DERIVING THE SIMPLEST GAUGE-STRING DUALITIES (ICMS, JUN'25) - SUPPLEMENTAR[Y]

(1) $\boxed{\text{p. 12}}$ Horizontal Trajectories Def.

$$\phi_s(z_H(t))\left(\frac{d z_H(t)}{dt}\right)^2 > 0.$$

t → real parameter along the curve.

Motivation $\phi_s(z) = dz^2$ on $\mathbb{C}$

$z_H(t) = at + ib$ w/ (a,b) real

Locally this is the behaviour near any non-singular point (i.e. zero or pole).

(a) Near a double pole $\displaystyle \phi_s \underset{u\to 0}{\sim} -\frac{L_k^2}{(2\pi)^2}\frac{du^2}{u^2}$. Then w/ $u = re^{i\theta}$.

L_k → real. +ve.

$$\sim -\frac{L_k^2}{(2\pi)^2}\, d\theta^2$$

→ Concentric circles

The Hor. trajectories (in the vicinity) of the pole are $u_H(t) = \rho e^{it}$

ρ ↳ real

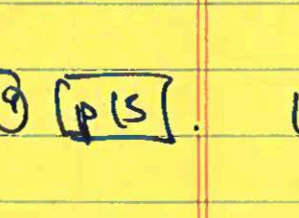
(2) $\boxed{\text{p. 13}}$

(b) Near a zero of order p. i.e. $\phi_s(z)\,dz^2 \sim z^p\, dz^2$.

we have $z_H^{(m)}(t) = ct\, e^{2\pi i m/p+2}$. $(m = 0, 1, \ldots, p+1)$.

i.e. → Hor. traj. i.e. a vertex of valence $(p+2)$. Thus cubic for a simple zero $(p=1)$

$(p=1)$



$\boxed{\text{p. 14}}$

(3) Note that $L_k = \oint \sqrt{\phi_s}\, dz > 0$ (Usual residue)

(4) $\boxed{\text{p. 15}}$. $\displaystyle L_{(ab)} = \int_{z_a}^{z_b} \sqrt{\phi_s(z)}\, dz > 0$. [Since there is a (critical) Hor. trajectory from z_a to z_b

5 [p16] CLAIM: The Strebel graphs (i.e. topologies + Strebel lengths l_{ab}) uniquely parametrise points on the decorated moduli space $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$.

At least, see that the counting of parameters / dimensions works. ~~$\mathcal{M}_{g,n} \times \mathbb{R}$~~ How many l_{ab} does one have (generically)? i.e. no. of Edges of the Strebel Graph. Generically, vertices are cubic. So $2E = 3V$. (Each edge is double counted since it begins & ends on a vertex). Also $F = n$ (# of Faces). $\therefore$ $V - E + F = 2 - 2g$

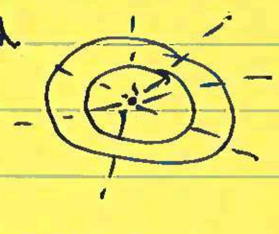
$$\Rightarrow \frac{2}{3}E - E + n = 2 - 2g \Rightarrow E = 6g - 6 + 3n.$$

Since $\dim_{\mathbb{R}}(\mathcal{M}_{g,n}) = 6g - 6 + 2n$, this counting matches that of $\dim_{\mathbb{R}}(\mathcal{M}_{g,n} \times \mathbb{R}_+^n)$. This $\mathbb{R}_+^n$ is nothing but all the L_k $(k = 1 \ldots n)$. Thus for fixed residues $\{L_k\}$, the Strebel lengths (subject to $\sum_{(ab) \in [k]} l_{ab} = L_k$) give a parametrisation of $\mathcal{M}_{g,n}$.

6 [p20] Vert. Trajectories $\phi \left(\frac{dz_v(t)}{dt}\right)^2 < 0$

0. On $\mathbb{C}$, these correspond to $z_v(t) = iat + b$. This is the local behaviour.

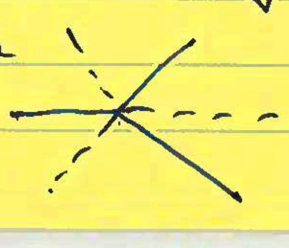
a) Near a Double pole. $u_v(t) \sim t e^{i\theta_0}$ (arbitrary; θ_0 fixed) — radial lines.



b) " a zero.

$$z_v^m(t) = c\,t\,e^{2\pi i m/p+2} \times e^{i\pi m/p+2}.$$

Bisect the angle between (critical) hor. traj.



$\boxed{p.21}$ Can use the Strebel Differential to define a metric on the world sheet.

$$ds^2 = |\phi(z)|\, dz d\bar{z} \qquad \text{— STREBEL GAUGE.}$$

A very special metric — locally flat.

All the curvature is delta-fn. localised at the zeroes + poles of the Strebel differential.

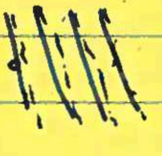
Contribution to $\frac{1}{4\pi}\sqrt{g}R$ of $\delta(v_k)$ at each pole v_k.

and of $-\frac{p}{2}\delta(z_a)$ for a zero of order p.

See examples of explicit Strebel Differentials.

7 [p22] Gluing rules:

(a) Each Strip for a Feynman diag. edge. Assign a local coord. $\phi(z_j)\,dz_j^2 = dz_j^2$. (Assign unit width $0 < \mathrm{Re}\, z_j < 1$)



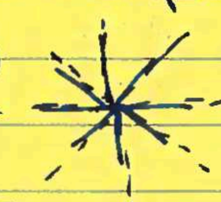
(b) Homotopic edges (multiple Wick contractions) glued together to make a bigger strip of width = Sum of the individual edges (i.e. 1) = # of strips.

$\phi_s(z_j)\,dz_j^2 = \phi_s(z_k)\,dz_k^2 = dz^2$ (i.e. $z_j = z_k + 1$ etc

(c) Vertices of Feynman Graph $\leftrightarrow$ Poles (Marked points

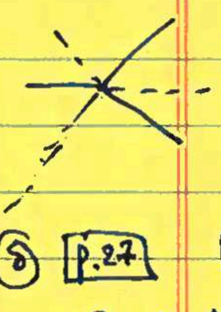
Thus for the op $\mathrm{Tr}\, M^l$, we will have

l strips come together to meet at the pole

($l=8$)  $U_k(z) = e^{2\pi i\left(\frac{j+z_j}{l}\right)}$.

(d). Faces of the Feynman Graph $\leftrightarrow$ Zeroes of the Strebel diff

$W(z_{aj}) = e^{\frac{(j-1)i\pi}{p+2}} (z_j)^{1/m}$.



8 [p.27] Each Wick contraction in terms of 3 permutations.

2 of them are specified by their conjugacy class.

k vertices: $(l_1)(l_2)\dots(l_k)$ - σ_k (Note $\sum l_i = \sum n_j \equiv d$

M " : $(n_1)(n_2)\dots(n_M)$ - σ_M for a non-vanishing corr.)

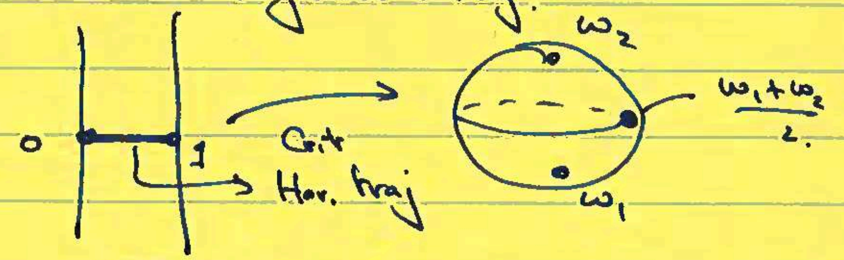
The third cycle carries the information of the specific Wick Contraction (diagram). It is given by $\sigma_F = \sigma_M^{-1}\sigma_K^{-1}$. It carries the information about the faces of the Feynman Diagram.

How do we get these permutations? ~~Assign an overall orientation~~ $\to = d$.
label the edges of the Graph $\{1 \ldots E\}$.
Break this up into cycles for each vertex (in some cyclic fashion). Similarly for all K, M vertices. These give an element of S_d – perm. group. Similarly, for all the faces $\to \sigma_F$.

These three permutations are the 3 branch points (or cycle struct

(9) $\boxed{\text{p. 28}}$ Belyi Maps from ~~[illegible]~~ Feynman Diag.

Can map each strip



$$X(z) = \frac{\omega_2 e^{2\pi i z} + \omega_1}{e^{-2\pi i z} + 1} \qquad \left[X(+i\infty) = \omega_2,\ X(-i\infty) = \omega_1\right]$$

$$X(0) = \frac{\omega_1 + \omega}{2} = X(1)$$

We can glue these strips together as before and the Belyi map is compatible with these gluings.

Interestingly $e^{-2\pi A_{NG}} = \frac{\varepsilon^2}{|\omega_1 - \omega_2|^2}$ (for each strip – Gives Feynman propagator!)

A_{NG} → Nambu-Goto action in Strebel Gauge