

- $\mathrm{Conn}(\mathcal{D}_n, G)$ denotes the category of all G-connections on the *formal n-dimensional disc*,
- $\mathrm{Rep}([\mathcal{L}(\mathbb{R}^n), \mathcal{L}(\mathbb{R}^n)], \mathfrak{g})$ denotes the category of $\mathfrak{g}$-representations of the commutator subalgebra $[\mathcal{L}(\mathbb{R}^n), \mathcal{L}(\mathbb{R}^n)]$.

Given an (analytic) connection $\alpha \in \Omega^1(\mathbb{R}^n) \otimes \mathfrak{g}$, we use the infinitesimal Riemann-Hilbert correspondence extract a homomorphism

$$\phi_\alpha : [\mathcal{L}(\mathbb{R}^n), \mathcal{L}(\mathbb{R}^n)] \to \mathfrak{g},$$

which we integrate to

$$\Phi_\alpha : H_1(\mathbb{R}^n) \to G.$$

Then the parallel transport of $\nabla = d - \alpha$ along γ is

$$P(\nabla) = \Phi_\alpha \circ S(\gamma).$$

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Thin path group

Recall: $\tau(V)$, the group of paths up to thin homotopy and translation. There is a homomorphism

$$t : \tau(V) \to V, \qquad \gamma \mapsto \gamma(1).$$

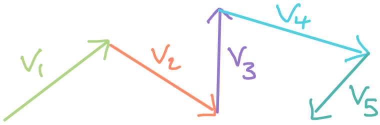
Then

$$\Pi^{\mathrm{thin}}(V) \cong \tau(V) \ltimes V.$$

Piecewise linear paths

There is an interesting subgroup of the thin group: the group of piecewise linear paths.

$$PL(V) \subset \tau_1(V).$$



Piecewise linear paths

The group $PL(V)$ is equipped with a map

$$\eta_V : V \to PL(V),$$

which restricts to a homomorphism on each 1-dimensional subspace. This property characterizes the group:

Universal property: Let V be a vector space, let G be a group, and let $f : V \to G$ be a map which restricts to a homomorphism on each line. Then there is a unique group homomorphism $F : PL(V) \to G$ such that $F \circ \eta_V = f$.

$$\begin{array}{ccc} V & \xrightarrow{\eta_V} & PL(V) \\ & \searrow^{f} & \downarrow{\scriptstyle F} \\ & & G \end{array}$$

Using the universal property we obtain:

- The identity map $id : V \to V$ is a homomorphism. Therefore, there is an induced map
$$t : \mathrm{PL}(V) \to V.$$
- Consider $r : V \to \tau(V)$ sending $v \in V$ to the path $\gamma_v(t) = tv$. This induces a realization homomorphism
$$R : \mathrm{PL}(V) \to \tau(V)$$
- Consider $\exp : V \to \hat{K}(V)$ sending $v \in V$ to $\sum_{n \geq 0} \frac{1}{n!} v^{\otimes n}$. This induces the signature
$$S_{PL} : \mathrm{PL}(V) \to \hat{K}(V).$$

Using the homomorphisms

$$t : \mathrm{PL}(V) \to V, \qquad t : \tau(V) \to V, \qquad \pi : \hat{K}(V) \to V,$$

we obtain action groupoids over V

$$\Pi_{PL}(V) = \mathrm{PL}(V) \ltimes V, \qquad \Pi^{\mathrm{thin}}(V) \cong \tau(V) \ltimes V, \qquad \Pi_K(V) = \hat{K}(V) \ltimes V,$$

and these define functors

$$\Pi_{PL}, \Pi^{\mathrm{thin}}, \Pi_K : \mathrm{Vect} \to \mathrm{Gpd}.$$

The homormorphisms for all $V \in \mathrm{Vect}$

$$S_{PL} : \mathrm{PL}(V) \ltimes V \xrightarrow{R} \tau(V) \ltimes V \xrightarrow{S} \hat{K}(V) \ltimes V$$

assemble together into natural transformations

$$S_{PL} : \Pi_{PL} \overset{R}{\Rightarrow} \Pi^{\mathrm{thin}} \overset{S}{\Rightarrow} \Pi_{K}.$$

Uniqueness

Theorem: The piecewise linear signature is the unique natural transformation:

$$S_{PL} : \Pi_{PL} \Rightarrow \Pi_K.$$

Proof If V is 1-dimensional, then $K(V) \cong V$ and so

$$K(V) \ltimes V \cong \mathrm{Pair}(V).$$

and therefore there is a unique homomorphism

$$S_{PL} : \mathrm{PL}(V) \ltimes V \to \mathrm{Pair}(V).$$

Since piecewise linear paths can be factored into 1-dimensional path, general uniqueness of S_{PL} follows.

Uniqueness

Using the fact that piecewise linear paths are dense in smooth paths (with Lipschitz topology), we immediately obtain:

Theorem: The signature is the unique continuous natural transformation:

$$S : \Pi^{\mathrm{thin}} \Rightarrow \Pi_K.$$

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Categorifying the parallel transport

In order to define the signature of a surface in $\mathbb{R}^n$, we must 'categorify' all concepts used in the section about paths:

- The thin group $\tau(V)$ and the group of non-commutative power series $\hat{K}(V)$ must be categorified to '2-groups'.
- Connections must be categorified to '2-connections'.
- The parallel transport needs to be categorified to a '2-functor'.

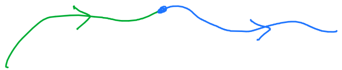
Crossed modules

Our model for '2-dimensional groups' is given by *crossed modules*, which were introduced by Whitehead. A crossed module

$$H = (\delta : H_1 \to H_0, \triangleright)$$

consists of the following data:

- A group H_0 of 'paths'



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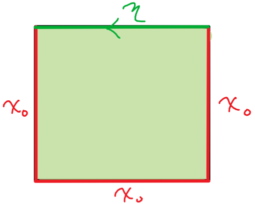

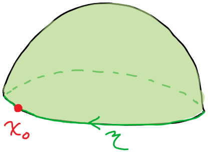

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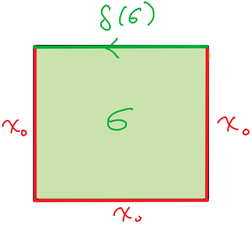

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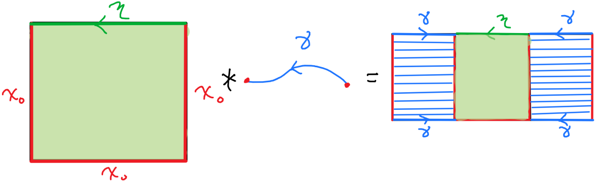
- An action by automorphisms $\triangleright : H_0 \to \mathrm{Aut}(H_1)$



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- A group H_1 of 'surfaces'
- A group homomorphism $\delta : H_1 \to H_0$
- An action by automorphisms $\triangleright : H_0 \to \mathrm{Aut}(H_1)$

such that

- The homomorphism δ is H_0-equivariant.
- The Peiffer identity holds:

$$\delta(\mathbf{E}) \triangleright \mathbf{F} = \mathbf{E}\mathbf{F}\mathbf{E}^{-1}.$$

The crossed module of surfaces

Our first example of a crossed module is the *crossed module of surfaces in V*

$$\tau(V) = (\delta : \tau_2(V) \to \tau_1(V), \triangleright)$$

- The 'group of paths' $\tau_1(V)$ is the group of paths modulo translation and thin homotopy, considered before.
- The 'group of surfaces' $\tau_2(V)$ is the group of smooth surfaces

  $$X : [0,1]^2 \to V$$

  such that $X(t,0) = X(0,s) = X(1,s) = 0$, taken module translation and thin homotopy.
- The group structure is given by horizontal concatenation
- δ is given by restriction to the top boundary $\delta(X)(t) = X(t,1)$.

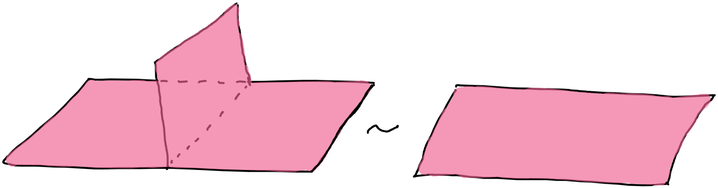
Thin homotopy

A thin homotopy between two surfaces $X, Y : [0,1]^2 \to V$ is a homotopy

$$H : [0,1]^3 \to V$$

which does not 'sweep out' any volume: $\mathrm{rank}(dH) \leq 2$.

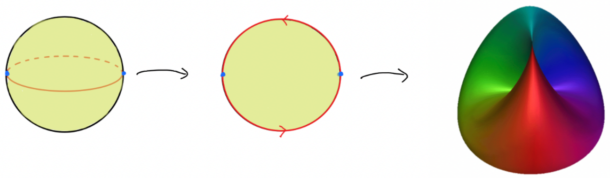
This includes reparametrization and cancellation of folds...



Thin homotopy

...but it also includes certain 'non-local' cancellations:

$$S^2 \to \mathbb{RP}^2 \to \mathbb{R}^n.$$



This map is thinly null-homotopic, even though there are no folds to cancel. There is no null-homotopy which stays completely within the image of this map: we must introduce new surfaces.

2-dimensional gauge theory

- Let $(\delta : H \to G, \triangleright)$ be a crossed module of Lie groups.
- Differentiating gives a crossed module of Lie algebras

$$(\delta : \mathfrak{h} \to \mathfrak{g}, \triangleright).$$

- A 2-connection on a vector space V valued in $(\delta : \mathfrak{h} \to \mathfrak{g}, \triangleright)$ consists in the data of
  - a $\mathfrak{g}$-valued 1-form $A \in \Omega^1_X \otimes \mathfrak{g}$,
  - a $\mathfrak{h}$-valued 2-form $B \in \Omega^2_X \otimes \mathfrak{h}$,

  such that $\delta(B) = F_A$.

2-dimensional gauge theory

Theorem (Schreiber, Waldorf, Martins, Picken, ...) A (translation invariant) 2-connection induces a parallel transport homomorphism between crossed modules

$$P : \tau(V) \to (\delta : H \to G, \triangleright).$$

In particular, it associates an element of H to every surface in X in V.

Kapranov's free crossed module

In order to define the surface signature, we need a certain free crossed module generated by a vector space

$$\mathfrak{k}(V) = (\partial : \mathfrak{k}_1(V) \to \mathfrak{k}_0(V), \triangleright)$$

- $\mathfrak{k}_0(V) = \mathcal{L}(V)$, the free Lie algebra generated by V.
- $\mathfrak{k}_1(V) = T(V) \otimes \wedge^2(V)/\mathrm{Peiff}$, where Peiff is the subspace generated by

  $$\langle v \otimes a \wedge b, w \otimes c \wedge e \rangle = ad_v([a, b])w \otimes c \wedge e + ad_w([c, e])v \otimes a \wedge b.$$

- The bracket on $\mathfrak{k}_1(V)$ is given by

  $$[v \otimes a \wedge b, w \otimes c \wedge e] = ad_v([a, b])w \otimes c \wedge e$$

Krapranov's free crossed module

We can formally integrate $\mathfrak{k}(V)$ to get the *crossed module of formal surfaces*

$$\hat{K}(V) = (\partial : \hat{K}_1(V) \to \hat{K}_0(V), \triangleright)$$

The tautological connection

The **tautological 2-connection** on a vector space V is the translation invariant $\mathfrak{k}(V)$-connection with

$$A = \mathrm{id}_V \subset \Omega^1(V) \otimes \mathcal{L}(V), \qquad B = \mathrm{id}_{\wedge^2 V} \subset \Omega^2(V) \otimes \mathfrak{k}_1(V).$$

More concretely, in coordinates

$$A = \sum_i z_i dx_i, \qquad B = \sum_{i<j} z_i \wedge z_j dx_i \wedge dx_j.$$

The surface signature

Definition (Kapranov, Lee, Chevyrev, Diehl, Ebrahimi-Fard, Tapia) The *surface signature* is the surface holonomy of the tautological connection:

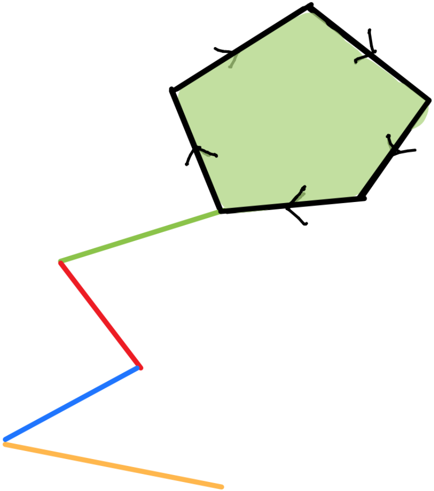
$$S = (S_1, S_0) : \tau(V) \to \hat{K}(V).$$

- Given a path $\gamma \in \tau_1(V)$, the component $S_0(V) \in \hat{K}_0(V)$ is the path signature.
- Given a surface $X \in \tau_2(V)$, the component $S_1(X) \in \hat{K}_1(V)$ is the surface signature.

Piecewise linear surfaces

There is an interesting sub-crossed module of $\tau(V)$: the crossed module of piecewise linear surfaces

$$\mathrm{PL}(V) = (\delta : \mathrm{PL}_1(V) \to \mathrm{PL}_0(V), \triangleright).$$

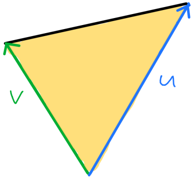


Piecewise linear surfaces

There are canonical functions

$$\eta_{V,0} : V \to \mathrm{PL}_0(V), \qquad \eta_{V,1} : V \times V \to \mathrm{PL}_1(V),$$

where $\eta_{V,0}(v)$ is the line segment path tv, and $\eta_{V,1}(v, u)$ is the triangular surface spanned by v and u.



Piecewise linear surfaces

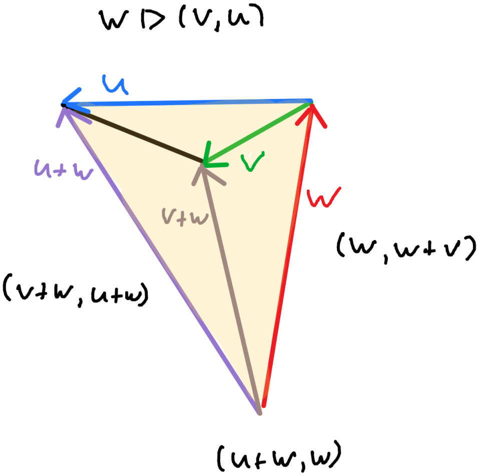
Universal property of $\mathrm{PL}(V)$: Let V be a vector space, let $\mathbf{H} = (\delta : H \to G, \triangleright)$ be a crossed module, and let $f_0 : V \to G$ and $f_1 : V \times V \to H$ be functions such that

- $f_0 : V \to G$ restricts to a homomorphism on every one-dimensional subspace of V,
- $f_1 : V \times V \to H$ is trivial on all linearly dependent pairs (v, u)
- **Triangle identity**: for every $v, u \in V$

$$\delta \circ f_1(v, u) = f_0(v) f_0(u - v) f_0(-u).$$

- **Tetrahedron identity**: For $v, w, u \in V$ which lie in a common two-dimensional subspace

$$f^0(w) \triangleright f^1(v, u) = f^1(w, w + v) f^1(w + v, w + u) f^1(w + u, w).$$


w ▷ (v, u)
u
v
w
u+w
v+w
(w, w+v)
(v+w, u+w)
(u+w, w)

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Then there is a unique morphism of crossed modules $F : \mathrm{PL}(V) \to \mathbf{H}$ such that $F_0 \circ \eta_{V,0} = f_0$ and $F_1 \circ \eta_{V,1} = f_1$.

Signature as a natural transformation

The crossed modules

$$\mathrm{PL}, \tau, \hat{K} : \mathrm{Vect} \to \mathrm{XMod}$$

are functors from the category of vector spaces to the category of crossed modules. The signature $S : \tau \Rightarrow \hat{K}$ is a natural transformation. There is a (uniquely defined) *realization* natural transformation

$$R : \mathrm{PL} \Rightarrow \tau(V).$$

We can define the piecewise linear signature to be the composition

$$S_{\mathrm{PL}} = S \circ R : \mathrm{PL} \Rightarrow \hat{K}.$$

The surface signature is unique

Theorem(B., Lee) The piecewise linear surface signature

$$\mathrm{S}_{PL} : \mathrm{PL} \Rightarrow \hat{K}$$

is the unique natural transformation extending the piecewise linear path signature.

Furthermore, the smooth surface signature

$$S : \tau \Rightarrow \hat{K}$$

is the unique *continuous* natural transformation extending the smooth path signature.

Computing the signature

Theorem(B., Lee) There is a canonical embedding

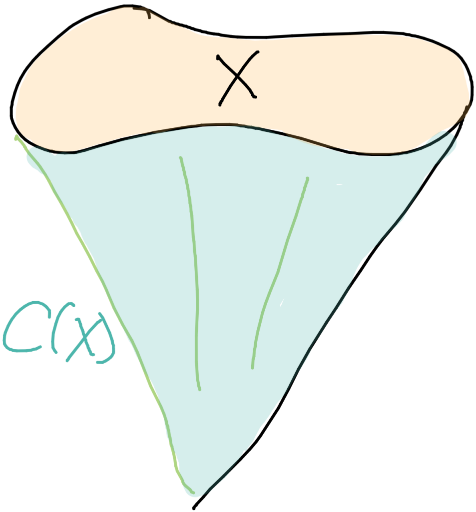
$$\hat{K}_1(V) \to \hat{\Gamma}_2(V) \times \hat{T}(V),$$

where $\hat{T}(V)$ is the algebra of formal non-commutative power series and $\hat{\Gamma}_2(V) = \hat{S}(V) \otimes \wedge^2 V$ is the vector space of formal 2-currents. The surface signature decomposes as follows:

$$S_1 = (S_1^{\Gamma}, S_1^{\mathcal{E}}) : \tau_2(V) \to \hat{\Gamma}_2(V) \times \hat{T}(V),$$

where, for a surface $X \in \tau_2(V)$,

- $S_1^{\mathcal{E}}(X) = S_0(\partial(X))$, the path signature of the boundary,
- $S_1^{\Gamma}(X) = \sum_{I, i<j} (\int_{\mathcal{C}(X)} x^I dx^i \wedge dx^j) z^I \otimes z_i \wedge z_j$, where $\mathcal{C}(X)$ is the closed surface obtained from conning off the boundary of X.


X
C(X)

The surface signature is injective

Theorem(B., Lee) The piecewise linear surface signature

$$S_{\mathrm{PL},1} : \mathrm{PL}_1(V) \to \hat{K}(V)$$

is injective.

In particular, if $X \in \mathrm{PL}_1(V)$ is a surface such that $S_{\mathrm{PL},1}(X) = 0$, then $R_1(X)$ is thinly null-homotopic.

Thank You