Heat equation with Dirichlet white noise boundary conditions

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My talk will be based on a joint work with Ben Goldys (Univ. Sydney). It is a continuation of Z. Brzeźniak, B. Goldys, S. Peszat, and F. Russo, *Second order PDEs with Dirichlet white noise boundary conditions*, J. Evol. Equ. 15 (2015), 1–26.

Given a possibly unbounded domain $\mathcal{O} \subset \mathbb{R}^d$ and an elliptic second order operator A we are concerned with the following parabolic boundary problem

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= Au(t,x) \qquad \text{on } \mathcal{O}, \\
u(0,x) &= u_0(x), \\
u(t,x) &= \sum_k e_k(x)\frac{dW_k}{dt}(t) \qquad \text{on } (0,T] \times \partial\mathcal{O},
\end{aligned}
$$

where $e_k\colon \partial\mathcal{O} \mapsto \mathbb{R}$ and (W_k) are independent real-valued Wiener processes.

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$$\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) \qquad t>0,\ x\in(0,1),\\
u(0,x) &= u_0(x), \qquad x\in(0,1),\\
u(t,0) &= \dot W_0(t), \qquad t>0,\\
u(t,1) &= \dot W_1(t), \qquad t>0,
\end{aligned}$$

where W_0 and W_1 are real-valued Brownian motions.

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Da Prato and Zabczyk (1993) used semigroup approach (see Lasiecka (1980), Balakrishnan (1981), Lasiecka and Triggiani (2000) for deterministic problem).
Alós and Bonaccorsi (2002) provided some formulae for the integral kernel.

The idea of Lasiecka, Balakrishnan, Da Prato and Zabczyk: let τ be the boundary operator

- Dirichlet

$$\tau\psi(x) = \psi(x), \qquad x \in \partial\mathcal{O},$$

- Neumann

$$\tau\psi(x) = \frac{\partial\psi}{\partial\mathbf{n}}(x), \qquad x \in \partial\mathcal{O}.$$

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where $\lambda \geq 0$ chosen properly.
D_τ - the Dirichlet map if τ Dirichlet.

After some calculation, for regular $\xi\colon [0,T]\times\partial\mathcal{O}\mapsto\mathbb{R}$, one obtains the so-called mild solution to the boundary problem

$$u(t) = e^{A_\tau t}u_0 + (\lambda - A_\tau)\int_0^t e^{A_\tau(t-s)}D_\tau\xi(s,\cdot)ds.$$

In fact the mild solution is a weak solution.

Thus formally

$$u(t) = e^{A_\tau t}u_0 + (\lambda - A_\tau)\int_0^t e^{A_\tau(t-s)}D_\tau \sum_k e_k dW_k(s),$$

or

$$du = A_\tau u dt + (\lambda - A_\tau)D_\tau \sum_k e_k dW_k(s).$$

In the example

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \delta_0'(x)\frac{\partial W_0}{\partial t}(t) - \delta_1'(x)\frac{\partial W_1}{\partial t}(t),$$
$$u(0,x) = u_0(x).$$

Our main goal is show that under some condition

$$du = A_\tau u dt + (\lambda - A_\tau) D_\tau \sum_k e_k dW_k(s).$$

has good meaning and its solution is a Markov family some function space ($E = L^p(\mathcal{O}, w(x)dx)$).

Let us first consider the problem on the scale of Sobolev spaces. Namely assume that

Hypothesis 1 There are $\lambda \geq 0$, $p > 1$ and $s_0 \geq 0$ such that the Dirichlet map D_λ is a well defined linear operator acting from $\mathrm{linspan}\,\{e_k\}$ into the Sobolev space $W^{-s_0,p}(\mathcal{O})$.

Hypothesis 2 Operator A with homogeneous Dirichlet boundary conditions generates an analytic C_0-semigroup S on each $W^{s,p}(\mathcal{O})$-spaces. For all $s, s' \in \mathbb{R}$, $p > 1$ and $t > 0$, $S(t)\colon W^{s,p}(\mathcal{O}) \mapsto W^{s',p}(\mathcal{O})$. Moreover, if $A_{s,p}$ denotes the generator of S on $W^{s,p}(\mathcal{O})$, then we assume that there is an s_1 such that $W^{-s_0,p}(\mathcal{O}) \hookrightarrow D(A_{-s_1,p})$.

The hypotheses hold in a number of cases; if $\mathcal{O}$ is a bounded Lipschitz domain and the operator A has Lipschitz coefficients, then $D_0\colon H^{1/2}(\partial\mathcal{O}) \to H^1(\mathcal{O})$ is well defined and bounded. In that case it is enough to assume that
$\mathrm{linspan}\,\{e_k\} \subset H^{1/2}(\partial\mathcal{O})$.
If $\mathcal{O}$ is a bounded C^∞ domain and the operator A has C^∞ coefficients, then

$$D_0\colon H^{-s-\frac{3}{2}}(\partial\mathcal{O}) \to H^{-s}(\mathcal{O})$$

is well defined and bounded for any $s \geq 0$, see Sections 6 and 7 in Chapter 2 of Lions-Magenes. In particular, if $s \leq -\frac{3}{2}$ then $\mathrm{linspan}\,\{e_k\} \subset H^{-s-\frac{3}{2}}(\partial\mathcal{O}) \subset L^2(\mathcal{O})$.
Very general conditions given in terms of capacities of $\mathcal{O}$ can be found in Chapter 15.7 of Mazya.

Our hypotheses enable us to reformulate the boundary problem into

$$du = Audt + \sum_k Be_k dW_k$$

considered on the state space $W^{-s_1,p}(\mathcal{O})$. In fact the map

$$B = (\lambda - A)\, D_\lambda := (\lambda - A_{-s_1,p})\, D_\lambda$$

is a linear operator from $\mathrm{linspan}\,\{e_k\}$ into $W^{-s_1,p}(\mathcal{O})$ and $A = A_{-s_1,p}$ generates a C_0-semigroup $S = S_{-s_1,p}$ on $W^{-s_1,p}(\mathcal{O})$.

Therefore, using some basic facts from the theory on stochastic integration in L^p-spaces (Brzezniak and van Neerven (2003), Brzezniak and Veraar (2012)) the boundary problem has the mild solution solution

$$u(t) = S(t)u_0 + \int_0^t S(t-s)BdW(s)$$

in $W^{-s_1,p}(\mathcal{O})$-space if and only if

$$\int_{\mathcal{O}} \left[\sum_k \int_0^T \left((I-\Delta)^{-s_1/2} S(t)Be_k \right)^2 (x)dt \right]^{p/2} dx < +\infty$$

for a certain or equivalently for any $T \in (0, +\infty)$.

Moreover, if there is an $\alpha > 0$ such that

$$\int_{\mathcal{O}} \left[\sum_k \int_0^T t^{-\alpha} \left((I - \Delta)^{-s_1/2} S(t) B e_k \right)^2 (x) dt \right]^{p/2} dx < +\infty$$

then the mild solution has continuous trajectories in $W^{-s_1,p}(\mathcal{O})$.

Alos and Bonaccorsi observed that the solution lives in a weigthed space $L^2((0,+\infty), x^\delta dx)$. In Brzezniak, Goldys, Peszat, Russo we proved that in many cases the solution is C^∞ in $t > 0$ and $x \in \mathcal{O}$.

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N.V. Krylov, *Weighted Sobolev spaces and Laplace's equation and the heat equations in a half space*, Comm. Partial Differential Equations **24** (1999), 1611–1653.
N.V. Kylov, *The heat equation in $L_q((0,T), L_p)$-space with weights, SIAM J. Math. Anal. (***32** (2001), 1117–1141.
N. Krylow showed that the heat semigroup in analytical on $L^p((0,+\infty), x^\delta dx)$, $\delta < p$.

We showed that under very mild assumptions on A and $\mathcal{O}$ the semigroup in C_0 on the weighted space $L^p_{\theta,\delta} := L^p(\mathcal{O}, w_{\theta,\delta}(x)dx)$, where

$$w_{\theta,\delta}(x) = \min\left\{ \mathrm{dist}\,(x, \partial\mathcal{O})^{\theta}, \left(1 + |\, x\,|^2\right)^{-\delta} \right\},$$

$p \in (1, +\infty)$, $\theta \in [0, 2p-1)$ and $\delta \geq 0$.

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$p \in (1,+\infty)$, $\theta \in [0, 2p-1)$ and $\delta \geq 0$. Lindemulder and Veraar (2020) showed this for $A = \Delta$ on half spaces.

Let $w^i\colon \mathcal{O}\mapsto (0,+\infty)$, $i=1,2$, be measurable weights. Let

$$\mathcal{L}^p_i := L^p\left(\mathcal{O},\mathcal{B}(\mathcal{O}),w^i(x)dx\right), \qquad i=1,2,$$

and let $\mathcal{L}^p := L^p(\mathcal{O},\mathcal{B}(\mathcal{O}),w(x)\,dx)$, where $w(x)=\min\{w^1(x),w^2(x)\}$. We will need the following elementary result.

FACT

Assume that T is a bounded linear operator from $\mathcal{L}^p_i$ to $\mathcal{L}^p_i$ for $i=1,2$. Then it is bounded from $\mathcal{L}^p$ to $\mathcal{L}^p$ and the operator norm satisfies the estimate

$$\|T\|_{L(\mathcal{L}^p)} \le 2^{(p-1)/p}\max\left\{\|T\|_{L(\mathcal{L}^p_1)},\|T\|_{L(\mathcal{L}^p_2)}\right\}.$$

Hypothesis 3)

$$|G(t,x,y)| \leq C m_t(y) g_{ct}(x-y), \qquad t \leq 1, \qquad x,y \in \mathcal{O},$$

and

$$|\nabla_x G(t,x,y)| \leq C \frac{m_t(y)}{\sqrt{t}} g_{ct}(x-y), \qquad t \leq 1, \qquad x,y \in \mathcal{O},$$

where

$$m_t(z) := \min\left\{1, \frac{\rho(z)}{\sqrt{t}}\right\}, \qquad \rho(z) := \operatorname{dist}(z, \partial\mathcal{O})$$

and

$$g_t(z) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|z|^2}{2t}}.$$

Hypotesis 4) For any $c > 0$ and $\kappa \in (-1, 0)$ there is a constant $C < +\infty$ such that

$$\sup_{x\in\mathcal{O}} \int_{\mathcal{O}} \rho^{\kappa}(y) g_{ct}(x-y) dy \leq C t^{\frac{\kappa}{2}}, \qquad \forall\, t \in (0,1].$$

The hypothesis is satisfied if $\mathcal{O}$ is a half space or if $\mathcal{O}$ is a bounded $C^{1+\alpha}$-domain.

Hypotesis 3) holds if $\mathcal{O}$ is bounded $C^{1+\alpha}$ domain, or is a graph above $C^{1+\alpha}$ function, or $\mathcal{O} = \mathbb{R}^d_+$ or

$$\mathcal{O} = \left\{(x_i) \in \mathbb{R}^{d+1} \colon a < x_{d+1} < b\right\}.$$

$\mathcal{A}$ is uniformly elliptic, the coefficients a_{ij} are Dini continuous, and μ^i are sign measures of the parabolic Kato class. In general $a_{i,j}$ and μ^i may depend on t and x variables. For more details see S. Cho, P. Kim, H. Park 2012.

Since the semigroup acts from $W^{-s_1,p}(\mathcal{O})$ into $L^p_{\theta,\delta}$ we need only to verify the integrability condition

$$\int_{\mathcal{O}} \left[\sum_k \int_0^T \left(S(t)Be_k\right)^2(x)dt \right]^{p/2} w_{\theta,\delta}(x)dx < +\infty$$

for a certain or equivalently for any $T \in (0, +\infty)$. Moreover, this guarantees that the boundary problem defines a Markov family on the state space $L^p_{\theta,\delta}$.

If for a certain $\alpha > 0$,

$$\int_{\mathcal{O}} \left[\sum_k \int_0^T t^{-\alpha} \left(S(t)Be_k\right)^2(x)dt \right]^{p/2} w_{\theta,\delta}(x)dx < +\infty,$$

then the mild solution has continuous trajectories in $L^p_{\theta,\delta}$.

Consider the simplest case of $\mathcal{O} = (0,1)$. Let $p \in (1, +\infty)$ and $\theta \in (p-1, 2p-1)$. Then the boundary problem

$$\frac{\partial u}{\partial t}(t,x) = Au(t,x), \quad x \in (0,1),$$
$$u(t,0) = \frac{dW_0}{dt}(t),$$
$$u(t,1) = \frac{dW_1}{dt}(t),$$

defines Markov family with continuous trajectories in the space $L^p_{\theta,0} = L^p(0,1,\min\{x^\theta,(1-x)^\theta\}dx)$.

Let $\mathcal{O}$ be a bounded domain $\mathbb{R}^d$. Assume that

$$\sum_k \sup_{y\in\partial\mathcal{O}} e_k^2(y) < +\infty.$$

Then for any $1 < p$ and $\theta \in (p-1, 2p-1)$, boundary problem defines a Markov family with continuous trajectories in $L^p_{\theta,0}$.

In the case of the so-called *white noise on* $\mathbb{S}^{d-1}$, $\{e_k\}$ is an orthonormal basis of $L^2(\mathbb{S}^{d-1}, d\mathrm{s})$. We have the following result

Theorem

Assume that W is a white noise on $\mathbb{S}^1$. Let $p > 1$ and $\theta \in \left(\frac{3p}{2} - 1, 2p - 1\right)$. Then the boundary problem defines a Markov family with continuous trajectories in $L^p_{\theta,0}$. If the semigroup is stable on unweighted space, then the Markov family defined by the boundary problem on $L^p_{\theta,0}$ for $p \in (1, +\infty)$ and $\theta \in \left(\frac{3p}{2} - 1, 2p - 1\right)$ has a unique invariant measure.

Consider the case of half space $\mathcal{O} = \mathbb{R}^m \times (0, +\infty)$. Let W be a spatially homogeneous Wiener process on $\mathbb{R}^m = \partial\mathcal{O}$. Recall that

$$\mathbb{E}W(t,x)W(s,y) = t \wedge s\widehat{\mu}(x-y),$$

μ is the so-called spectral measure of W.

Theorem

Assume that: the spectral measure of W is finite, $\delta > (m+1)/2$, $p \in (1, +\infty)$ and $\theta \in (p-1, 2p-1)$. Then boundary problem defines Markov family with continuous trajectories in $L^p_{\theta,\delta}$.

If $\mu(d\mathbf{y}) = d\mathbf{y}$, then W is the co-called *cylindrical Wiener process on* $L^2(\mathbb{R}^m)$ or equivalently $\frac{\partial W}{\partial t}(t, \mathbf{y})$, $t \geq 0$, $\mathbf{y} \in \mathbb{R}^m$, is the *space-time white noise*.

Theorem

Let $m = 1$, $\delta > 1$, and let W be a cylindrical Wiener process on $L^2(\mathbb{R})$. Then the boundary problem defines a Markov family with continuous trajectories in the space $L^p_{\theta,\delta}$ for $p > 1$ and $\theta \in \left(\frac{3p}{2} - 1, 2p - 1\right)$.

Let

$$\mu_\kappa(d\mathbf{y}) := \left(1 + |\mathbf{y}|^2\right)^{-\kappa/2} d\mathbf{y}.$$

Theorem

Let W be a Wiener process on $\mathbb{R}^m$ with the spectral measure μ_κ, $0 \le \kappa$. (i) If $\kappa \ge m$, then boundary problem defines a Markov family with continuous trajectories in the space $L^p_{\theta,\delta}$ for $p > 1$ and $\theta \in (p-1, 2p-1)$.
(ii) If $m - 2 < \kappa < m$, then boundary problem defines a Markov family with continuous trajectories in the space $L^p_{\theta,\delta}$ for

$$1 < p < +\infty, \qquad p + \frac{p}{2}(m - \kappa) - 1 < \theta < 2p - 1.$$

Let

$$K_t\psi(x) := \int_{\mathcal{O}} k_t(x,y)\psi(y)\,\frac{dy}{\rho(y)},$$

$$k_t(x,y) := \left(\frac{\rho(x)}{\rho(y)}\right)^{\frac{\theta+1}{p}} m_t(y) g_{ct}(x-y)\rho(y).$$

C_0-property of S will be established as soon as we show that for each $0 < t \leq 1$, K_t is a bounded linear operator from $L^p\left(\mathcal{O}, \frac{dy}{\rho(y)}\right)$ into $L^p\left(\mathcal{O}, \frac{dy}{\rho(y)}\right)$ and that $\sup_{0<t\leq 1} \|K_t\| < +\infty$, where $\|\cdot\|$ is the operator norm on $L\left(L^p\left(\mathcal{O}, \frac{dy}{\rho(y)}\right), L^p\left(\mathcal{O}, \frac{dy}{\rho(y)}\right)\right)$.

Taking into account the Schur test, it is enough to show that

$$\sup_{0<t\leq 1}\sup_{x\in\mathcal{O}}\int_{\mathcal{O}} k_t(x,y)\,\frac{dy}{\rho(y)}+\sup_{0<t\leq 1}\sup_{y\in\mathcal{O}}\int_{\mathcal{O}} k_t(x,y)\,\frac{dx}{\rho(x)}<+\infty\,.$$

Note that our assumption $\theta<2p-1$ is necessary for the application of the Schur test.

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