Diophantine Approximation, Fractal Geometry, and Shrinking Targets.

Introduction

Diophantine Approximation, Fractal Geometry, and the study of Shrinking Targets, are three important, and at first sight seemingly unrelated, areas of mathematics. The purpose of these notes is to introduce these topics and to study some problems that lie at their intersection. In the first lecture we will introduce the shrinking target problem and prove a simple result for the doubling map. This result and its proof exhibit some of the key ideas in this area. In the second lecture our focus will turn to Diophantine Approximation on self-similar sets. The highlight of this section will be a proof of a result due to Levesley, Salp, and Velani ([17]) on the existence of well-approximable numbers in the middle third Cantor set that are not Liouville. The argument we give is based upon the extremely powerful mass transference principle due to Beresnevich and Velani [5]. In the final lecture we will outline some of the authors work from [3]. This work applies ideas from Diophantine Approximation and Shrinking Targets to the study of overlapping iterated function systems.

Notation. In these notes we will adopt the following notational conventions. Given a set S and two functions $f, g : S \to \mathbb{R}$ we write $f \ll g$ if there exists $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \in S$. We write $f \asymp g$ if $f \ll g$ and $g \ll f$. We will also use $f = \mathcal{O}(g)$ interchangeably to mean $f \ll g$.

Lecture 1

Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ be a measure preserving map. Given a sequence of measurable sets $E = (E_n)_{n=1}^{\infty}$ we can associate a limsup set as follows

$$W(E) := \{x \in X : T^n(X) \in E_n \text{ for i.m. n}\}.$$

Here and throughout we use i.m. as a shorthand for infinitely many. Note that we can also write $W(E)$ as follows

$$W(E) = \limsup_{n\to\infty} T^{-n}(E_n) = \bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} T^{-n}(E_n).$$

The sequence of sets (E_n) can be very general. Often they are taken to be a nested sequence of balls centred at a point. As such the study of the sets $W(E)$ is informally known as the shrinking target problem. As well as being a natural problem from the perspective of Dynamical Systems and Ergodic Theory, shrinking target problems arise from many other areas of Mathematics, most notably Number Theory.

Often we are interested in understanding the metric properties of the sets $W(E)$. Two avenues of research naturally arise:

1. What is $\mu(W(E))$? Moreover, do there exist simple conditions for determining $\mu(W(E))$?

2. What is the Hausdorff dimension of $W(E)$?

In these notes we will consider both of these questions. In the second lecture we will see a deep result which shows that in many cases, if we know the answer to the first question then we can deduce the answer to the second.

There is a natural criteria which is expected to provide a solution to the first question. Before stating it we recall the first and second Borel-Cantelli lemmas.

Lemma 1. *Let $(X, \mathcal{B}, \mu)$ be a probability space and (F_n) be a sequence of measurable sets. Then the following statements are true:*

1. *Suppose $\sum_{n=1}^{\infty} \mu(F_n) < \infty$. Then $\mu(\{x \in X : x \in F_n \text{ for i.m. } n\}) = 0$.*
2. *Suppose $\sum_{n=1}^{\infty} \mu(F_n) = \infty$ and the events (F_n) are independent. Then $\mu(\{x \in X : x \in F_n \text{ for i.m. } n\}) = 1$.*

Proof of item 1. Let $\epsilon > 0$ be arbitrary and $N \in \mathbb{N}$ be a sufficiently large natural number for which $\sum_{n=N}^{\infty} \mu(F_n) < \epsilon$. Now using the fact $\{x \in X : x \in F_n \text{ for i.m. } n\} \subset \cup_{n=N}^{\infty} F_n$ we have

$$\mu(\{x \in X : x \in F_n \text{ for i.m. } n\}) \leq \mu\left(\bigcup_{n=N}^{\infty} F_n\right) \leq \sum_{n=N}^{\infty} \mu(F_n) < \epsilon.$$

Because ϵ was arbitrary this completes our proof. □

Taking $(F_n) = (T^{-n}(E_n))$ and using the fact T is measure preserving, we see that Lemma 1 implies that if $\sum_{n=1}^{\infty} \mu(E_n) < \infty$ then $\mu(W(E)) = 0$. For a general dynamical system the events $(T^{-n}(E_n))$ are certainly not independent. As such we cannot directly use Lemma 1 to deduce $\mu(W(E)) = 1$. Nevertheless it is natural to wonder whether an analogue of Lemma 1 holds. More specifically, it is natural to wonder whether we see the following general phenomenon:

$$\mu(W(E)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mu(E_n) < \infty; \\ 1 & \text{if } \sum_{n=1}^{\infty} \mu(E_n) = \infty. \end{cases} \tag{1}$$

We emphasise that (1) does not hold for all dynamical systems. See the following exercise.

Exercise 1. *Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an irrational rotation of the circle given by $T(x) = x + \alpha \mod 1$ for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.*

1. *Prove that there exists $E = (E_n)_{n=1}^{\infty}$ a sequence of intervals such that $\sum_{n=1}^{\infty} \mu(E_n) = \infty$ and $W(E)$ is a singleton.*
2. *Prove that for any $b \in (0, 1)$ there exists $E = (E_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \mu(E_n) = \infty$ and $\mu(W(E)) = b$.*

As a general principle, if a dynamical system is mixing sufficiently quickly then (1) will hold for sufficiently nice (E_n). A particular instance of this is the following theorem.

Theorem 1. *Let $T : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be given by $T(x) = 2x \mod 1$. Then (1) holds for $(E_n)_{n=1}^{\infty}$ a sequence of intervals.*

The key to proving this result is the following generalised Borel-Cantelli lemma.

Lemma 2 (Quasi-independence on average)**.** *Let $(X, \mathcal{B}, \mu)$ be a finite measure space and $(F_n)_{n=1}^\infty$ be a sequence of measurable sets satisfying $\sum_{n=1}^\infty \mu(F_n) = \infty$. Then we have*

$$\mu\left(\limsup_{n\to\infty} F_n\right) \geq \limsup_{Q\to\infty} \frac{\left(\sum_{n=1}^Q \mu(F_n)\right)^2}{\sum_{n=1}^Q \sum_{m=1}^Q \mu(F_n \cap F_m)}.$$

Proof. For $n \in \mathbb{N}$ let χ_n be the characteristic function for F_n. Similarly for $l, n \in \mathbb{N}$ we let $\chi_{l,n}$ denote the characteristic function for $\cup_{m=l}^n F_m$. By the Cauchy-Schwartz inequality we have

$$\begin{aligned}\sum_{n=m}^Q \mu(F_n) = \int \sum_{n=m}^Q \chi_n \, d\mu = \int \chi_{m,Q} \cdot \sum_{n=m}^Q \chi_n \, d\mu \\ &\leq \left(\int \chi_{m,Q} \, d\mu\right)^{1/2} \left(\int \left(\sum_{n=m}^Q \chi_n\right)^2 d\mu\right)^{1/2} \\ &= \mu\left(\bigcup_{n=m}^Q F_n\right)^{1/2} \left(\sum_{n=m}^Q \sum_{l=m}^Q \mu(F_n \cap F_l)\right)^{1/2}.\end{aligned}$$

Taking squares and rearranging yields

$$\mu\left(\bigcup_{n=m}^Q F_n\right) \geq \frac{(\sum_{n=m}^Q \mu(F_n))^2}{\sum_{n=m}^Q \sum_{l=m}^Q \mu(F_n \cap F_l)} = \frac{(\sum_{n=1}^Q \mu(F_n) + \mathcal{O}(m))^2}{\sum_{n=1}^Q \sum_{l=1}^Q \mu(F_n \cap F_l) + \mathcal{O}(m \sum_{n=1}^Q \mu(F_n))}.$$

Now using the fact that $\sum_{n=1}^\infty \mu(F_n) = \infty$ we see that

$$\mu\left(\bigcup_{n=m}^\infty F_n\right) \geq \limsup_{Q\to\infty} \frac{\left(\sum_{n=1}^Q \mu(F_n)\right)^2}{\sum_{n=1}^Q \sum_{m=1}^Q \mu(F_n \cap F_m)}. \tag{2}$$

The result now follows from the observation $\mu(\cap_{m=1}^\infty \cup_{n=m}^\infty F_n) = \lim_{m\to\infty} \mu(\cup_{n=m}^\infty F_n)$, and the fact that the right hand side of (2) does not depend upon m. □

Lemma 2 is an important tool for bounding the measure of limsup sets. Notice that if the sequence of sets (F_n) were independent then this lemma would imply $\mu(\limsup F_n) = 1$. As such it can be viewed as a strengthening of the second Borel-Cantelli lemma. Recent results by Beresnevich and Velani [6] effectively demonstrate that the only way for a limsup set to have positive measure is for the lower bound provided by Lemma 2 to be positive.

Proof of Theorem 1. Here we will prove Theorem 1 for (E_n) of the form $E_n = [0, \frac{1}{2^{f(n)}})$ for some $f : \mathbb{N} \to \mathbb{N}$. So our divergence assumption now reads $\sum_{n=1}^\infty 2^{-f(n)} = \infty$.

We begin our proof by observing that

$$T^{-n}(E_n) = \bigcup_{(a_i)\in\{0,1\}^n} \left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n-f(n)}\right)$$

for each $n \in \mathbb{N}$. Let $(a_i) \in \{0,1\}^n$ be arbitrary. We will now bound

$$\mu\left(\left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n-f(n)}\right) \cap T^{-m}(E_m)\right)$$

for $m > n$. We do this via a case analysis.

Case 1 $n < m \leq n + f(n)$**.** If $n < m \leq n + f(n)$ then there is one word $(b_i)_{i=1}^m \in \{0,1\}^m$ such that

$$\left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n-f(n)}\right) \cap \left[\sum_{i=1}^m b_i 2^{-i}, \sum_{i=1}^m b_i 2^{-i} + 2^{-m-f(m)}\right) \neq \emptyset,$$

namely the word $b_1 \ldots b_m = a_1 \ldots a_n (0)^{m-n}$. Therefore

$$\begin{aligned}\mu\left(\left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n-f(n)}\right) \cap T^{-m}(E_m)\right) &\leq \mu\left(\left[\sum_{i=1}^m b_i 2^{-i}, \sum_{i=1}^m b_i 2^{-i} + 2^{-m-f(m)}\right)\right) \\ &= 2^{-m-f(m)}.\end{aligned}$$

Case 2 $m > n+f(n)$**.** If $m > n+f(n)$ then only words $(b_i) \in \{0,1\}^m$ that begin with $a_1 \ldots a_n (0)^{f(n)}$ satisfy

$$\left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n-f(n)}\right) \cap \left[\sum_{i=1}^m b_i 2^{-i}, \sum_{i=1}^m b_i 2^{-i} + 2^{-m-f(m)}\right) \neq \emptyset.$$

Moreover for any such word we have

$$\left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n-f(n)}\right) \cap \left[\sum_{i=1}^m b_i 2^{-i}, \sum_{i=1}^m b_i 2^{-i} + 2^{-m-f(m)}\right) = \left[\sum_{i=1}^m b_i 2^{-i}, \sum_{i=1}^m b_i 2^{-i} + 2^{-m-f(m)}\right).$$

There are $2^{m-f(n)-n}$ words in $\{0,1\}^m$ that begin with $a_1 \ldots a_n (0)^{f(n)}$. Therefore we have

$$\mu\left(\left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n-f(n)}\right) \cap T^{-m}(E_m)\right) = 2^{m-f(n)-n} \cdot 2^{-m-f(m)} = 2^{-n-f(n)-f(m)}.$$

Combining the bounds provided by Case 1 and Case 2, we see that for any $m > n$ we have

$$\mu\left(\left[\sum_{i=1}^n a_i 2^{-i}, \sum_{i=1}^n a_i 2^{-i} + 2^{-n-f(n)}\right) \cap T^{-m}(E_m)\right) \leq 2^{-m-f(m)} + 2^{-n-f(n)-f(m)}.$$

Now summing over all $(a_i) \in \{0,1\}^n$ we obtain

$$\mu(T^{-n}(E_n) \cap T^{-m}(E_m)) \leq 2^{n-m-f(m)} + 2^{-f(n)-f(m)}.$$

Using this estimate we have

$$\sum_{n=1}^Q \sum_{m=1}^Q \mu(T^{-n}(E_n) \cap T^{-m}(E_m)) = \sum_{n=1}^Q \mu(E_n) + 2\sum_{n=1}^{Q-1} \sum_{m=n+1}^Q \mu(T^{-n}(E_n) \cap T^{-m}(E_m))$$

$$
\begin{aligned}
&\le \sum_{n=1}^{Q}\mu(E_n) + 2\sum_{n=1}^{Q-1}\sum_{m=n+1}^{Q} 2^{n-m-f(m)} + 2\sum_{n=1}^{Q-1}\sum_{m=n+1}^{Q} 2^{-f(n)-f(m)} \\
&= \sum_{n=1}^{Q}\mu(E_n) + 2\sum_{m=2}^{Q}\sum_{n=1}^{m-1} 2^{n-m-f(m)} + \sum_{\substack{1\le n,m\le Q\\ n\neq m}} \mu(E_n)\mu(E_m) \\
&\le \sum_{n=1}^{Q}\mu(E_n) + 2\sum_{m=2}^{Q} 2^{-f(m)} + \sum_{n=1}^{Q}\sum_{m=1}^{Q}\mu(E_n)\mu(E_m) \\
&= \left(\sum_{n=1}^{Q}\mu(E_n)\right)^2 + \mathcal{O}\left(\sum_{n=1}^{Q}\mu(E_n)\right).
\end{aligned}
$$

Now substituting this upper bound into Lemma 2 and using the invariance of μ we have

$$
\mu(W(E)) = \mu\left(\limsup_{n\to\infty} T^{-n}(E_n)\right) \ge \limsup_{Q\to\infty} \frac{\left(\sum_{n=1}^{Q}\mu(E_n)\right)^2}{\left(\sum_{n=1}^{Q}\mu(E_n)\right)^2 + \mathcal{O}\left(\sum_{n=1}^{Q}\mu(E_n)\right)} = 1.
$$

This completes our proof. In the final line we used that $\sum_{n=1}^{\infty}\mu(E_n) = \infty$ to ensure that the $\left(\sum_{n=1}^{Q}\mu(E_n)\right)^2$ term dominates in the denominator. □

It is not always the case that the lower bound provided by Lemma 2 is 1. Sometimes all it provides us with is that the limsup set has positive measure. To upgrade this statement from positive measure to full measure we need an extra tool/idea. In many cases a sufficient tool is the following result known as the Lebesgue density theorem.

Theorem 2 (Lebesgue density theorem)**.** *Let $A \subset \mathbb{R}^n$ be a measurable set. Then for Lebesgue almost every $x \in A$ we have*

$$
\lim_{r\to 0} \frac{\mu(A \cap B(x,r))}{\mu(B(x,r))} = 1.
$$

Notice that Theorem 2 only says something meaningful if $\mu(A) > 0$. A useful corollary of the Lebesgue density theorem is the following statement whose proof we leave as an exercise.

Exercise 2. *Let $A, B \subset \mathbb{R}^n$ be measurable sets both with positive Lebesgue measure. Suppose that that there exists $c > 0$ such that for any $x \in A$ we have*

$$
\mu(B \cap B(x,r)) \ge c\mu(B(x,r))
$$

for all r sufficiently small. Then $\mu(A \cap B) = \mu(A)$.

Exercise 3. *Prove Theorem 1 for an arbitrary sequence of intervals.*

Lecture 2

In this section we will explore some connections between Diophantine Approximation and Fractal Geometry. We start with an extremely brief introduction to these topics.

Diophantine Approximation

Diophantine Approximation is the study of how well real numbers can be approximated by rational numbers. With this in mind, the following setup is natural: Given a function $\psi : \mathbb{N} \to [0, \infty)$ one associates the set

$$W_\psi := \{x \in \mathbb{R} : |x - p/q| \leq \psi(q) \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N}\}.$$

Notice that W_ψ has a simple dynamical interpretation:

$$\begin{aligned} x \in W_\psi &\iff |x - p/q| \leq \psi(q) \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \\ &\iff d(qx, \mathbb{Z}) \leq q\psi(q) \text{for i.m. } q \in \mathbb{Z} \\ &\iff T_x^q(0) \in [0, q\psi(q)] \cup [1 - q\psi(q), 1) \text{for i.m. } q \in \mathbb{Z}. \end{aligned}$$

Where in the last line $T_x : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the rotation of the circle given by $z \to z + x \mod 1$. This means that x being contained in W_ψ can be reinterpreted in terms of 0 being contained in a shrinking target set where the underlying dynamics are provided by T_x. Adapting this argument, it can be shown that the shrinking target problem for the doubling map can be reinterpreted in terms of restricted Diophantine Approximation where the denominators of the rational approximations are contained in the set $\{2^n\}_{n=1}^\infty$.

The following result gives a simple criteria for the Lebesgue measure of W_ψ.

Theorem 3. *The following statements are true:*

1. *Suppose* $\sum_{q=1}^\infty q\psi(q) < \infty$. *Then* $\mu(W_\psi) = 0$.
2. *Suppose* ψ *is decreasing and* $\sum_{q=1}^\infty q\psi(q) = \infty$. *Then Lebesgue almost every* $x \in \mathbb{R}$ *is contained in* W_ψ.

The first of these statements is straightforward and left as an exercise. The second statement is a deeper result due to Khintchine [15]. Interestingly, it is not possible to remove the monotonicity assumption from the second statement. This follows from an example due to Duffin and Schaeffer [8]. This example led to the so called Duffin-Schaeffer conjecture which was proved recently by Koukoulopoulos and Maynard [16].

Exercise 4. *Prove statement* 1 *in the above.*

Given $\tau \geq 2$ we define

$$W_\tau := \{x \in \mathbb{R} : 0 < |x - p/q| \leq q^{-\tau} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N}\}.$$

We refer to an element of $\cup_{\tau>2} W_\tau$ as well approximable. By Theorem 3 it follows that $\mu(W_\tau) = 0$ for any $\tau > 2$ and therefore $\mu(\cup_{\tau>2} W_\tau) = 0$. As we will see, despite being of measure 0 the sets W_τ are still significant in terms of Hausdorff dimension. We define the set of Liouville numbers to be

$$L = \bigcap_{\tau \geq 2} W_\tau.$$

Exercise 5. 1. *Prove that* L *is nonempty.*

2. *Prove that any element of L is a transcendental number, i.e. it is not the zero of any integer polynomial.*[1]

3. *Prove that $W_2 = \mathbb{R}$.*

An extension of part 2. in the exercise above was obtained by Roth in [19]. He proved that if $x \in W_\tau$ for some $\tau > 2$ then x is transcendental. This result significantly contributed to Roth being awarded the Fields medal in 1958.

Fractal Geometry

We call a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ a similarity if there exists $r \in (0,1)$ such that $\|\varphi(x)-\varphi(y)\| = r\|x-y\|$ for all $x, y \in \mathbb{R}^n$. We call a finite set of similarities an iterated function system of IFS for short. Given an IFS $\{\varphi_i\}_{i\in\mathcal{I}}$ a well known result due to Hutchinson [13] states that there exists a unique non-empty compact set X satisfying

$$X = \bigcup_{i\in\mathcal{I}} \varphi_i(X).$$

X is called the self-similar set of the IFS. Many well known fractal sets can be realised as self-similar sets for appropriate choices of IFS. For instance the middle third Cantor set

$$C := \left\{ \sum_{i=1}^{\infty} \frac{a_i}{3^i} : (a_i) \in \{0,2\}^{\mathbb{N}} \right\}$$

is the self-similar set for $\{\varphi_0(x) = x/3, \varphi_2(x) = (x+2)/3\}$.

To describe the metric properties of fractal sets we require the notions of Hausdorff measure and Hausdorff dimension. These are defined as follows: Let $X \subset \mathbb{R}^n$. Given $\gamma > 0$ and $\epsilon > 0$ we define

$$\mathcal{H}^{\gamma}_{\epsilon}(X) := \inf \left\{ \sum_{i=1}^{\infty} Diam(U_i)^{\gamma} : \{U_i\} \text{ is an } \epsilon\text{-cover of } X \right\}.$$

Clearly $\mathcal{H}^{\gamma}_{\epsilon}(X)$ is decreasing with ϵ. As such we can define the γ-dimensional Hausdorff measure to be

$$\mathcal{H}^{\gamma}(X) := \lim_{\epsilon\to 0} \mathcal{H}^{\gamma}_{\epsilon}(X).$$

We then define the Hausdorff dimension of X to be

$$\dim_H(X) := \sup\{\gamma > 0 : \mathcal{H}^{\gamma}(X) = \infty\} = \inf\{\gamma > 0 : \mathcal{H}^{\gamma}(X) = 0\}.$$

Exercise 6. *Prove the equality* $\sup\{\gamma > 0 : \mathcal{H}^{\gamma}(X) = \infty\} = \inf\{\gamma > 0 : \mathcal{H}^{\gamma}(X) = 0\}$ *stated above.*

Exercise 7. *Let (E_n) be a sequence of sets in $\mathbb{R}^n$ such that*

$$\sum_{n=1}^{\infty} Diam(E_n)^{\gamma} < \infty.$$

Prove that

$$\dim_H\left(\limsup_{n\to\infty} E_n\right) \leq \gamma.$$

[1] This fact was first established in 1844 by Joseph Liouville. Before this point it was not actually known whether transcendental numbers existed.

For more on Fractal Geometry and self-similar sets we refer the reader to the books by Falconer [9, 10].

Diophantine Approximation on Fractals

Often a self-similar set X will have empty interior and be of zero Lebesgue measure. As such determining metric information for sets of the form $X \cap W_\psi$ is a non trivial task. Some natural questions one might ask are:

1. Does an analogue of Theorem 3 hold for self-similar sets? Here the relevant measure could be the $\dim_H(X)$-dimensional Hausdorff measure restricted to X or a self-similar measure (see [9, 10] for the definition of a self-similar measure).
2. What is $\dim_H(X \cap W_\tau)$ for $\tau > 2$?

Many important results have been obtained in recent years that provide partial answers to these questions (see [1, 2, 7, 11, 12, 14, 20, 22, 23] and the references therein). The methods used in these papers combine ideas from Dynamical Systems, Fourier Analysis, and Number Theory.

In this section we will discuss some work of Levesley, Salp, and Velani [17] which gives a lower bound for $\dim_H(C \cap W_\tau)$ where C is the middle third Cantor set. This lower bound verifies an assertion attributed to Mahler who claimed without proof that the middle third Cantor set contains well-approximable numbers that are not Liouville. Our proof of this bound will exhibit an extremely powerful result in this area known as the mass transference principle. This result was proved by Beresnevich and Velani in [5]. It can be applied to obtain metric information for many limsup sets. We give a formulation of this result below that is well suited to our purposes.

We consider compact sets $X \subset \mathbb{R}^n$ such that for any sufficiently small ball $B = B(x, r)$ centred at a point in X we have

$$c_1 r^{\dim_H(X)} \leq \mathcal{H}^{\dim_H(X)}(B \cap X) \leq c_2 r^{\dim_H(X)}. \tag{3}$$

Here $c_1, c_2 > 0$. Given $s \in (0, \dim_H(X))$ and a ball $B = B(x, r)$ we define

$$B^s := B(x, r^{s/\dim_H(X)}).$$

We have the following result.

Theorem 4 (Mass transference principle)**.** *Let $X \subset \mathbb{R}^n$ be a compact set satisfying* (3)*. Let $s \in (0, \dim_H(X))$ and $\{B_i\}$ be a sequence of balls centred at points in X such that $radius(B_i) \to 0$. Suppose that for any ball B centred in X we have*

$$\mathcal{H}^{\dim_H(X)}(B \cap \limsup_{i\to\infty} B_i^s) = \mathcal{H}^{\dim_H(X)}(B \cap X).$$

Then for any ball B centred in X we have

$$\mathcal{H}^s(B \cap \limsup_{i\to\infty} B_i) = \infty.$$

In applications the mass transference principle is often applied as follows:

- Suppose you have a sequence of balls $\{B_i\}$ for which you want to calculate a lower bound for $\dim_H(\limsup_{i\to\infty} B_i)$.
- Often there is a natural guess s for a lower bound for $\dim_H(\limsup_{i\to\infty} B_i)$[2]. Using this value of s we scale the balls to obtain $\{B_i^s\}$.
- We then show that $\mathcal{H}^{\dim_H(X)}(B\cap \limsup_{i\to\infty} B_i^s)=\mathcal{H}^{\dim_H(X)}(B\cap X)$ for any ball B centred at X. By Theorem 4 this implies that $\dim_H(\limsup_{i\to\infty} B_i)\geq s$. Importantly, it is often a much simpler task to establish $\mathcal{H}^{\dim_H(X)}(B\cap \limsup_{i\to\infty} B_i^s)=\mathcal{H}^{\dim_H(X)}(B\cap X)$. This is because by (3) we known that $\mathcal{H}^{\dim_H(X)}|_X$ is a relatively nice measure. This approach will often in fact yield the exact value for $\dim_H(\limsup_{i\to\infty} B_i)$.

Exercise 8. *Using Theorem 4 prove that* $\dim_H(W_\tau)=2/\tau$ *for all* $\tau\geq 2$.

Exercise 9. *Let* $T:\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ *be the doubling map. Using Theorems 1 and 4 prove that*

$$\dim_H\left(\left\{x : T^n(x)\in\left[0,\frac{1}{\gamma^n}\right) \text{ for i.m. } n\in\mathbb{N}\right\}\right)=\frac{\log 2}{\log 2\gamma}$$

for all $\gamma\geq 1$.

We will now use the mass transference principle to prove Mahler's assertion.

Theorem 5. *The middle third Cantor set contains well approximable numbers that are not Liouville. In fact, for any* $\tau>2$ *we have* $\dim_H(C\cap W_\tau)\geq \frac{\log 2}{\tau\log 3}$.

Proof. We begin our proof by asserting without proof that C satisfies (3)[3] and $\dim_H(C)=\log 2/\log 3$. Notice that

$$\limsup_{n\to\infty}\bigcup_{(a_i)\in\{0,2\}^n}\left(\sum_{i=1}^n\frac{a_i}{3^i}-1/3^n+\sum_{i=1}^n\frac{a_i}{3^i}+1/3^n\right)$$

equals C modulo a countable set of endpoints. Therefore for any ball B centred at X we have

$$\mathcal{H}^{\log 2/\log 3}\left(B\cap\limsup_{n\to\infty}\bigcup_{(a_i)\in\{0,2\}^n}\left(\sum_{i=1}^n\frac{a_i}{3^i}-1/3^n+\sum_{i=1}^n\frac{a_i}{3^i}+1/3^n\right)\right)=\mathcal{H}^{\log 2/\log 3}(B\cap X). \quad (4)$$

For $\tau>2$ consider the set

$$C_\tau=\limsup_{n\to\infty}\bigcup_{(a_i)\in\{0,2\}^n}\left(\sum_{i=1}^n\frac{a_i}{3^i}-1/3^{\tau n}+\sum_{i=1}^n\frac{a_i}{3^i}+1/3^{\tau n}\right).$$

We emphasise that $C_\tau\subset C\cap W_\tau$. Now notice that

$$B^{\frac{\log 2}{\tau\log 3}}\left(x,\frac{1}{3^{\tau n}}\right)=B\left(x,\left(\frac{1}{3^{\tau n}}\right)^{\frac{\log 2/\tau\log 3}{\log 2/\log 3}}\right)=B\left(x,\frac{1}{3^n}\right).$$

Now by (4) and Theorem 4 we have $\mathcal{H}^{\frac{\log 2}{\tau\log 3}}(C_\tau\cap B)=\infty$ for any ball B centred at a point in C. In particular this means $\dim_H(C_\tau)\geq\frac{\log 2}{\tau\log 3}$ and therefore $\dim_H(C\cap W_\tau)\geq\frac{\log 2}{\tau\log 3}$. □

[2]This guess is often given by the divergence of some natural sum. See Exercise 7.

[3]This assertion can be checked by using the facts $0<\mathcal{H}^{\dim_H(C)}(C)<\infty$ and $\mathcal{H}^{\dim_H(C)}(r\cdot C+t)=r^{\dim_H(X)}\cdot\mathcal{H}^{\dim_H(C)}(C)$ for any $r>0$ and $t\in\mathbb{R}$.

Lecture 3

In this section we will discuss some recent work of the author from [3]. This work is motivated in part by the results of the previous sections, but also by the study of overlapping iterated function systems. Such systems are difficult to understand and we are far from a complete picture for how they behave.

Throughout this section we will focus on one particular family of overlapping IFSs: Given $\lambda \in (1/2, 1)$ let

$$\Phi_\lambda := \{\varphi_0(x) = \lambda x, \varphi_1(x) = \lambda x + \lambda\}.$$

It is a simple exercise to show that for this IFS the self-similar set is $I_\lambda := [0, \frac{\lambda}{1-\lambda}]$. The goal for this section will be to provide some insight into how to prove the following statement.

Theorem 6. *For Lebesgue almost every $\lambda \in (1/2, 0.668$*[4] *$\ldots)$, Lebesgue almost every $x \in [0, \frac{\lambda}{1-\lambda}]$ is contained in the set*

$$W_\lambda := \left\{ x \in \mathbb{R} : \left| x - \sum_{i=1}^{n} a_i \lambda^i \right| \leq \frac{1}{2^n \cdot n} \text{ for i.m. } (a_i)_{i=1}^n \in \cup_{m=1}^{\infty} \{0,1\}^m \right\}.$$

Before discussing the proof we make some remarks.

- Let $T_0 : \mathbb{R} \to \mathbb{R}$ and $T_1 : \mathbb{R} \to \mathbb{R}$ be given by $T_0(x) = x/\lambda$ and $T_1(x) = x/\lambda - 1$. It is easy to prove that $x \in W_\lambda$ if and only if x belongs to the set

  $$\left\{ x \in \mathbb{R} : (T_{a_1} \circ \cdots \circ T_{a_n})(x) \in B\left(0, \frac{1}{2^n \cdot \lambda^n \cdot n}\right) \text{ for i.m. } (a_i)_{i=1}^n \in \cup_{m=1}^{\infty} \{0,1\}^m \right\}.$$

  As such the set W_λ has a natural dynamical interpretation. The maps T_0 and T_1 might seem artificial but they arise naturally from the study of β-transformations.

- Notice that

  $$\left\{ \sum_{i=1}^{n} a_i \lambda^i : (a_i)_{i=1}^n \in \{0,1\}^n \right\} = \{(\varphi_{a_1} \circ \cdots \circ \varphi_{a_n})(0) : (a_i)_{i=1}^n \in \{0,1\}^n\}.$$

  This observation shows that the definition of W_λ can be generalised to arbitrary IFSs by fixing a point and drawing balls around its level n images. This generalisation is considered in depth in [3].

- It is not possible to replace Lebesgue almost every $\lambda \in (1/2, 0.668\ldots)$ with for every $\lambda \in (1/2, 0.668\ldots)$ in the statement of Theorem 6. There is a dense set of exceptions for which the conclusion of this theorem does not hold. This dense set of exceptions makes the analysis much more difficult. The overarching picture is very different to that seen in the previous section where we analysed the middle third Cantor set. The left end points of the middle third Cantor set are well separated. This makes studying the corresponding limsup set an easier task.

[4] $0.668\ldots$ is a constant that naturally arises from our analysis.

Exercise 10. *1. Prove that the set of* $\lambda \in (1/2, 1)$ *for which there exists* $(b_i)_{i=1}^n \in \bigcup_{m=1}^{\infty}(\{-1,0,1\}^m \setminus \{0^m\})$ *satisfying* $\sum_{i=1}^n b_i\lambda^i = 0$ *is dense in* $(1/2,1)$*.*

2. Prove that for any such λ *we have that the Lebesgue measure of* W_λ *is zero.*

The proof of Theorem 6 relies upon the following three important steps:

1. Show that for almost every $\lambda \in (1/2, 0.668\ldots)$, there exists $c_1, c_2 > 0$ and a large infinite set $A \subset \mathbb{N}$ such that the following holds:

   For each $n \in A$ there exists $S_n \subset \{0,1\}^n$ satisfying $\#S_n \geq c_1 \cdot 2^n$, and for any distinct $(a_i), (b_i) \in S_n$ we have
   $$\left|\sum_{i=1}^n a_i\lambda^i - \sum_{i=1}^n b_i\lambda^i\right| \geq \frac{c_2}{2^n}.$$

2. For each of these typical λ we associate the limsup set
   $$\bigcap_{m=1}^{\infty}\bigcup_{n\in A: n\geq m}\bigcup_{(a_i)\in S_n}\left(\sum_{i=1}^n a_i\lambda^i - \frac{1}{2^n\cdot n}, \sum_{i=1}^n a_i\lambda^i + \frac{1}{2^n \cdot n}\right).$$
   Using statement 1 together with Lemma 2 we prove that this set has positive measure. This implies that W_λ has positive measure.

3. Use the IFS Φ_λ and a measure theory argument to upgrade this conclusion from positive measure to full measure.

We outline these steps in the next three sections. Before giving this argument it is useful to make a minor simplification. To prove Theorem 6 it is sufficient to show that its conclusion holds for almost every $\lambda \in [\lambda_0, \lambda_1]$ where $1/2 < \lambda_0 < \lambda_1 < 0.668\ldots$. In what follows we fix such a λ_0 and λ_1, and show that the three steps outlined above hold in the interval $[\lambda_0, \lambda_1]$.

Step 1. Constructing a well separated set

To any $\lambda \in [\lambda_0, \lambda_1]$, $n \in \mathbb{N}$, and $s > 0$ we associate the following set:

$$P(\lambda, s, n) := \left\{((a_i),(b_i)) \in \{0,1\}^n \times \{0,1\}^n : (a_i) \neq (b_i) \text{ and } \left|\sum_{i=1}^n a_i\lambda^i - \sum_{i=1}^n b_i\lambda^i\right| < \frac{s}{2^n}\right\}$$

The significance of $P(\lambda, s, n)$ in constructing a well separated set follows from the following lemma.

Lemma 3. *Suppose* $\#P(\lambda, s, n) \leq c_1 \cdot 2^n$*. Then there exists* $S_n \subset \{0,1\}^n$ *such that* $\#S_n \geq (1-c_1)\cdot 2^n$ *and* $|\sum_{i=1}^n a_i\lambda^i - \sum_{i=1}^n b_i\lambda^i| \geq \frac{s}{2^n}$ *for all distinct* $(a_i), (b_i) \in S_n$*.*

Proof. Define $f : P(\lambda, s, n) \to \{0,1\}^n$ by $f((a_i),(b_i)) = (a_i)$. This is a surjective map onto those (a_i) for which there exists a distinct (b_i) satisfying $|\sum_{i=1}^n a_i\lambda^i - \sum_{i=1}^n b_i\lambda^i| < \frac{s}{2^n}$. By our assumptions this set has cardinality less than or equal to $c_1 \cdot 2^n$. Taking S_n to be the compliment of this set our result now follows. $\square$

It follows from Lemma 3 that to construct a well separated set at level n it suffices to bound $\#P(\lambda,s,n)$ from above. This is achieved in the following lemmas. The first of these lemmas is a result due to Solomyak that was proved in [21]. The second lemma was proved in [4] by Benjamini and Solomyak and is based upon an argument given in [18] by Peres and Solomyak.

Lemma 4 ([21]). *There exists $C_1>0$ such that for any $g:[\lambda_0,\lambda_1]\to\mathbb{R}$ of the form*

$$g(x)=1+\sum_{i=1}^{\infty}a_ix^i$$

for some $(a_i)\in\{-1,0,1\}^{\infty}$, we have

$$\mu\left(x\in[\lambda_0,\lambda_1]:|g(x)|\leq\delta\right)\leq C_1\delta$$

for any $\delta>0$.

Lemma 4 does not hold throughout $(1/2,1)$. It is possible to have double roots. This is why we need to restrict our attention to $(1/2,0.668\ldots)$.

Lemma 5 ([4]). *There exists $C>0$ such that for all $n\in\mathbb{N}$ and $s>0$ we have*

$$\int_{\lambda_0}^{\lambda_1}\#P(\lambda,s,n)\,d\lambda\leq Cs2^n.$$

Proof. We have the following:

$$\begin{aligned}
\int_{\lambda_0}^{\lambda_1}\#P(\lambda,s,n)\,d\lambda &= \sum_{(a_i)\in\{0,1\}^n}\sum_{(b_i)\in\{0,1\}^n,(b_i)\neq(a_i)}\int_{\lambda_0}^{\lambda_1}\chi_{[-s/2^n,s/2^n]}\left(\sum_{i=1}^{n}a_i\lambda^i-\sum_{i=1}^{n}b_i\lambda^i\right)d\lambda\\
&=\sum_{(a_i)\in\{0,1\}^n}\sum_{k=1}^{n}\sum_{\substack{(b_i)\in\{0,1\}^n\\ \inf\{i:a_i\neq b_i\}=k}}\int_{\lambda_0}^{\lambda_1}\chi_{[-s/2^n,s/2^n]}\left(\sum_{i=1}^{n}a_i\lambda^i-\sum_{i=1}^{n}b_i\lambda^i\right)d\lambda\\
&=\sum_{(a_i)\in\{0,1\}^n}\sum_{k=1}^{n}\sum_{\substack{(b_i)\in\{0,1\}^n\\ \inf\{i:a_i\neq b_i\}=k}}\int_{\lambda_0}^{\lambda_1}\chi_{[-s/2^n,s/2^n]}\left(\sum_{i=k}^{n}a_i\lambda^i-\sum_{i=k}^{n}b_i\lambda^i\right)d\lambda\\
&\leq\sum_{(a_i)\in\{0,1\}^n}\sum_{k=1}^{n}\sum_{\substack{(b_i)\in\{0,1\}^n\\ \inf\{i:a_i\neq b_i\}=k}}\int_{\lambda_0}^{\lambda_1}\chi_{[-s\lambda^{-k}/2^n,s\lambda^{-k}/2^n]}\left(\sum_{i=k}^{n}(a_i-b_i)\lambda^{i-k}\right)d\lambda\\
&\leq\sum_{(a_i)\in\{0,1\}^n}\sum_{k=1}^{n}\sum_{\substack{(b_i)\in\{0,1\}^n\\ \inf\{i:a_i\neq b_i\}=k}}\int_{\lambda_0}^{\lambda_1}\chi_{[-s\lambda_0^{-k}/2^n,s\lambda_0^{-k}/2^n]}\left(\sum_{i=k}^{n}(a_i-b_i)\lambda^{i-k}\right)d\lambda\\
&\overset{\text{Lemma 4}}{\ll}\sum_{(a_i)\in\{0,1\}^n}\sum_{k=1}^{n}\sum_{\substack{(b_i)\in\{0,1\}^n\\ \inf\{i:a_i\neq b_i\}=k}}\frac{s\lambda_0^{-k}}{2^n}\\
&=s\sum_{(a_i)\in\{0,1\}^n}\sum_{k=1}^{n}2^{n-k}\cdot\frac{\lambda_0^{-k}}{2^n}
\end{aligned}$$

$$
\begin{aligned}
&= s \sum_{(a_i)\in\{0,1\}^n} \sum_{k=1}^{n} \frac{\lambda_0^{-k}}{2^k} \\
&\ll s \sum_{(a_i)\in\{0,1\}^n} 1 \\
&= s2^n.
\end{aligned}
$$

□

Lemma 6. *Given $\epsilon > 0$ there exists $c_1 > 0$ depending upon ϵ such that for each $n \in \mathbb{N}$, for all $\lambda \in [\lambda_0, \lambda_1]$ outside of a set of measure ϵ there exists $S_n \subset \{0,1\}^n$ satisfying:*

- *$\#S_n \geq 2^{n-1}$.*
- *For distinct $(a_i), (b_i) \in S_n$ we have*

$$\left| \sum_{i=1}^{n} a_i \lambda^i - \sum_{i=1}^{n} b_i \lambda^i \right| \geq \frac{c_1}{2^n}.$$

Proof. Let $\epsilon > 0$ and $n \in \mathbb{N}$. By Markov's inequality and taking $s = \frac{\epsilon}{2C}$ in Lemma 4, we have

$$2^{n-1}\mu(\lambda \in [\lambda_0, \lambda_1] : \#P(\lambda, \epsilon/2C, n) \geq 2^{n-1}) \leq \epsilon 2^{n-1}.$$

Therefore

$$\mu(\lambda \in [\lambda_0, \lambda_1] : \#P(\lambda, \epsilon/2C, n) \geq 2^{n-1}) \leq \epsilon.$$

It now follows by Lemma 3 that for all $\lambda \in [\lambda_0, \lambda_1]$ outside of a set of measure ϵ there exists $S_n \subset \{0,1\}^n$ satisfying

- $\#S_n \geq 2^{n-1}$.
- For distinct $(a_i), (b_i) \in S_n$ we have

$$\left| \sum_{i=1}^{n} a_i \lambda^i - \sum_{i=1}^{n} b_i \lambda^i \right| \geq \frac{\epsilon}{2C \cdot 2^n}.$$

Our result follows now by taking $c_1 = \frac{\epsilon}{2C}$. □

To prove Theorem 6 it is necessary to strengthen the conclusion of Lemma 6. By applying additional probabilistic arguments and Fatou's lemma, it is possible to show that the following result holds.

Lemma 7. *Given $\epsilon > 0$ there exists $c_1 > 0$ depending upon ϵ, such that for all $\lambda \in [\lambda_0, \lambda_1]$ outside of a set of measure ϵ there exists $A \subset \mathbb{N}$ for which the following properties are satisfied:*

1. *$\sum_{n\in A} 1/n = \infty$*
2. *For each $n \in A$ there exists $S_n \subset \{0,1\}^n$ satisfying:*
   (a) *$\#S_n \geq 2^{n-1}$.*

(b) For distinct $(a_i),(b_i)\in S_n$ we have

$$\left|\sum_{i=1}^{n} a_i\lambda^i - \sum_{i=1}^{n} b_i\lambda^i\right| \geq \frac{c_1}{2^n}.$$

We do not prove Lemma 7 in these notes. The following exercise is a little easier.

Exercise 11. *Using Lemma 6 prove the following statement:*

Given $\epsilon>0$ there exists $c_1>0$ depending upon ϵ, such that for all $\lambda\in[\lambda_0,\lambda_1]$ outside of a set of measure ϵ there exists infinitely many $n\in\mathbb{N}$ for which the following properties are satisfied:

1. *$\#S_n \geq 2^{n-1}$.*
2. *For distinct $(a_i),(b_i)\in S_n$ we have*

$$\left|\sum_{i=1}^{n} a_i\lambda^i - \sum_{i=1}^{n} b_i\lambda^i\right| \geq \frac{c_1}{2^n}.$$

Exercise 12. *Using the previous exercise prove the following statement:*

For Lebesgue almost every $\lambda\in(1/2, 0.668\ldots)$, the following set has positive measure

$$W_\lambda := \left\{x\in\mathbb{R} : \left|x-\sum_{i=1}^{n} a_i\lambda^i\right| \leq \frac{1}{2^n} \text{ for i.m. } (a_i)_{i=1}^n \in \cup_{m=1}^{\infty}\{0,1\}^m\right\}.$$

Note that this set has zero measure for a dense set of parameters.

Exercise 13. *We can define a measure μ_λ on $[0,\frac{\lambda}{1-\lambda}]$ to be the weak star limit of the measures*

$$\mu_{\lambda,n} := \frac{1}{2^n}\sum_{(a_i)\in\{0,1\}^n} \delta_{\sum_{i=1}^n a_i\lambda^i}.$$

The measure μ_λ is known as the Bernoulli convolution and is a well studied object in analysis. Using Exercise 11 prove that for Lebesgue almost every $\lambda\in(1/2,0.668\ldots)$ the measure μ_λ is absolutely continuous. Hint: You may use here without proof the fact that μ_λ is of pure type, i.e. it is either absolutely continuous with respect to the Lebesgue measure or it is singular.[5]

Step 2. Applying Lemma 2.

Throughout this section we let $\lambda\in[\lambda_0,\lambda_1]$ be such that there exists $c_1>0$ and $A\subset\mathbb{N}$ such that the conclusion of Lemma 7 holds. Now for each $n\in A$ we define

$$E_n := \bigcup_{(a_i)\in S_n} B\left(\sum_{i=1}^{n} a_i\lambda^i, \frac{c_1}{n\cdot 2^n}\right).$$

[5] An important result due to Solomyak states that for Lebesgue almost every $\lambda\in(1/2,1)$ the measure μ_λ is absolutely continuous [21]. With a little more work it is possible to upgrade the conclusion of this exercise to conclude Solomyak's result.

By Lemma 7, for each $n \in A$ we have

$$\mu(E_n) \asymp \frac{1}{n}.$$

Therefore by the first statement in Lemma 7 we know that $\sum_{n\in A} \mu(E_n) = \infty$. This is important because it shows that the divergence assumption of Lemma 2 is satisfied. The following lemma will allow us to use Lemma 2 to conclude that W_λ has positive measure.

Lemma 8. *Let $n, m \in A$ be such that $n < m$. Then we have*

$$\mu(E_n \cap E_m) \ll \left(\frac{1}{nm} + \frac{1}{m \cdot 2^{m-n}}\right).$$

Proof. Let $(a_i) \in S_n$ and consider

$$\mu\left(B\left(\sum_{i=1}^{n} a_i \lambda^i, \frac{c_1}{n \cdot 2^n}\right) \cap E_m\right).$$

Each ball in E_m is separated by a factor at least $c_1/2^m$. Therefore by a volume argument $B\left(\sum_{i=1}^{n} a_i\lambda^i, \frac{c_1}{n\cdot 2^n}\right)$ can only intercept $\ll \frac{2^m}{n\cdot 2^n} + 1$ balls in E_m. Each of these balls has measure of the order $\frac{1}{m\cdot 2^m}$. Therefore

$$\mu\left(B\left(\sum_{i=1}^{n} a_i \lambda^i, \frac{c_1}{n \cdot 2^n}\right) \cap E_m\right) \ll \frac{1}{2^n \cdot n \cdot m} + \frac{1}{m \cdot 2^m}$$

Now summing over all elements in S_n we have

$$\mu(E_n \cap E_m) \ll \frac{1}{n.m} + \frac{1}{m \cdot 2^{m-n}}.$$

□

Proof that W_λ has positive measure. For the λ we have fixed in this section we have the following:

$$\begin{aligned}
&\mu(W_\lambda)\\
&\geq \mu(\limsup_{n\to\infty} E_n)\\
&\geq \limsup_{Q\to\infty} \frac{\left(\sum_{n\in A\cap\{1,\ldots,Q\}} \mu(E_n)\right)^2}{\sum_{n\in A\cap\{1,\ldots,Q\}} \sum_{m\in A\cap\{1,\ldots,Q\}} \mu(E_n\cap E_m)}\\
&\overset{Lemma\, 8}{\gg} \limsup_{Q\to\infty} \frac{\left(\sum_{n\in A\cap\{1,\ldots,Q\}} \frac{1}{n}\right)^2}{\sum_{n\in A\cap\{1,\ldots,Q\}} \frac{1}{n} + \sum_{n\in A\cap\{1,\ldots,Q\}} \sum_{m\in A\cap\{1,\ldots,Q\}:m>n} \left(\frac{1}{nm} + \frac{1}{m\cdot 2^{m-n}}\right)}\\
&= \limsup_{Q\to\infty} \frac{\left(\sum_{n\in A\cap\{1,\ldots,Q\}} \frac{1}{n}\right)^2}{\sum_{n\in A\cap\{1,\ldots,Q\}} \frac{1}{n} + \sum_{n\in A\cap\{1,\ldots,Q\}} \sum_{m\in A\cap\{1,\ldots,Q\}:m>n} \frac{1}{nm} + \sum_{m\in A\cap\{1,\ldots,Q\}} \sum_{n\in A\cap\{1,\ldots,Q\}:n<m} \frac{1}{m\cdot 2^{m-n}}}\\
&\gg \limsup_{Q\to\infty} \frac{\left(\sum_{n\in A\cap\{1,\ldots,Q\}} 1/n\right)^2}{\sum_{n\in A\cap\{1,\ldots,Q\}} \frac{1}{n} + \left(\sum_{n\in A\cap\{1,\ldots,Q\}} 1/n\right)^2}
\end{aligned}$$

$= 1.$

In the last line we used that $(\sum_{n\in A\cap\{1,\dots,Q\}} 1/n)^2$ dominates $\sum_{n\in A\cap\{1,\dots,Q\}} 1/n$ as $Q \to \infty$. This is a consequence of $\sum_{n\in A} \frac{1}{n} = \infty$. We've shown that $\mu(W_\lambda) \gg 1$ and therefore W_λ has positive Lebesgue measure. □

Step 3. Upgrading from positive measure to full measure.

By the arguments given in the previous sections, we have established that W_λ has positive Lebesgue measure for almost every $\lambda \in [\lambda_0, \lambda_1]$. The goal of this section is to upgrade this statement to Lebesgue almost every element of $[0, \frac{\lambda}{1-\lambda}]$ is in W_λ. The first step in our proof is the following lemma.

Lemma 9. *Let $B(x_n, r_n)$ be a sequence of balls in $\mathbb{R}^d$ such that $r_n \to 0$ as $n \to \infty$. Then*

$$\mu(\{x : x \in B(x_n, r_n) \text{ for i.m. } n\}) = \mu\left(\bigcap_{0<c<1} \{x : x \in B(x_n, cr_n) \text{ for i.m. } n\}\right)$$

This lemma is essentially a consequence of the Lebesgue density theorem. For a proof see Lemma 1 from [6]. Equipped with this lemma we can now finish our proof of Theorem 6.

Proof. Let $\lambda \in [\lambda_0, \lambda_1]$ be such that the set W_λ has positive measure. By Lemma 9 it follows that the following set has positive measure

$$W_\lambda^* := \bigcap_{0<c<1} \left\{ x \in \mathbb{R} : \left| x - \sum_{i=1}^{n} a_i \lambda^i \right| \leq \frac{c}{2^n \cdot n} \text{ for i.m. } (a_i)_{i=1}^n \in \cup_{m=1}^\infty \{0,1\}^m \right\}.$$

Now consider the set

$$V_\lambda := \bigcup_{n=1}^\infty \bigcup_{(a_i)\in\{0,1\}^n} (\varphi_{a_1} \circ \cdots \circ \varphi_{a_n})(W_\lambda^*).$$

We claim that Lebesgue almost every $x \in [0, \frac{\lambda}{1-\lambda}]$ is contained in V_λ. To see this, let $x \in [0, \frac{\lambda}{1-\lambda}]$ be arbitrary and $(a_i) \in \{0,1\}^{\mathbb{N}}$ be a sequence[6] such that $x = \sum_{i=1}^\infty a_i \lambda^i$. Then for any $n \in \mathbb{N}$ we have

$$\mu\left(B\left(x, \frac{\lambda^{n+1}}{1-\lambda}\right) \cap V_\lambda\right) \geq \mu((\varphi_{a_1} \circ \cdots \circ \varphi_{a_n})(W_\lambda^*)) = \lambda^n \mu(W_\lambda^*).$$

It follows from the above that there are no $x \in [0, \frac{\lambda}{1-\lambda}]$ satisfying

$$\lim_{r\to 0} \frac{\mu(V_\lambda^c \cap B(x,r))}{\mu(B(x,r))} = 1.$$

Therefore by the Lebesgue density theorem almost every $x \in [0, \frac{\lambda}{1-\lambda}]$ is contained in V_λ.

To complete our proof it suffices to show that any $x \in V_\lambda$ is contained in W_λ. As such let us fix $x \in V_\lambda$. By the definition of V_λ there exists $(b_i)_{i=1}^m$ and $y \in W_\lambda^*$ such that $(\varphi_{b_1} \circ \cdots \circ \varphi_{b_m})(y) = x$. Now let

$$c = \frac{1}{(m+1)(2\lambda)^m}.$$

[6] It is a simple exercise to show that every element of $[0, \frac{\lambda}{1-\lambda}]$ admits such a sequence.

Let us suppose $(a_i) \in \{0,1\}^n$ is such that

$$\left| y - \sum_{i=1}^{n} a_i \lambda^i \right| \leq \frac{c}{2^n \cdot n}. \tag{5}$$

Then applying $(\varphi_{b_1} \circ \cdots \circ \varphi_{b_m})$ to both y and $\sum_{i=1}^{n} a_i \lambda^i$ we have

$$\left| x - \sum_{i=1}^{m} b_i \lambda^i - \lambda^m \sum_{i=1}^{n} a_i \lambda^i \right| \leq \frac{c\lambda^m}{2^n \cdot n} = \frac{1}{2^{n+m} n(m+1)} \leq \frac{1}{2^{n+m}(n+m)}.$$

Therefore if we let $(c_i)_{i=1}^{n+m} = (b_1 \ldots b_m a_1 \ldots a_n)$ then we have

$$\left| x - \sum_{i=1}^{n+m} c_i \lambda^i \right| \leq \frac{1}{2^{n+m}(n+m)}. \tag{6}$$

$y \in W_\lambda^*$ and so (5) has infinitely many solutions. It follows that (6) has infinitely many solutions and therefore $x \in W_\lambda$. This completes our proof. □

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