Deriving the Simplest Gauge-String Dualities

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Based on:

R.G. and E. Mazenc (hep-th/2212.05999)

R. Kaushik, S. Komatsu, E. Mazenc & D. Sarkar (hep-th/2412.13397)

Also WIP - M. Gaberdiel, R. G., W. Li, E. Mazenc

Outline

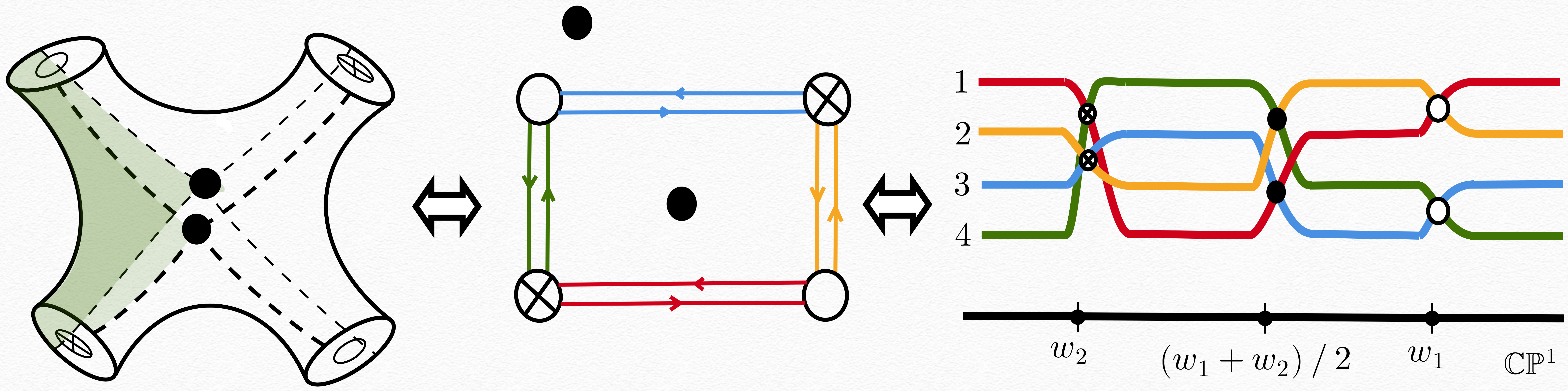
- Motivation and Broad Goals.
- Worldsheets from Feynman Diagrams.
- Target Space from Feynman Diagrams (Belyi Maps).
- The Simplest Gauge-String Dualities (and embedding in AdS/CFT).
- [Open-Closed-Open Trialities]

Motivation and Goals:
The Big Picture(s)

Motivation

- Gauge-String duality a more general phenomenon than AdS/CFT.
- Need to understand its nuts and bolts. Derive it from first principles, at least in some limits e.g. perturbative ($\lambda \ll 1$) large N QFT.
- Simple examples capture some of the bare bones features.
- Aim to extract general lessons for gauge-string duality.
- Often embedded inside more complex AdS/CFT dualities as well.

Big Picture - I



Closed String Worldsheet

Gauge Theory Feynman Diagram

Target Space Covering Map

Feynman diagrams know a lot about the dual closed strings.

Simple Gauge-String Duality

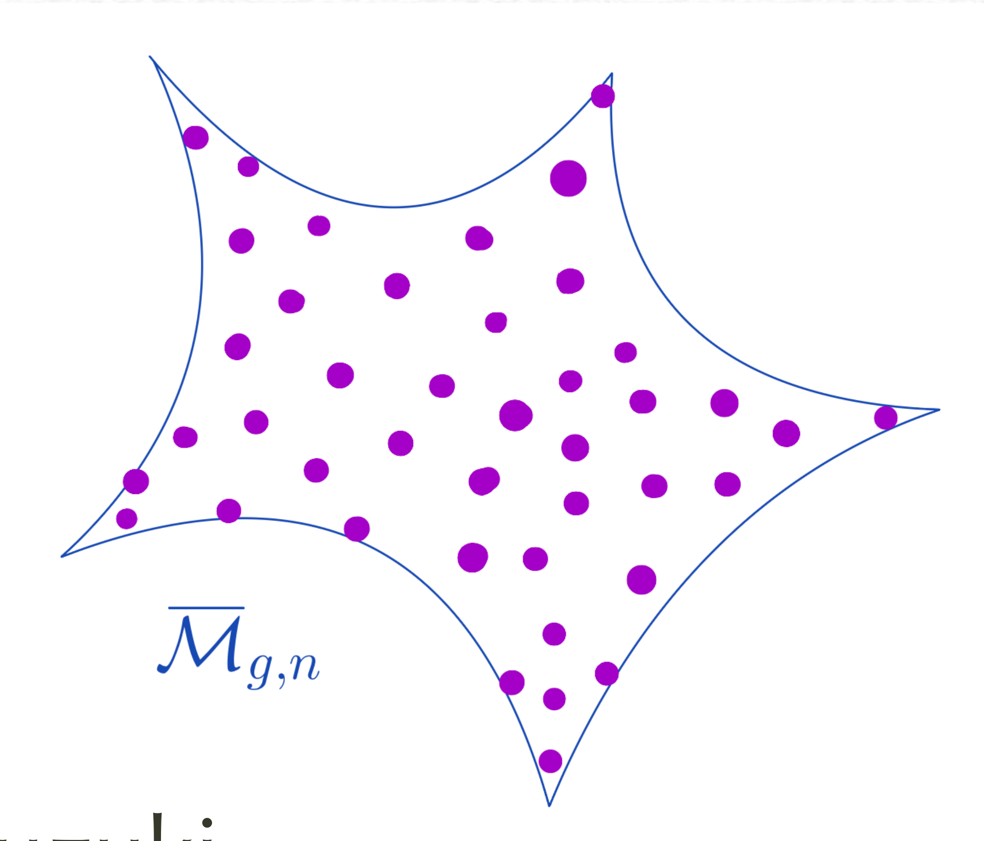
$$\frac{1}{Z_N}\int dKdM_{N\times N}e^{-NTr(KM)}\prod_i Tr(K^{l_i})\prod_j Tr(M^{n_j})$$

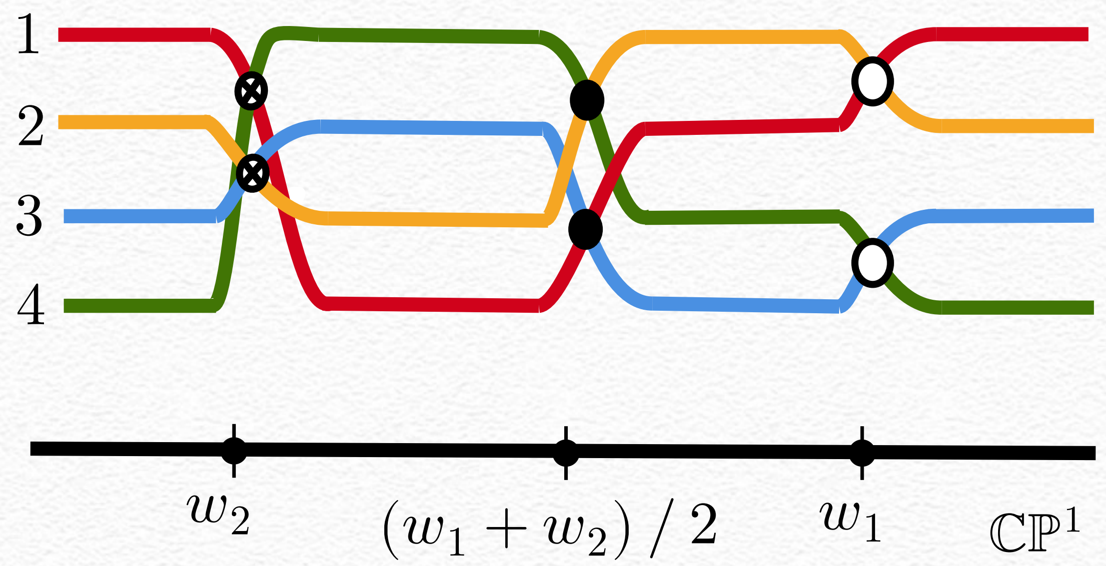
Reconstruct

[R.G, Kaushik, Komatsu, Mazenc, Sarkar '24]

Spacetime: Feynman Diagrams count special holomorphic maps to $\mathbb{P}^1$.

Worldsheet: Feynman Diagrams = Lattice points on moduli sp.





Reproduce

Closed String Dual: $\mathfrak{sl}(2,\mathbb{R})_{k=1}/\mathfrak{u}(1)$ Kazama-Suzuki (Cigar) exhibits these unusual features.

[Gaberdiel, R.G, Li, Mazenc (To appear)]

$$\langle\prod_i \mathscr{V}_{+l_i}\prod_j \mathscr{V}_{-n_j}\rangle$$

[Also relation to c=1 at R=1.]

[R.G, Mazenc '22]

The Simplest Duality Chain Complex

Two Matrix Model:

$$\langle \prod_i Tr(K^{l_i}) \prod_j Tr(M^{n_j}) \rangle_g$$

Open-Closed Duality

[Gaberdiel-R.G. Li-Mazenc]

A-Model: $\frac{SL(2,\mathbb{R})_1}{U(1)}$

$$\langle \prod_i \mathscr{V}_{+l_i} \prod_j \mathscr{V}_{-n_j} \rangle_g$$

Mirror Symmetry

B-Model: Top.L-G theory $W(Z) = \frac{1}{Z}$

$$\langle \prod_i \mathscr{T}_{l_1}(Z) \prod_j \mathscr{T}_{-n_j}(Z) \rangle_g$$

Open-Closed-Open Triality

[R.G.-Mazenc]

[Mukhi-Vafa]

[Nakamura-Niarchos]

[Ashok-Murthy -Troost]

[Ghoshal-Mukhi; Hanany-Oz-Plesser]

Imbimbo-Mukhi Matrix Model

[Cf. Kontsevich-Penner Model Chekhov et.al.; Bonora-Xiong]

[W_∞ identities]

$c = 1$ String theory at self dual radius

[Gross-Klebanov] [Moore-Plesser-Ramgoolam]

[Ghoshal-Vafa]

[Ooguri-Vafa; Aganagic et.al]

B-Model on deformed conifold [Dijkgraaf-Vafa]

The Embedding: $\frac{1}{2}$SUSY Correlators

$$\frac{1}{Z_{free}}\int D(Fields)_{N\times N}\; e^{-\frac{N}{\lambda}S_{\mathcal{N}=4}[\psi,A,\Phi^I]}\prod_{a=1}^{Q}\det(x_a - Y_1\cdot\Phi(w_1))\prod_{\mu=1}^{R}\det(y_a - Y_2\cdot\Phi(w_2))$$

Giant Graviton Operators

$$= \frac{1}{Z_N}\int dKdM_{N\times N}\; e^{-\frac{N}{\lambda}|w_1-w_2|^2\; Tr(KM)}\prod_{a=1}^{Q}\det(x_a - K)\prod_{\mu=1}^{R}\det(v_u - M)$$

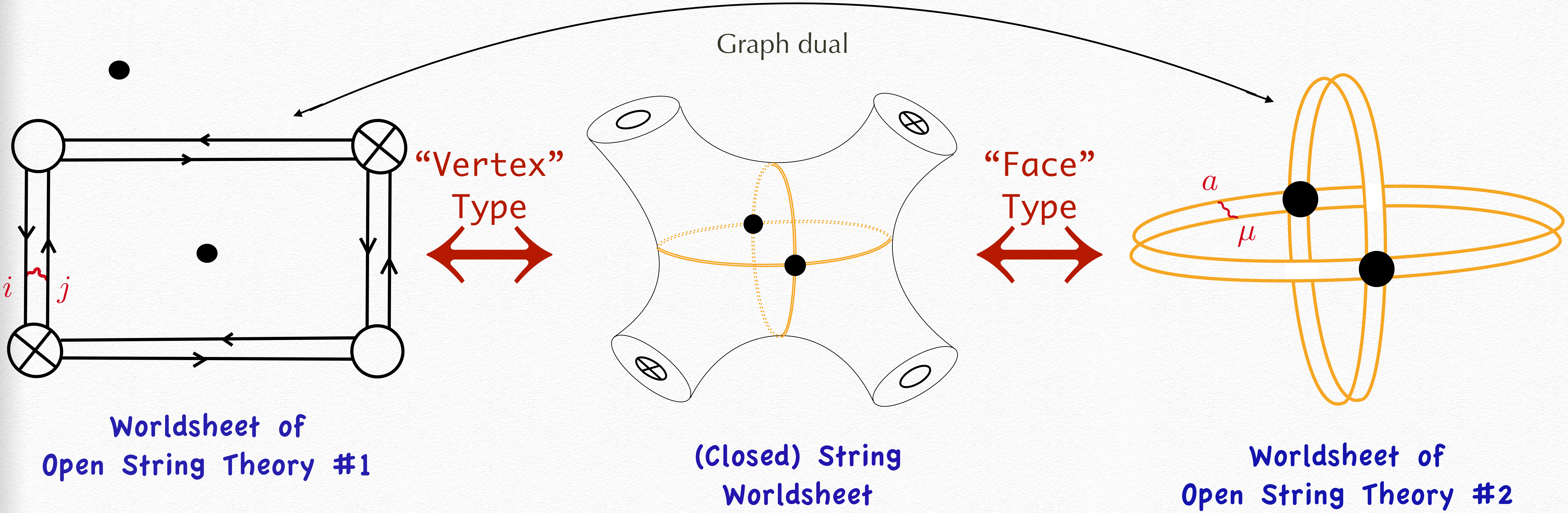
$$\boxed{\begin{aligned} K_{ij} &= Y_1\cdot\Phi_{ij}(w_1) \\ M_{ij} &= Y_2\cdot\Phi_{ij}(w_2)\end{aligned}}$$

$$\propto \int dKdM_{N\times N}\; e^{-\frac{N}{\lambda}|w_1-w_2|^2\; Tr(KM)+\sum_n t_n Tr(K^n)+\sum_n \tilde{t}_n Tr(M^n)} \equiv Z(t_n,\tilde{t}_n)$$

Generating Function of $\left\langle \prod_{i=1}^{n_1}\frac{1}{k_i}Tr\left[\left(Y_1\cdot\Phi(w_1)\right)^{k_i}\right]\prod_{j=1}^{n_2}\frac{1}{l_j}Tr\left[\left(Y_2\cdot\Phi(w_2)\right)^{l_j}\right]\right\rangle^c_{\mathcal{N}=4}$

= correlators in a Two Matrix Model.

Big Picture - II



Open-Closed-Open Triality

LEGO Riemann Surfaces

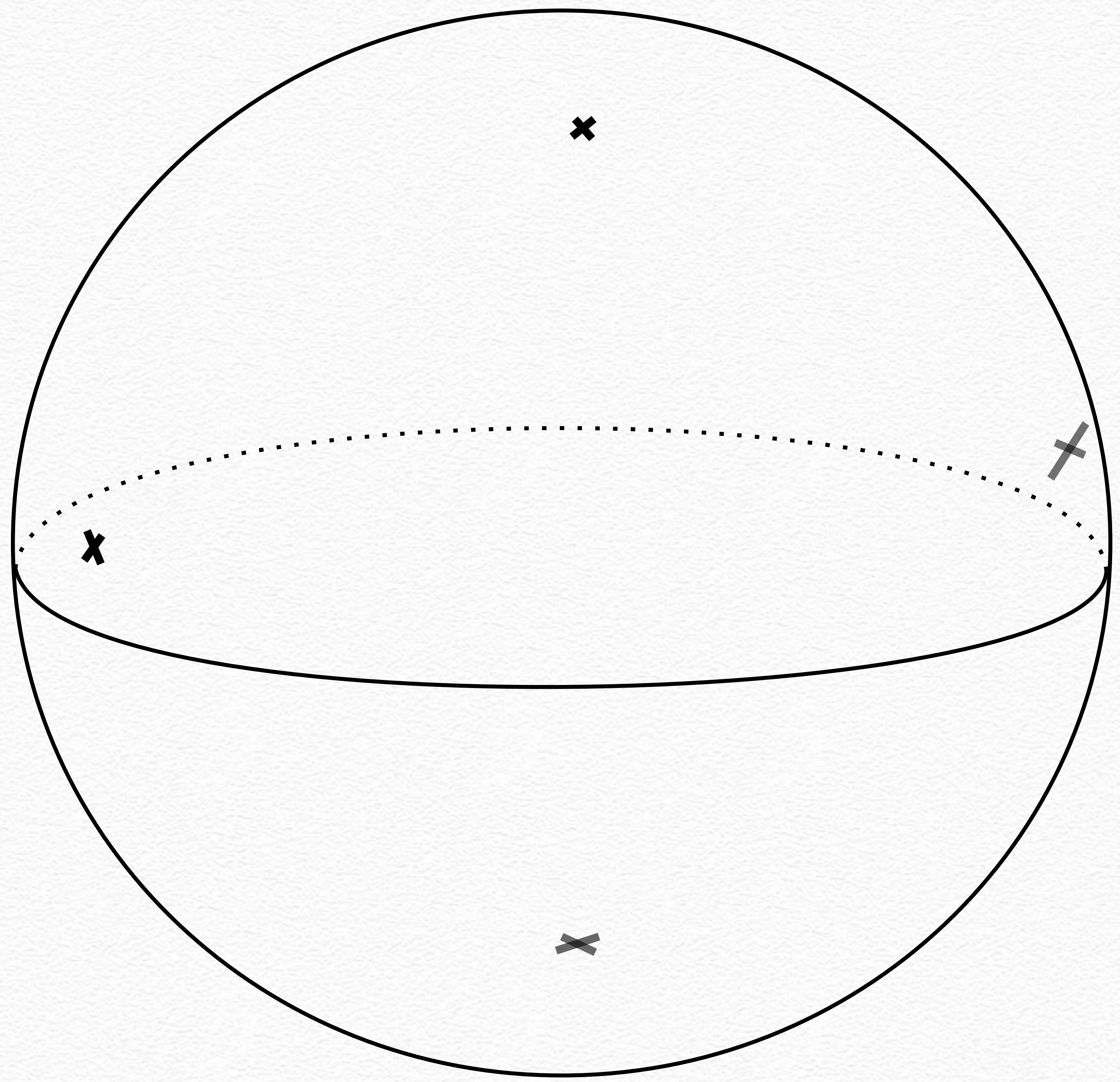
Some Pretty Maths Pictures

The key object: The Strebel Differential for $\Sigma_{g,n}$

[Strebel '84]

"Meromorphic
Quadratic Differential"

$$\phi_S = \phi_S(z) dz \otimes dz$$



Only Double-poles
at marked points

$$\phi_S \sim \frac{-L_k{}^2}{(2\pi)^2} \frac{du^2}{u^2}$$

(L_k = Residue)

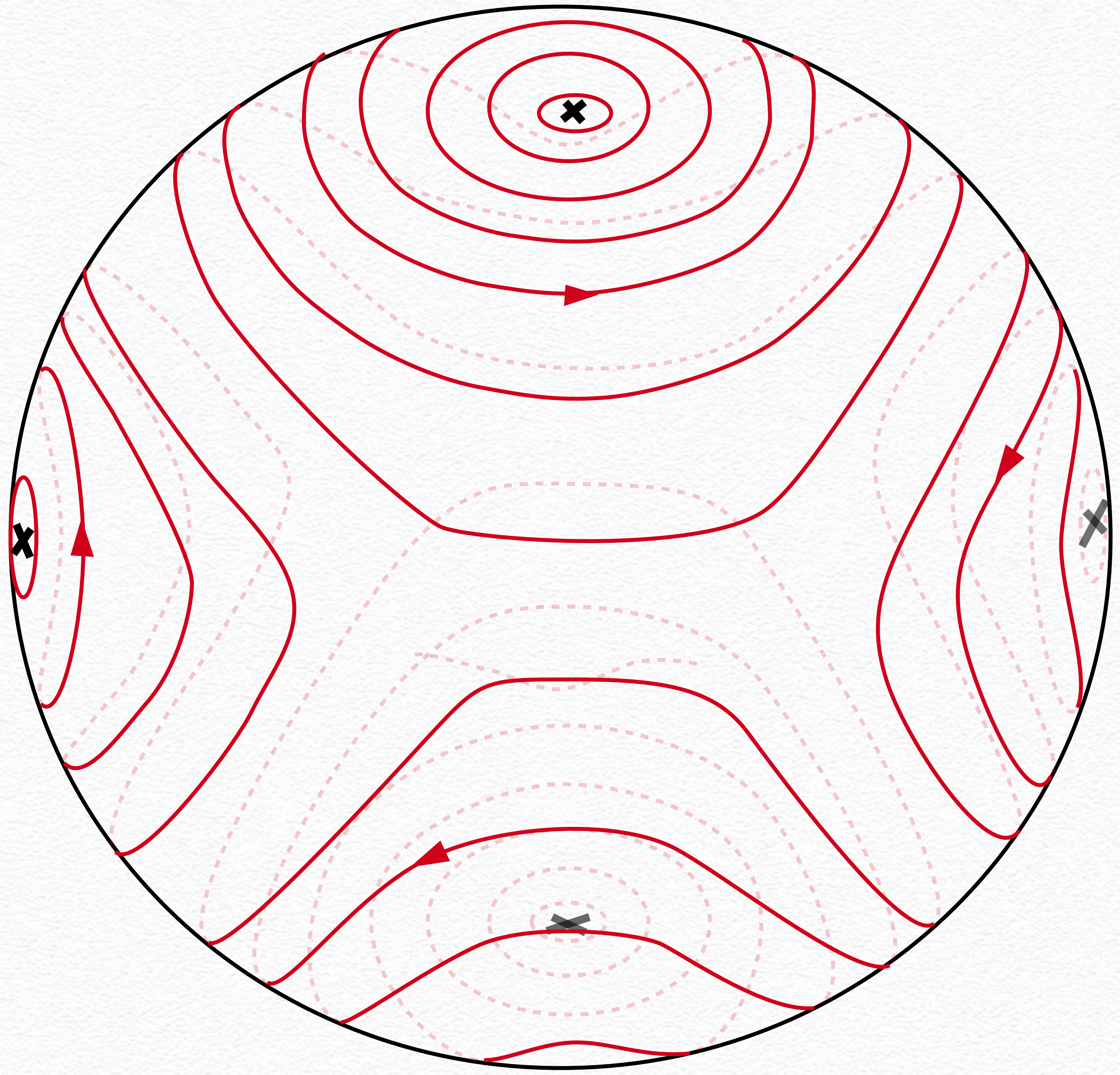
Some Pretty Maths Pictures

The key object: The Strebel Differential for $\Sigma_{g,n}$

[Strebel '84]

Horizontal trajectories:

$$\phi_S(z(t))\left(\frac{dz(t)}{dt}\right)^2 > 0$$



Concentric closed curves around each double pole.

$$\phi_S \sim \frac{L_k^2}{(2\pi)^2} d\theta^2$$

$$(u = re^{i\theta})$$

Special to Strebel Differentials: Horizontal trajectories are like isotherms/equipotentials.

Some Pretty Maths Pictures

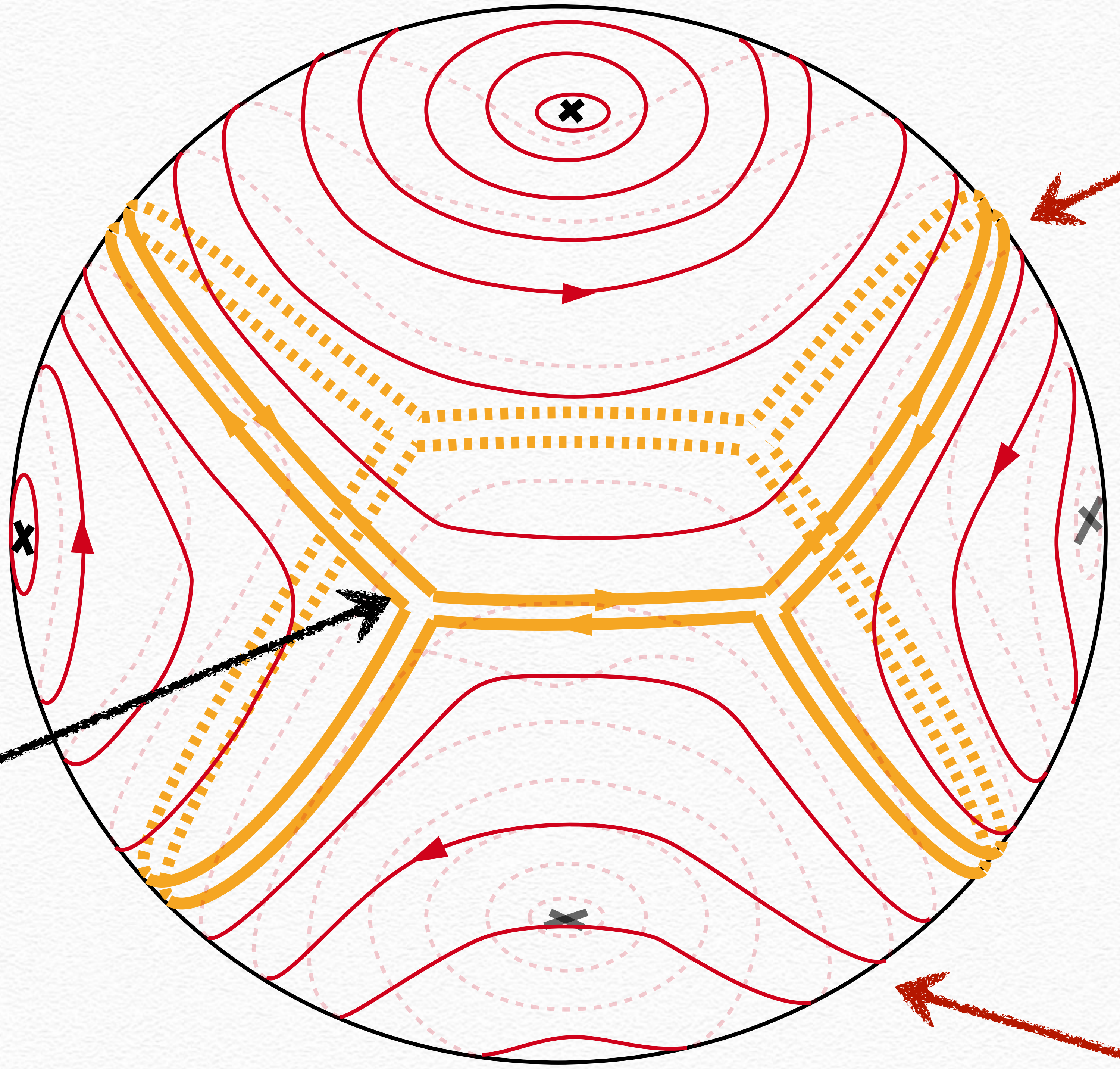
The key object: The Strebel Differential

[Strebel '84]

Horizontal trajectories:

$$\phi_S(z(t))\left(\frac{dz(t)}{dt}\right)^2 > 0$$

`Critical' Horizontal trajectories are graphs - not closed curves.
Strebel Graph

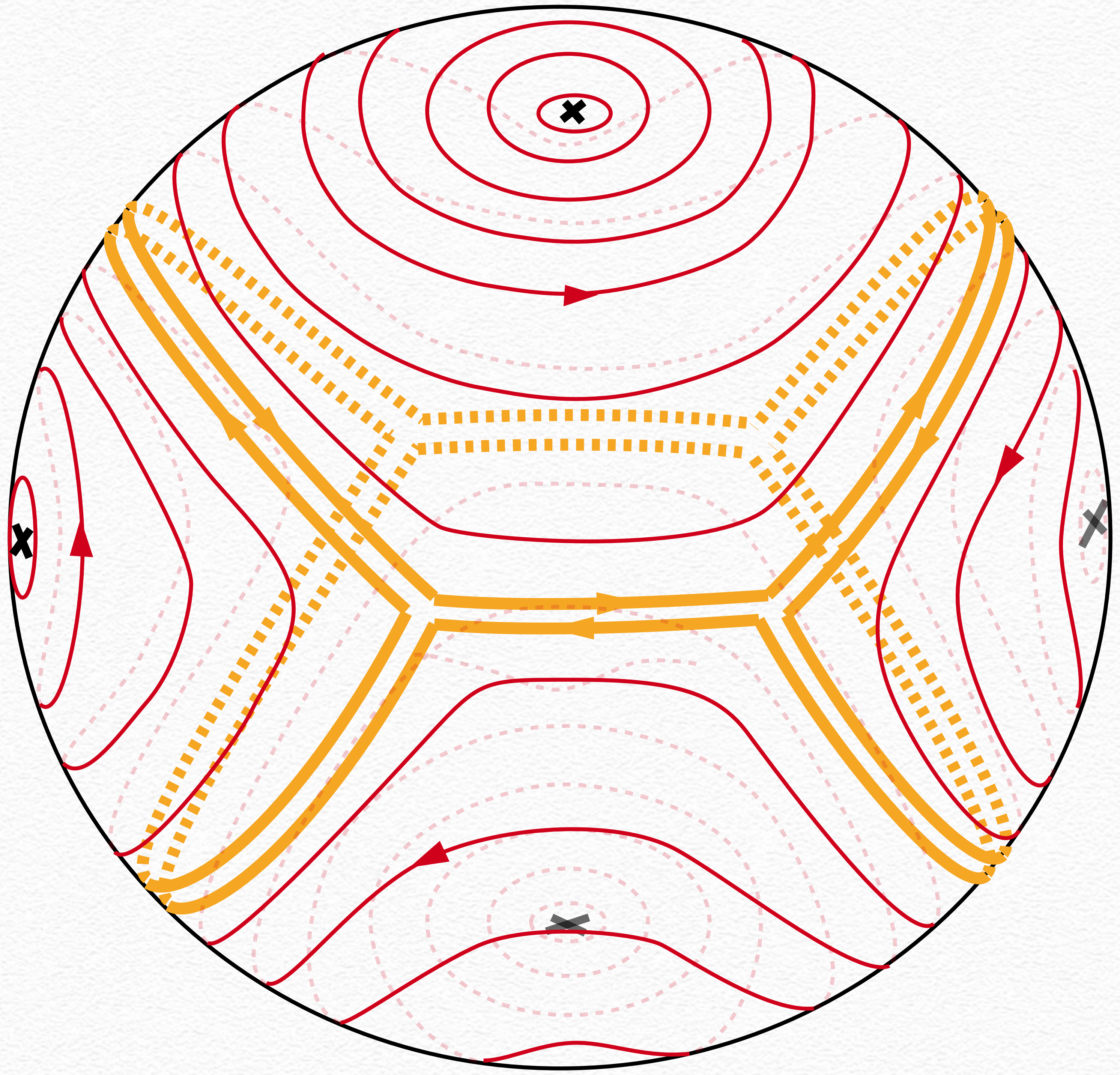


Each Face has a single double pole

Vertices are **zeroes** of the Strebel Differential:
Valence of vertex=order of zero+2

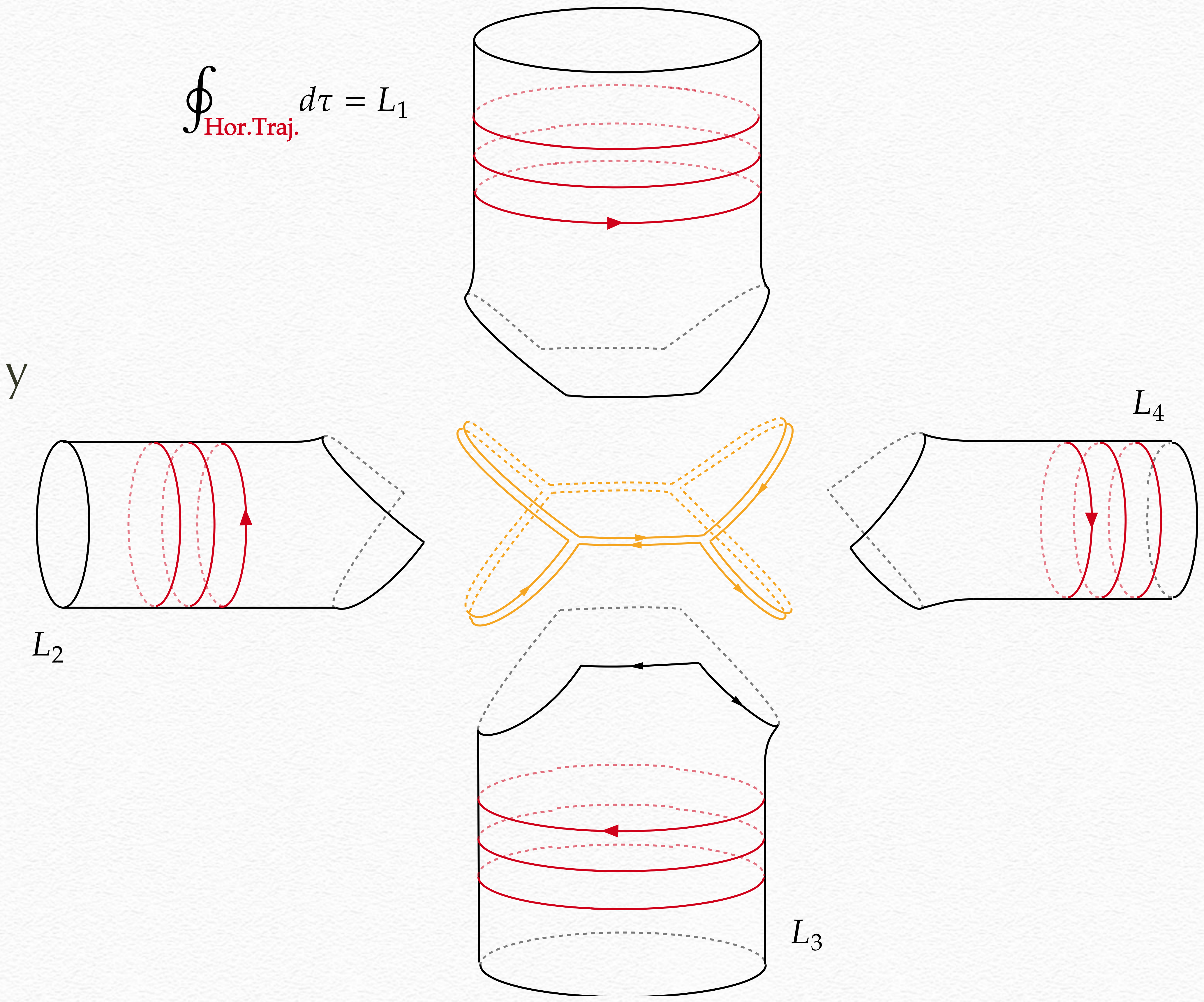
Generic Horizontal trajectories

Building Riemann Surfaces



$$L_k = \oint_k \sqrt{\phi_S(z)}dz > 0$$

Conformally Equivalent



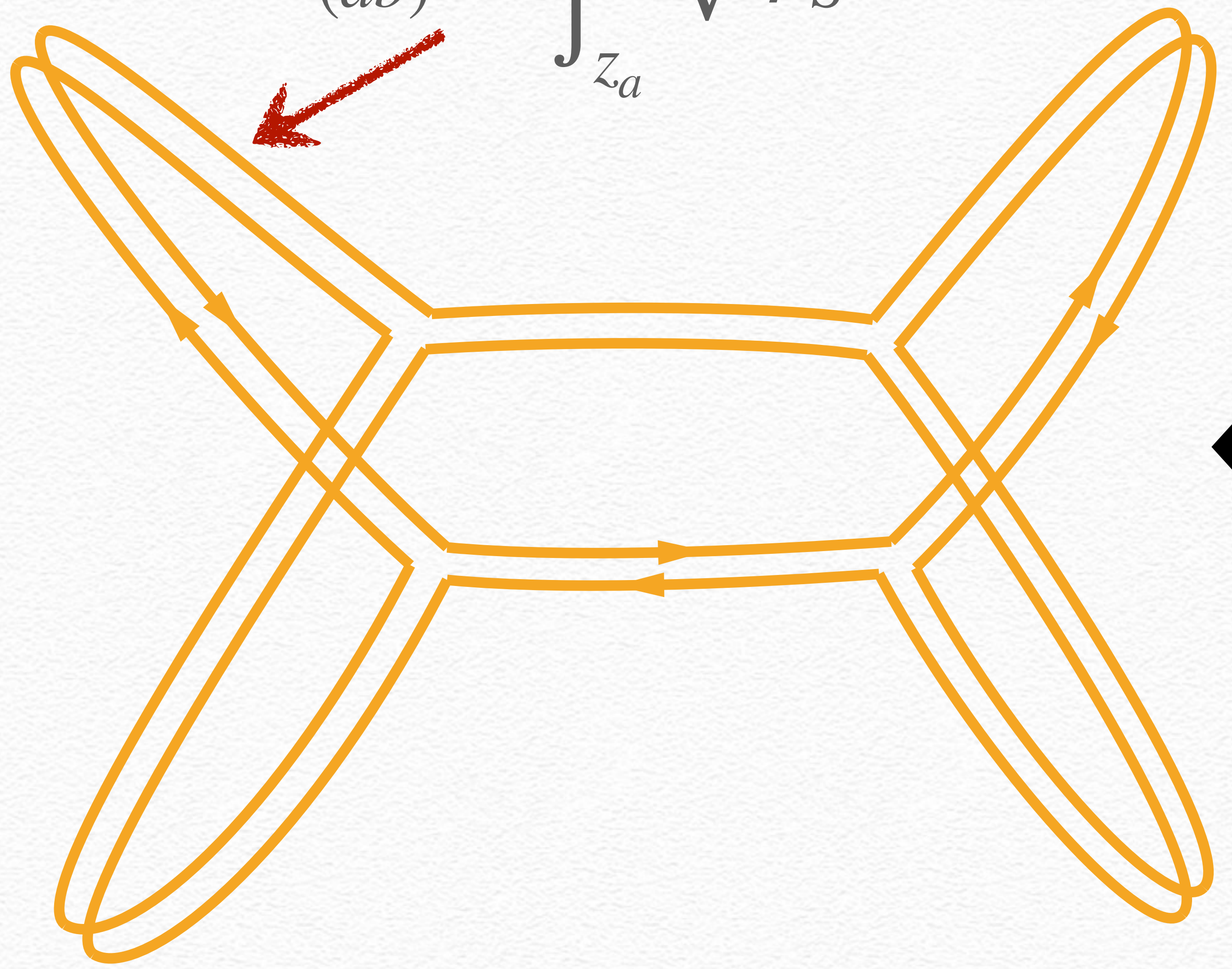
Build a punctured RS from gluing (half-)cylinders of circumference L_k along the Strebel Graph. Similar to light-cone gauge; no internal propagators unlike in closed SFT.

Strebel Parametrisation of $\mathscr{M}_{g,n}$

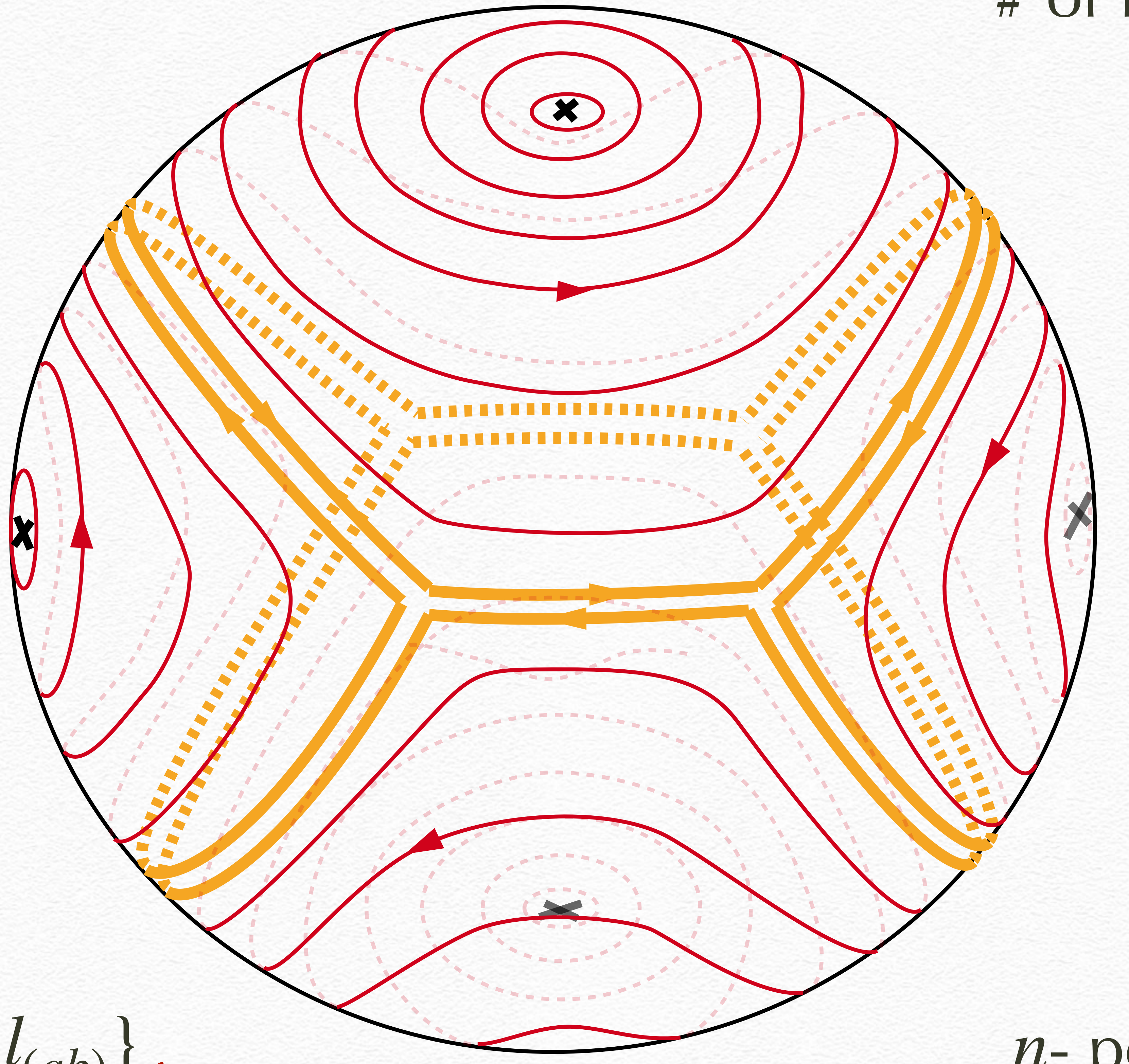
of Double Poles

Edge lengths $l_{(ab)}$ = proper distance between zeroes (a, b) of the Strebel Graph

$$l_{(ab)} = \int_{z_a}^{z_b} \sqrt{\phi_S(z)}dz$$



$\Leftrightarrow$

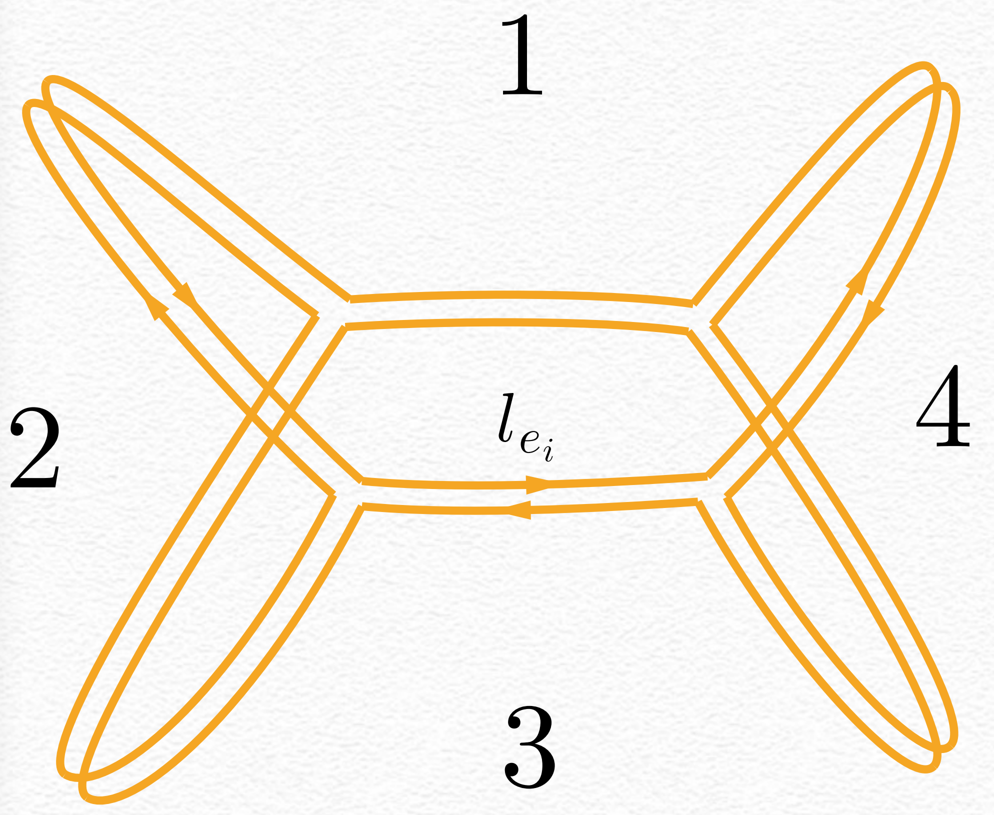


Parametrised by $\{l_{(ab)}\}$

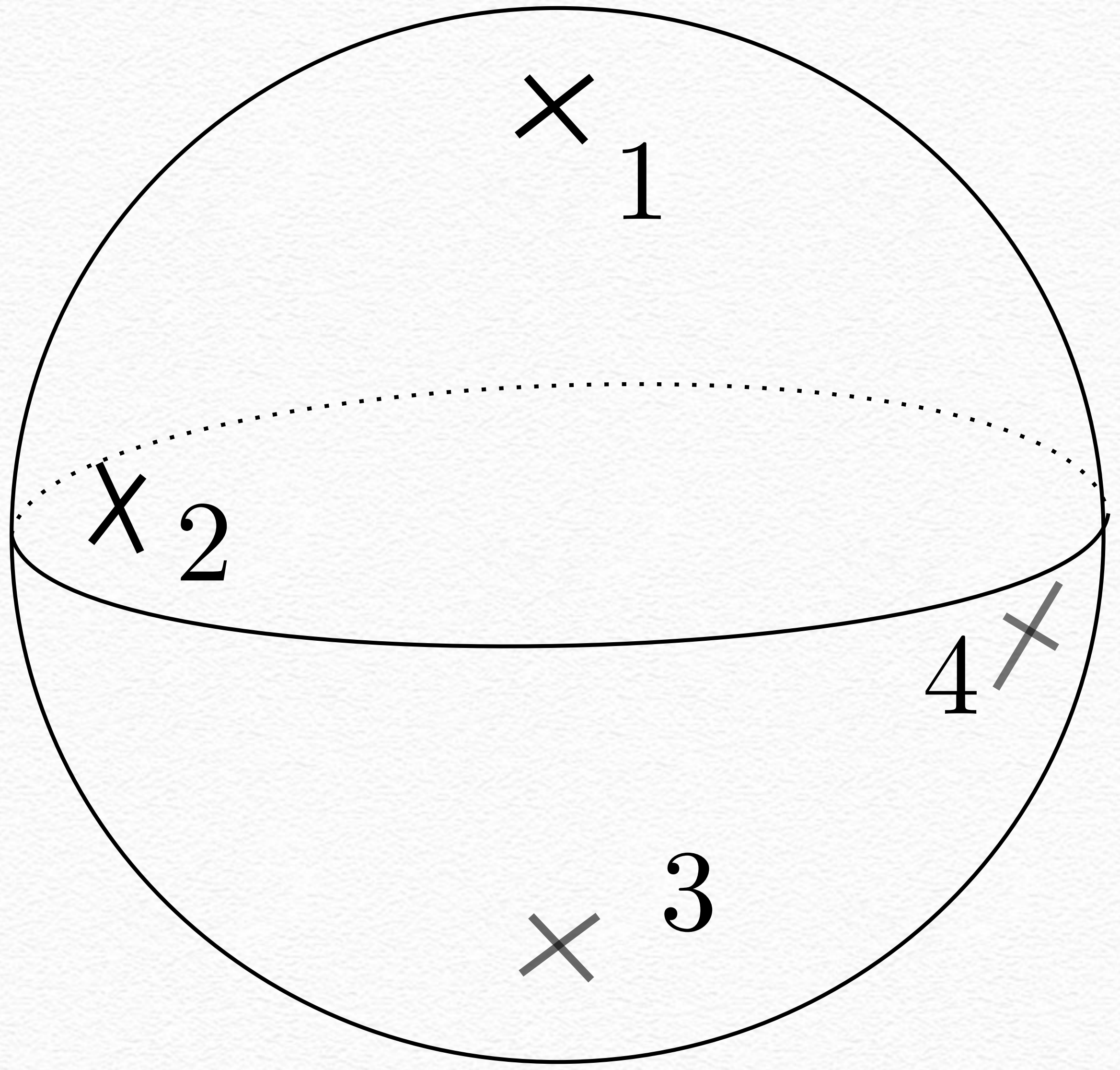
n- perimeters $\{L_k\}$

Theorem: Strebel graph $\leftrightarrow p \in \mathscr{M}_{g,n} \times \mathbb{R}^n_+$

Strebel Parametrisation of $\mathscr{M}_{g,n}$



$\Leftrightarrow$



$V - E + F = 2 - 2g$

$n \;=\; \#\text{Faces}$

edge lengths l_{e_i}

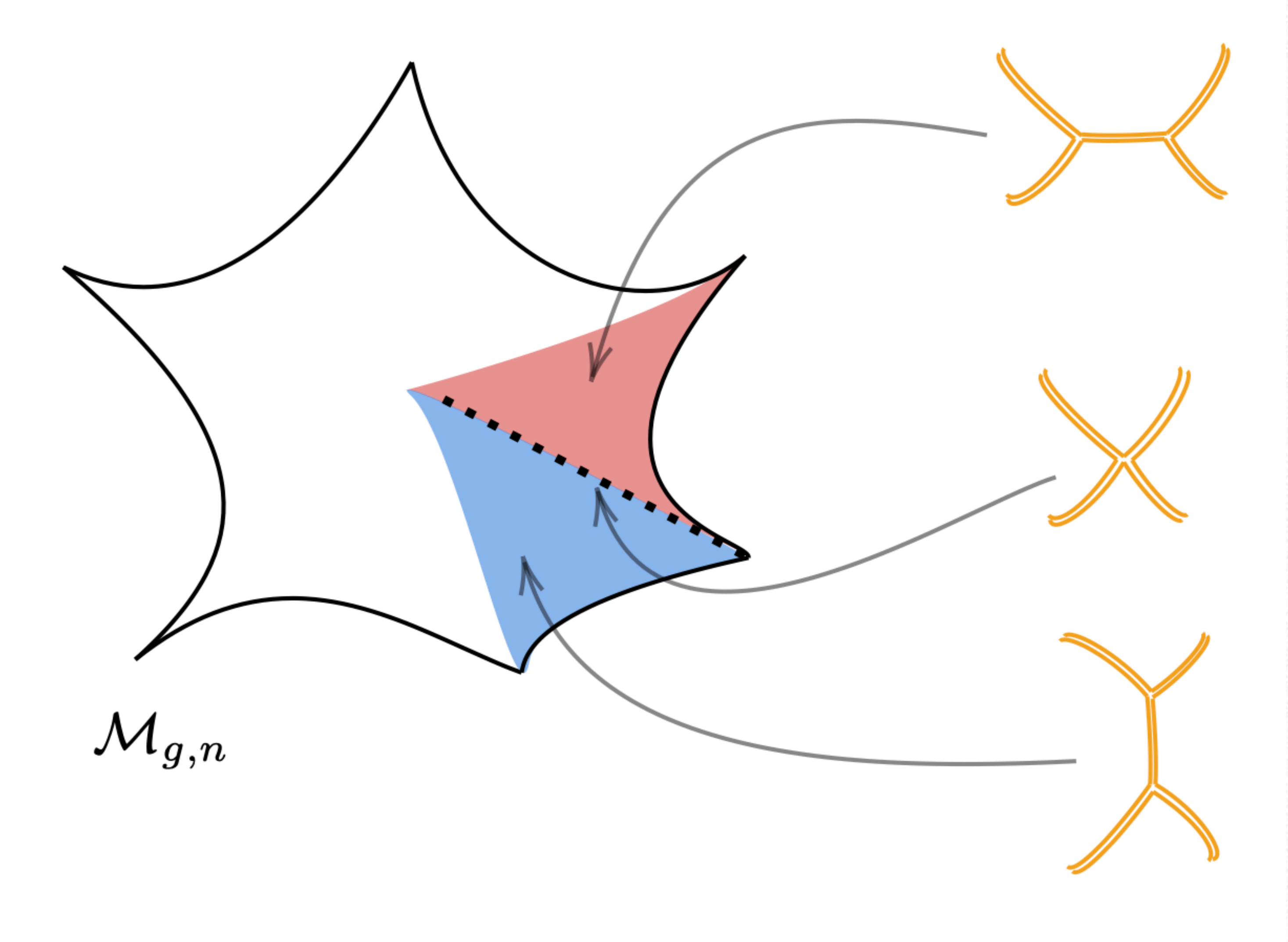
perimeters $L_k \;=\; \sum_{e_i \; around \; face \; k} l_{e_i}$

genus g Riemann surface

$n \;=\; \#$ marked points

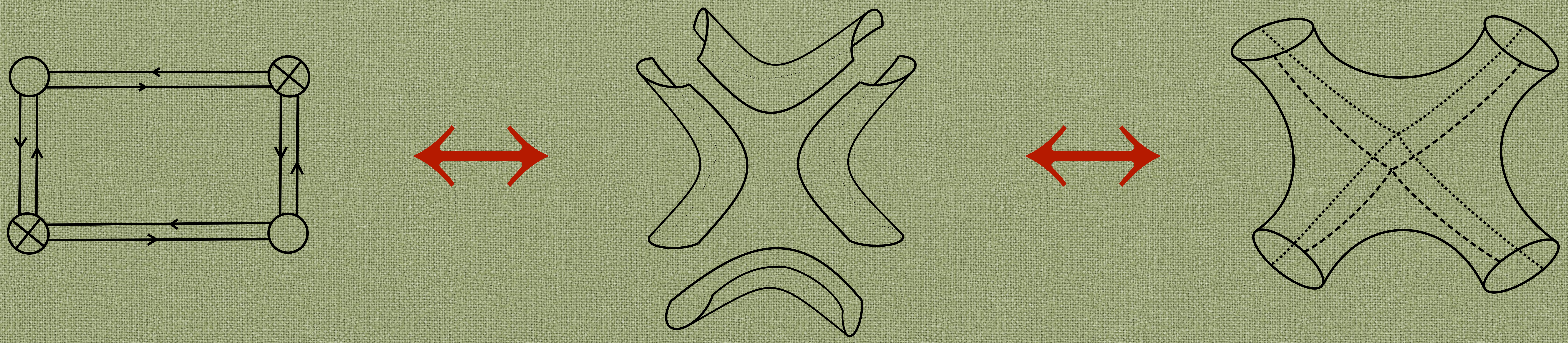
coordinates on $\mathcal{M}_{g,n} \times R_+^n$

coordinates on $\mathbb{R}_+^n$ $of\ \mathcal{M}_{g,n} \times \mathbb{R}_+^n$

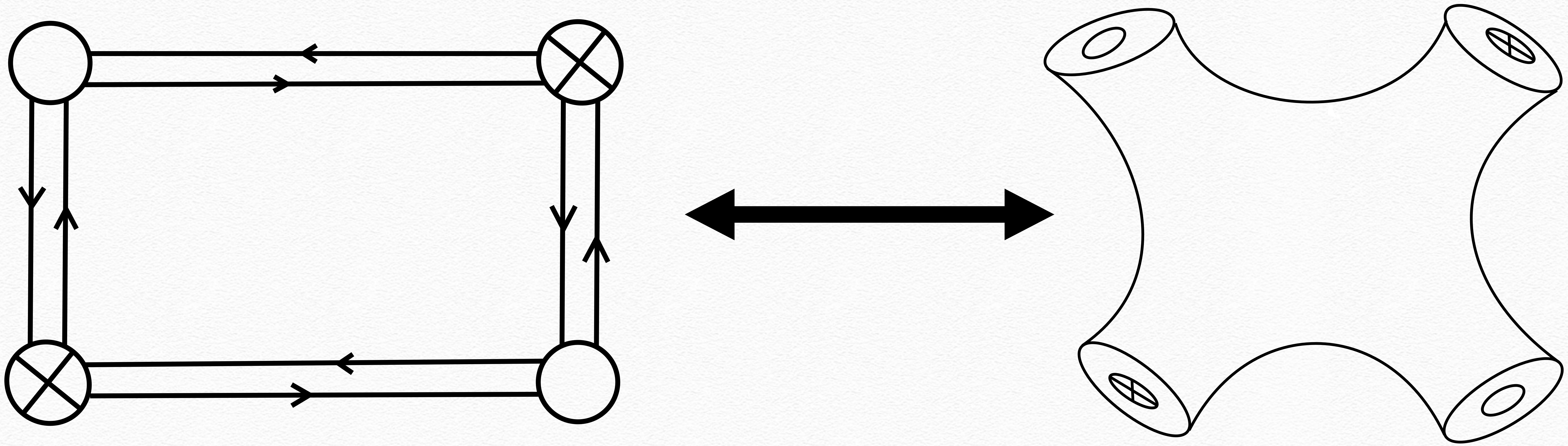


Each ribbon graph (topology)
+ all possible edge lengths
parametrize one cell of $\mathscr{M}_{g,n}$

Feynman Diagrams as Worldsheets



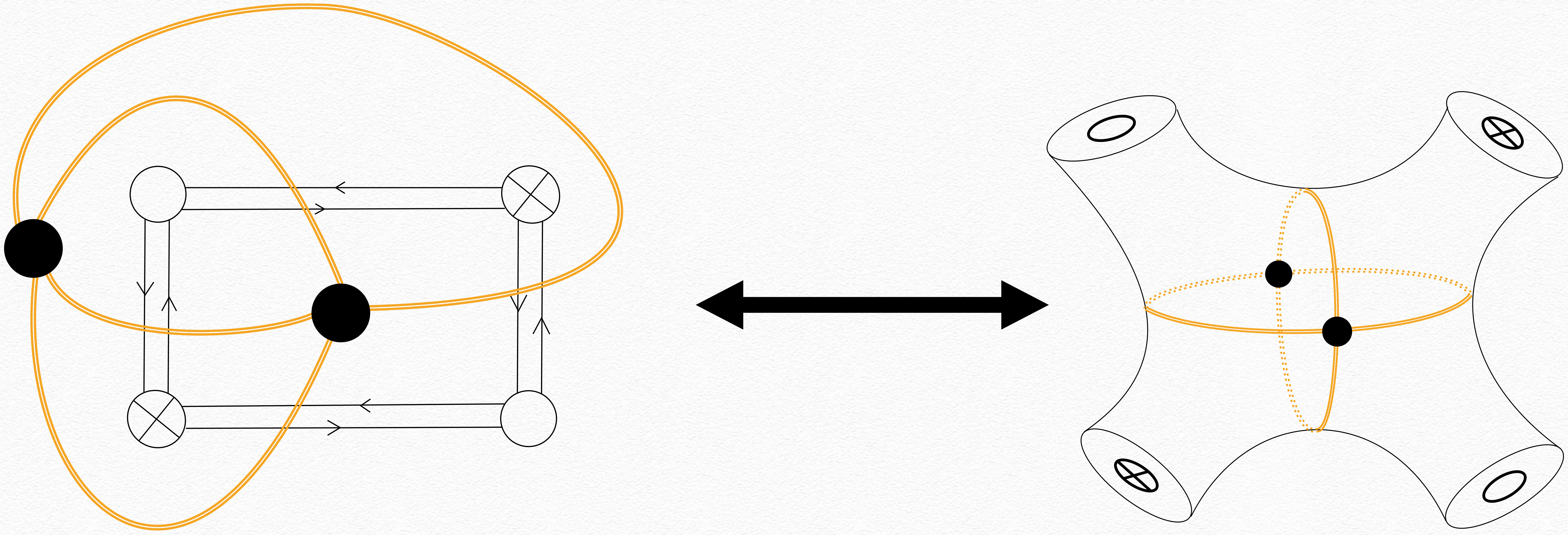
What about the Feynman Graphs?



$\left\langle (Tr(K^2))^2(Tr(M^2))^2 \right\rangle_{KM}$ $\langle (\mathscr{V}_{+2})^2(\mathscr{V}_{-2})^2 \rangle$

Expect **Vertices** of Feynman Graph $\leftrightarrow$ Punctures on dual closed RS

What about the Feynman Graphs?



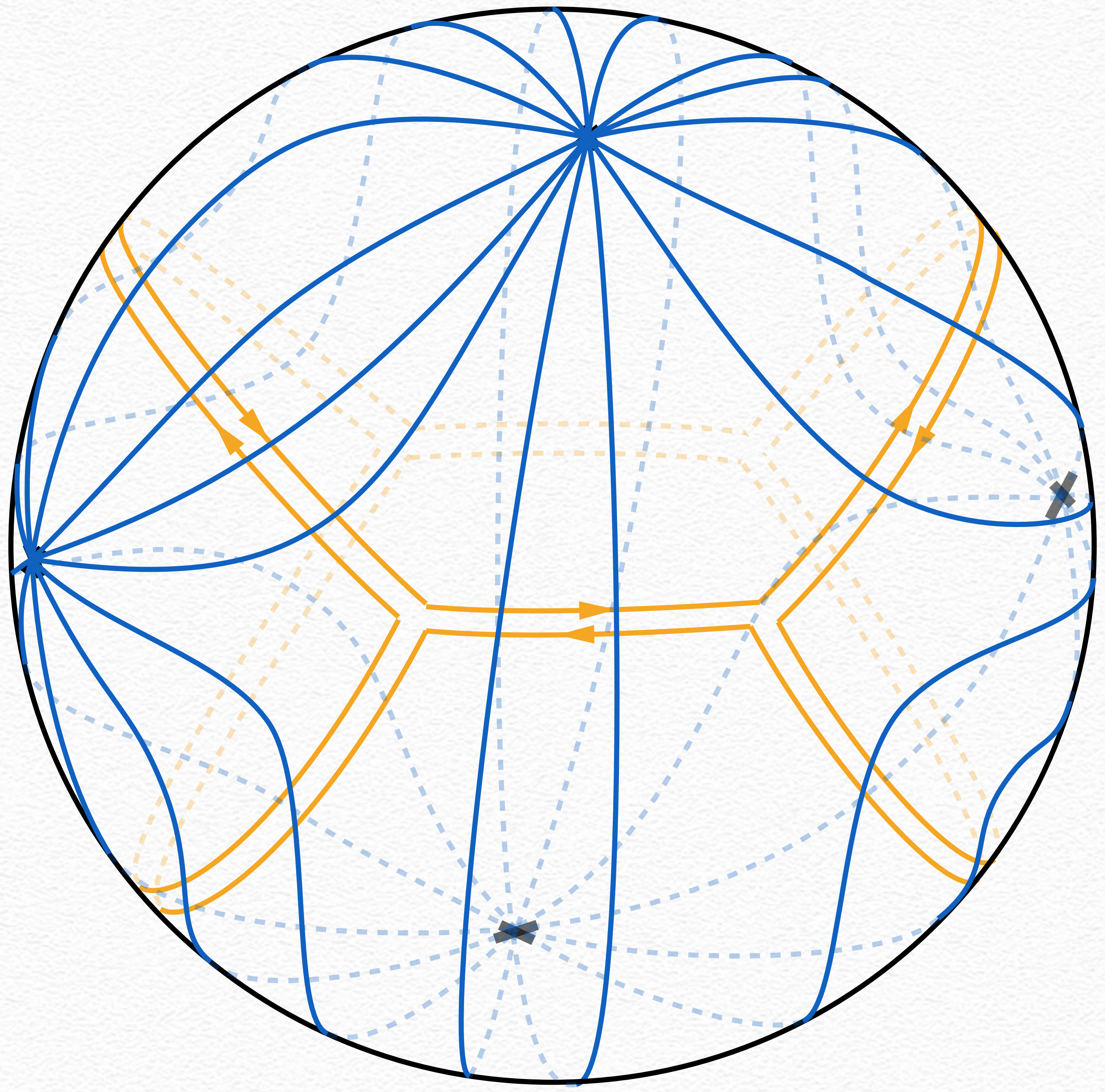
$\left\langle (Tr(K^2))^2 (Tr(M^2))^2 \right\rangle_{KM}$ $\qquad$ $\langle (\mathscr{V}_{+2})^2 (\mathscr{V}_{-2})^2 \rangle$

Whereas **Faces** of Strebel Graph $\leftrightarrow$ Punctures on dual closed RS

Feynman Diagram identified as **graph dual** to the Strebel Graph

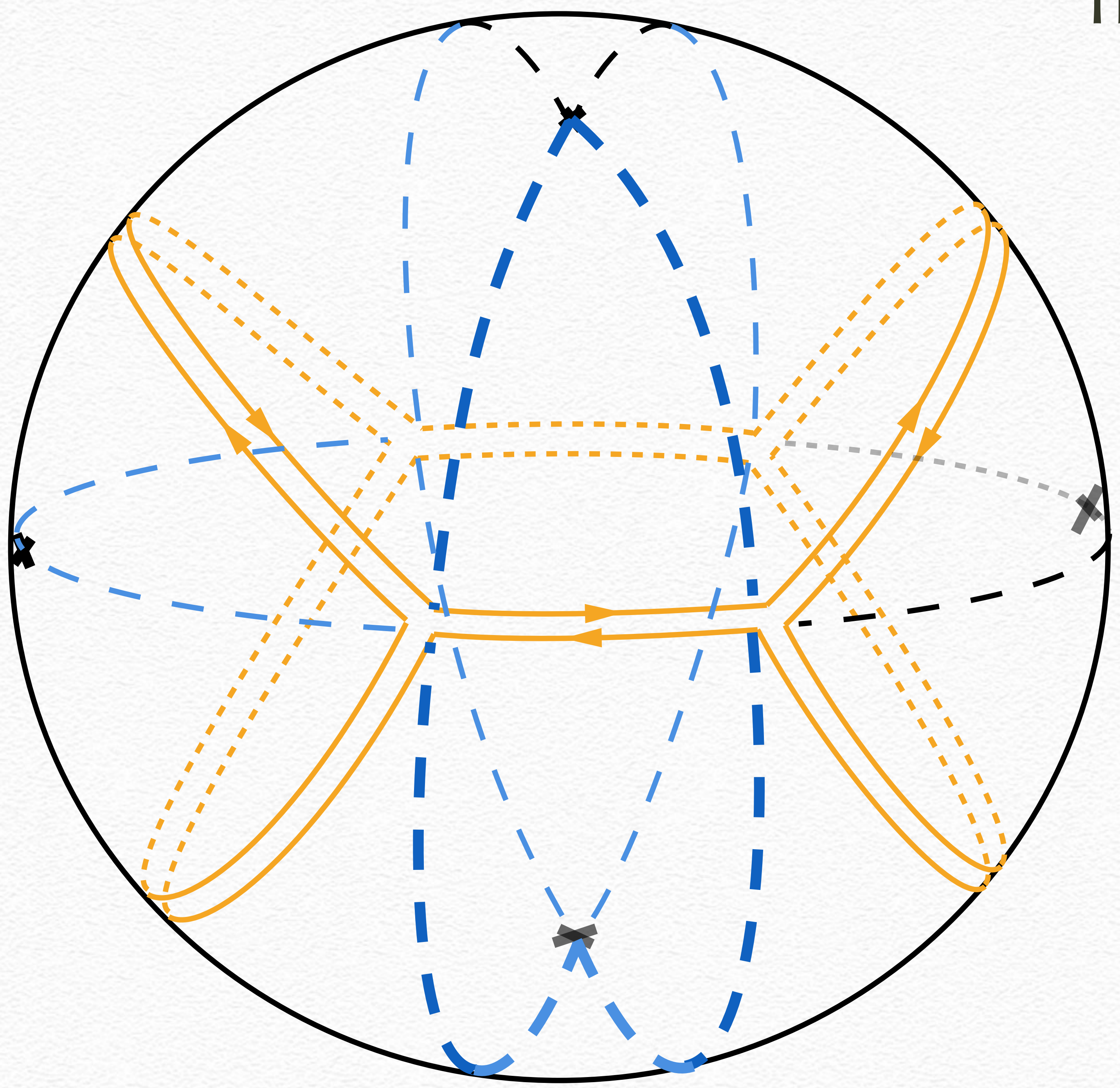
[R.G. '04-'06]

Worldsheets From Feynman Diagrams

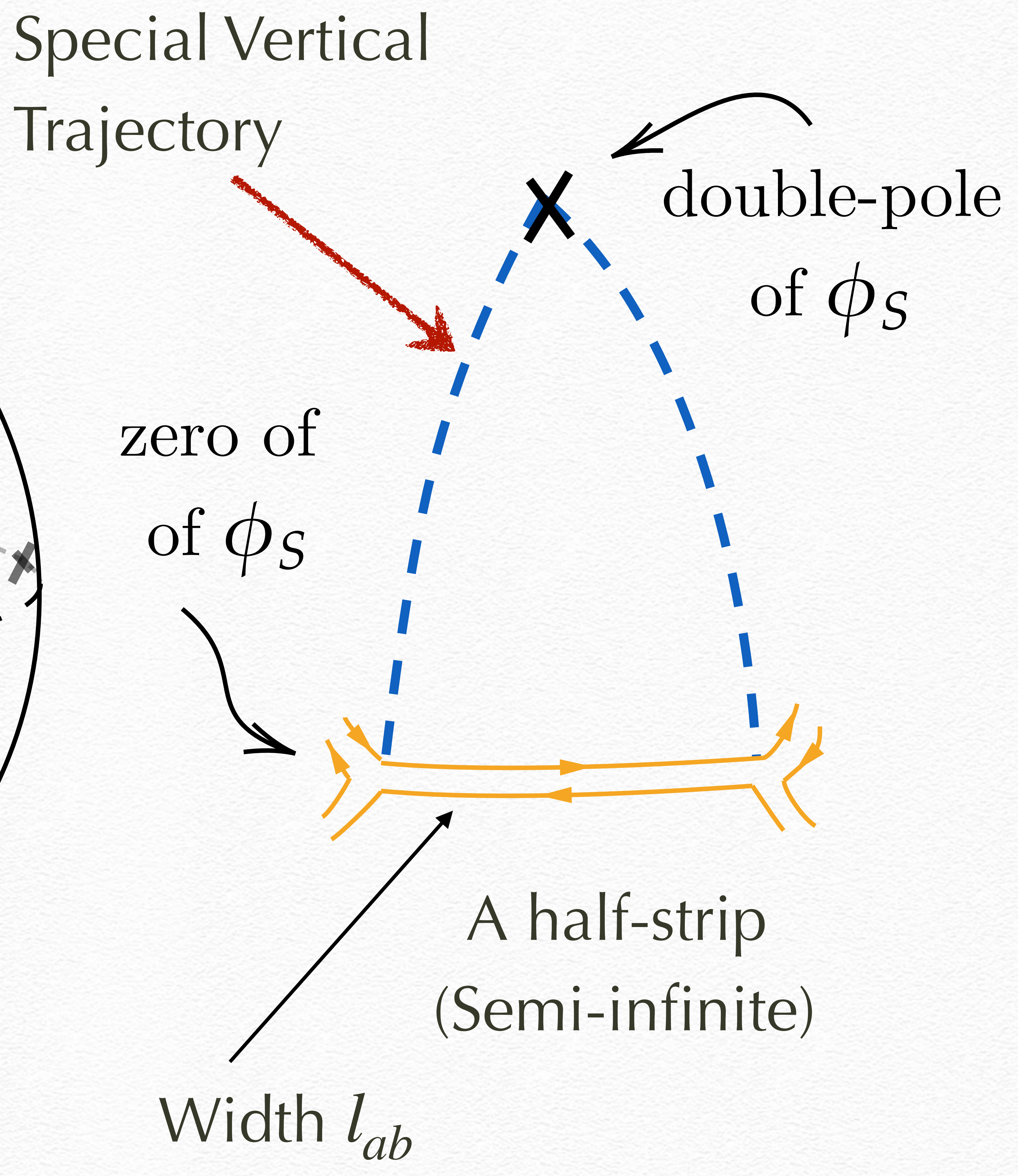


$$\phi_S(z(t))\left(\frac{dz(t)}{dt}\right)^2 < 0$$

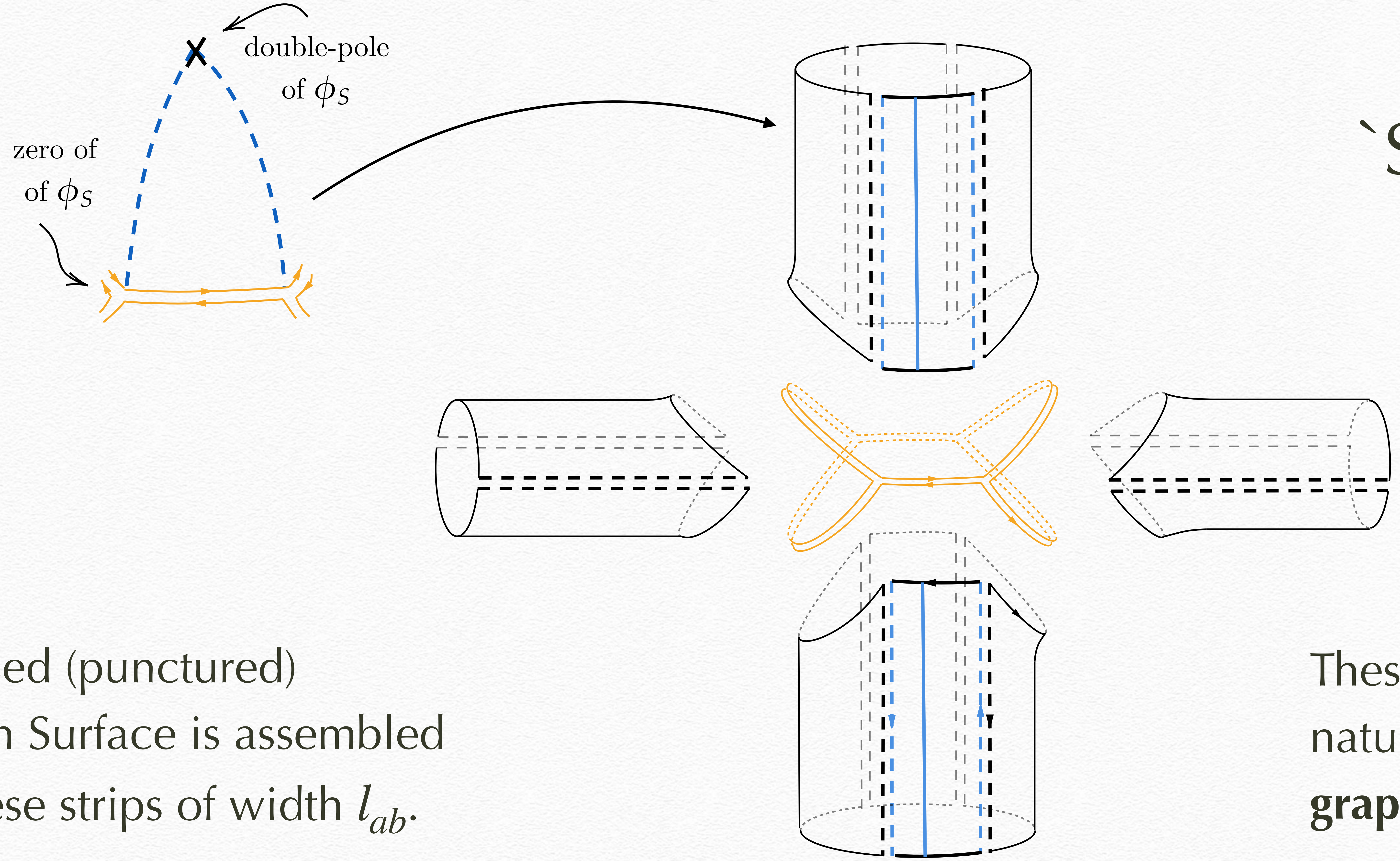
Vertical Trajectories



Carves the Riemann Surface into (infinite) strips



Worldsheets From Feynman Diagrams

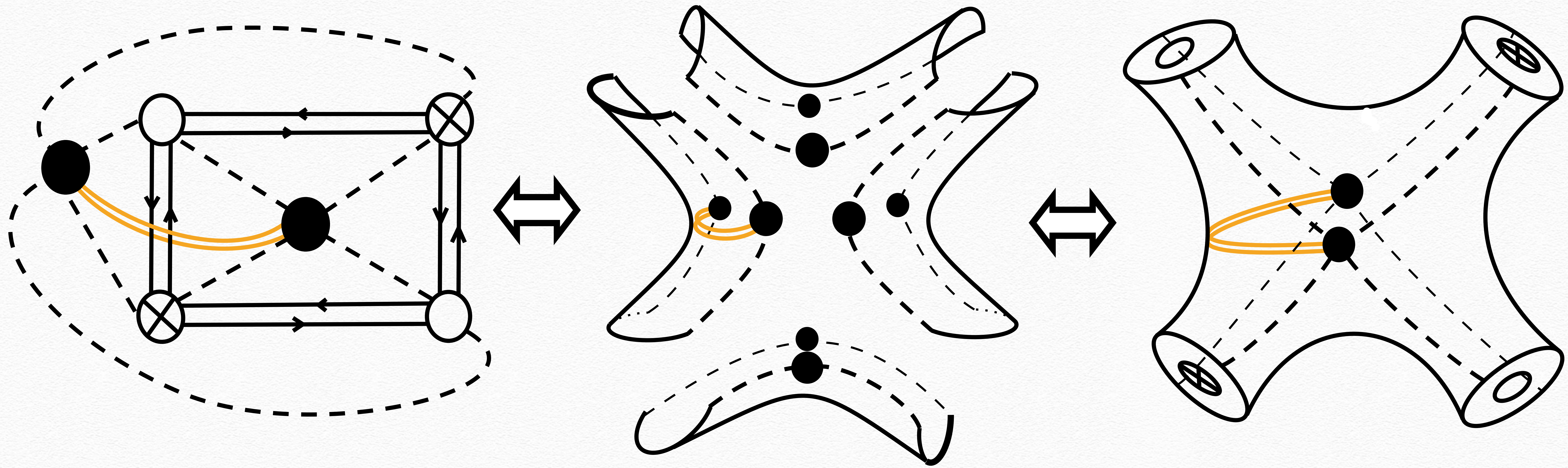


`String Bit' picture.

The closed (punctured) Riemann Surface is assembled from these strips of width l_{ab}.

These strips between double poles naturally identified with the **ribbon graphs** of the large N gauge theory.

Worldsheets from Feynman Diagrams

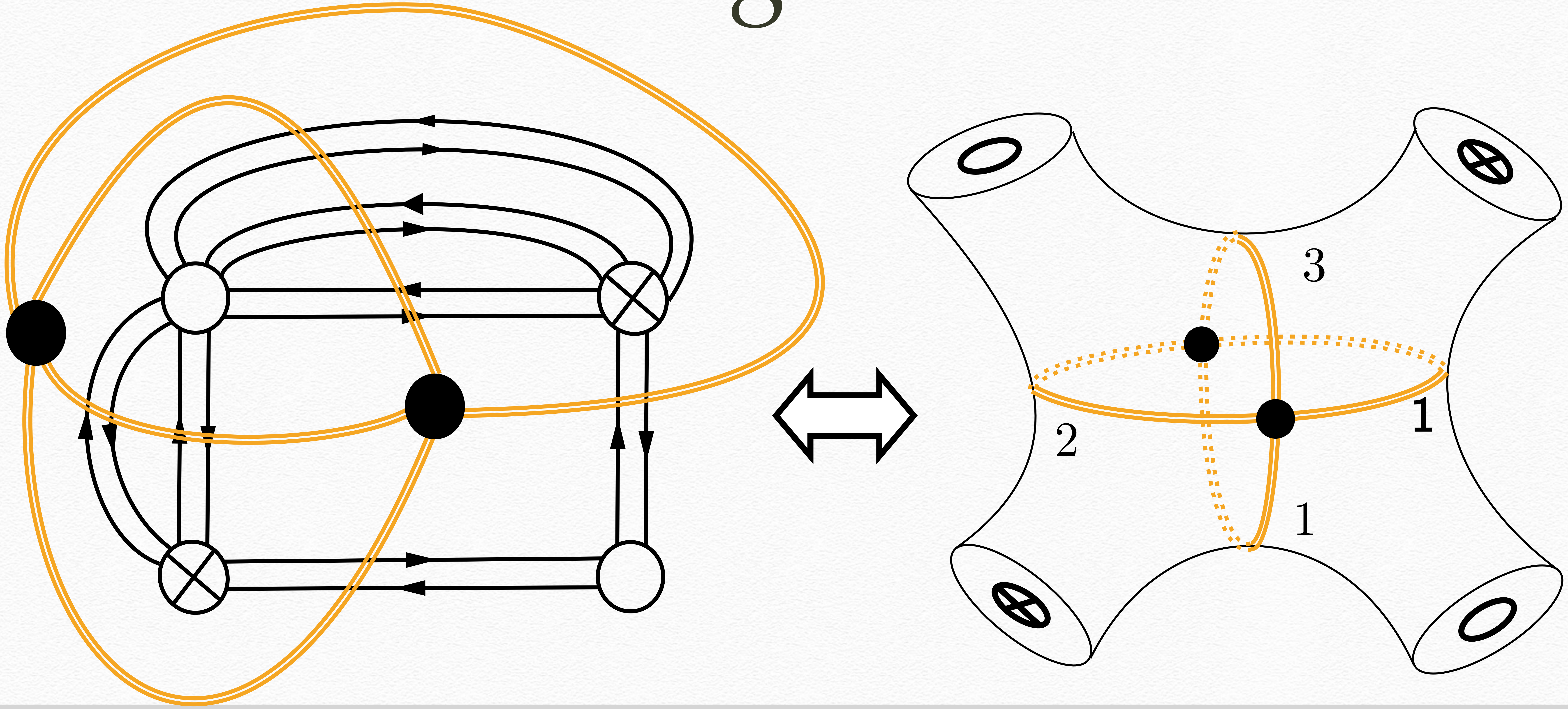


Can explicitly implement this picture of gluing by holomorphically patching together the strips.

The Strebel length assignment l_{ab} now made to the dual Feynman edge. [R. G. '06, Razamat'08]

For matrix integrals l_{ab} simply **number of homotopic Feynman edges** i.e. **lengths are +ve integers**.

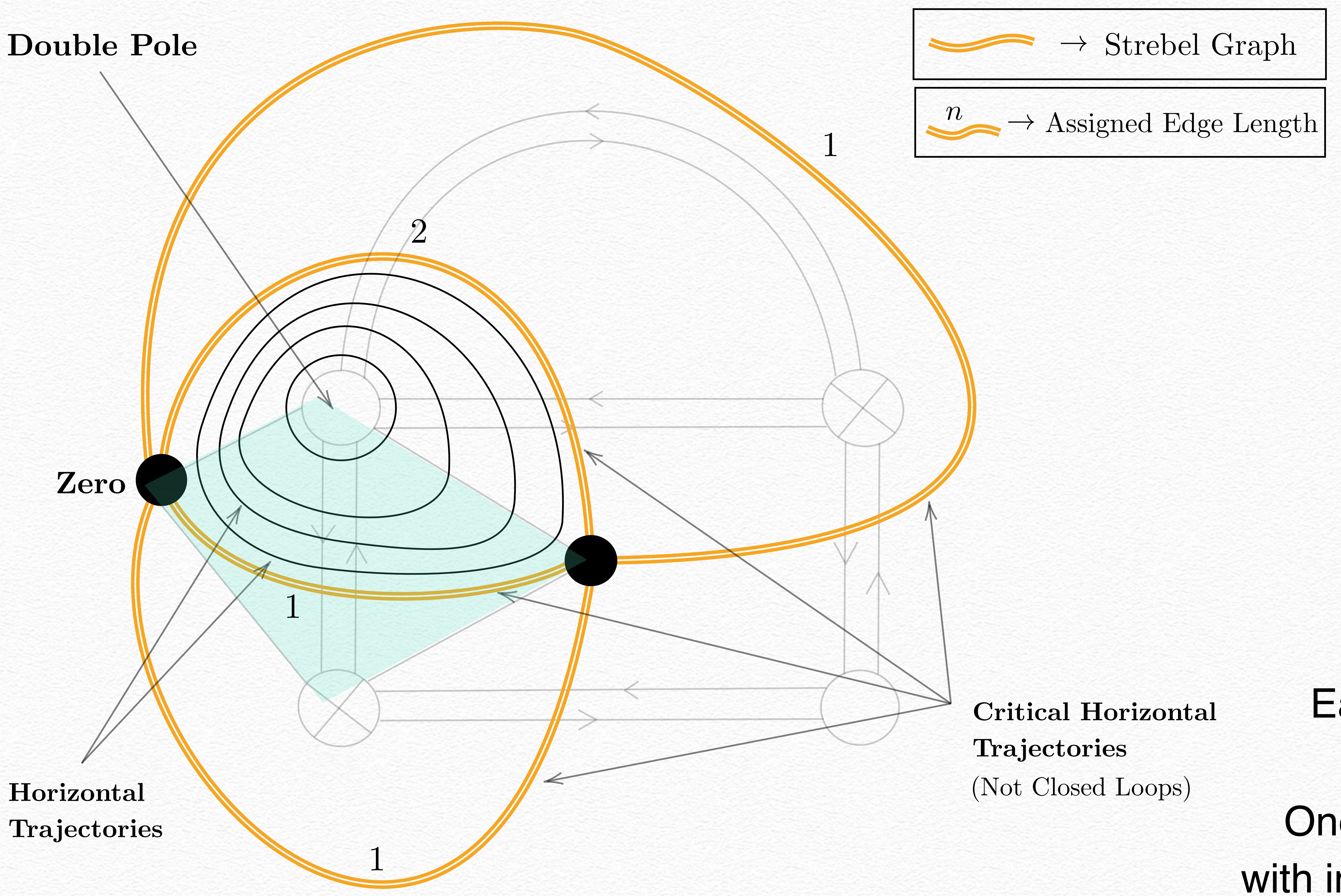
Worldsheets from Feynman Diagrams



Strebel Edge Lengths = # of (homotopic) Feynman Edges crossed

e.g. FD contributing to $\langle Tr[K^2]Tr[K^5]Tr[M^4]Tr[M^3]\rangle_{KM}$

Worldsheets From Feynman Diagrams



Integer Strebel Lengths are special in $\mathscr{M}_{g,n} \leftrightarrow$ Arithmetic Riemann Surfaces

[Mulase-Penkava'98]

Gives a natural **latticisation of $\mathscr{M}_{g,n}$**.

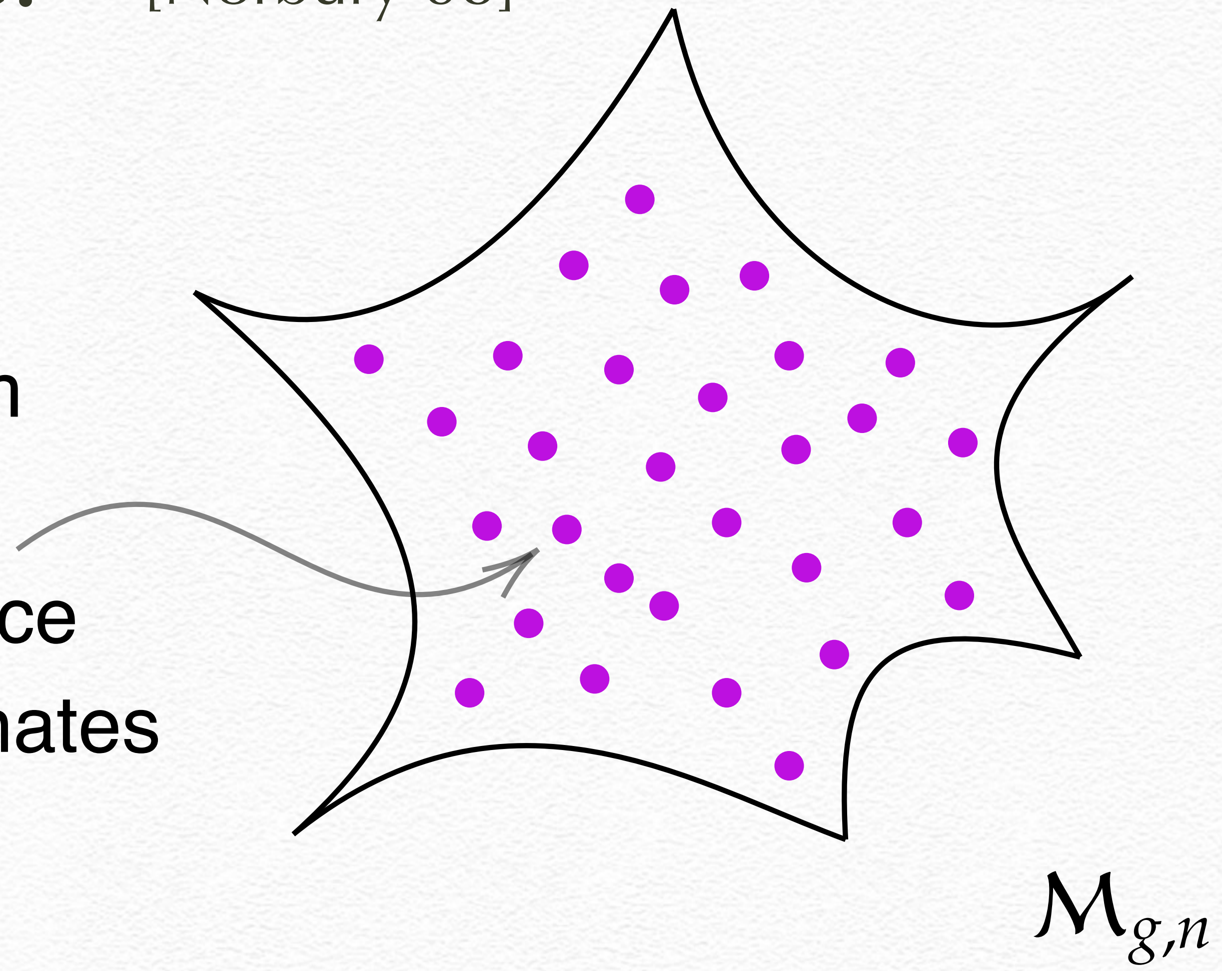
Not of the worldsheet.

Counting gives discretised volumes of moduli space. [Norbury'08]

Each Feynman diagram

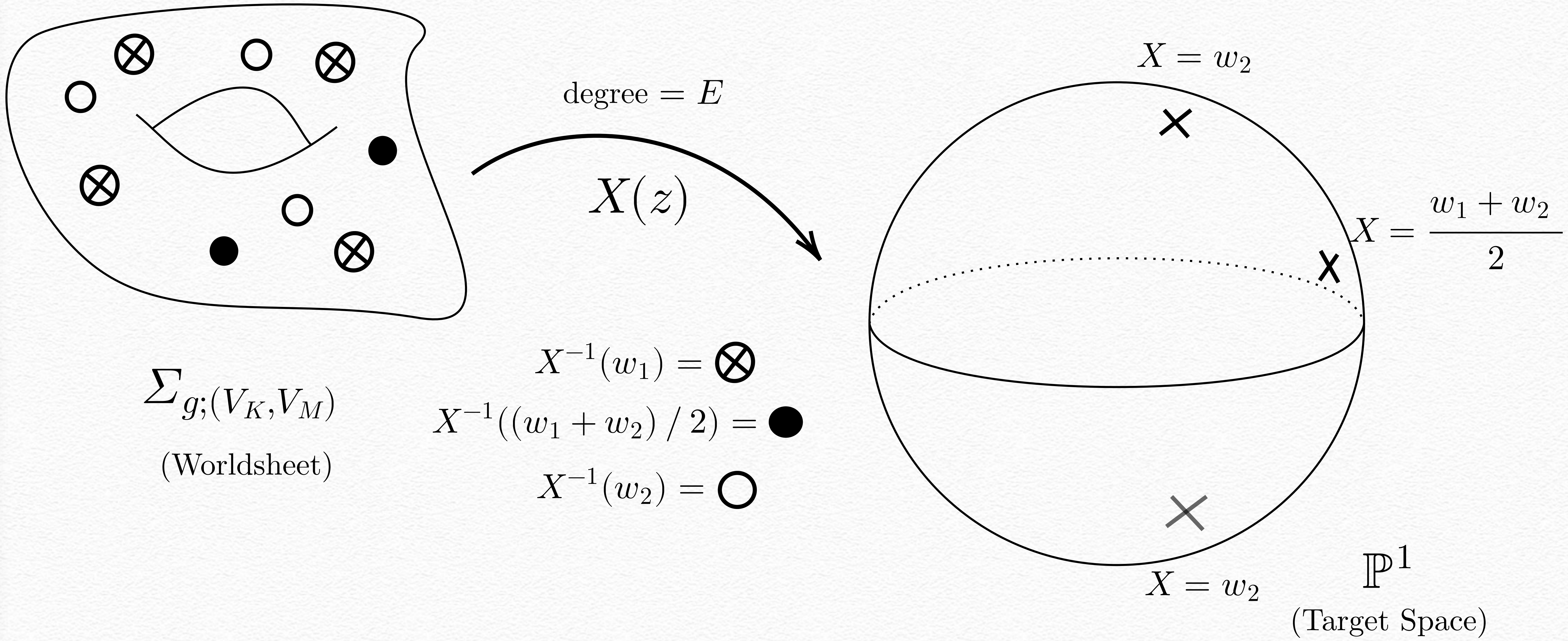
=

One point on moduli space with integer Strebel coordinates



Reconstructing the Target Space

Belyi Maps



Matrix model correlators

$$\left\langle \prod_i Tr(K^{l_i}) \prod_j Tr(M^{n_j}) \right\rangle_{KM}$$

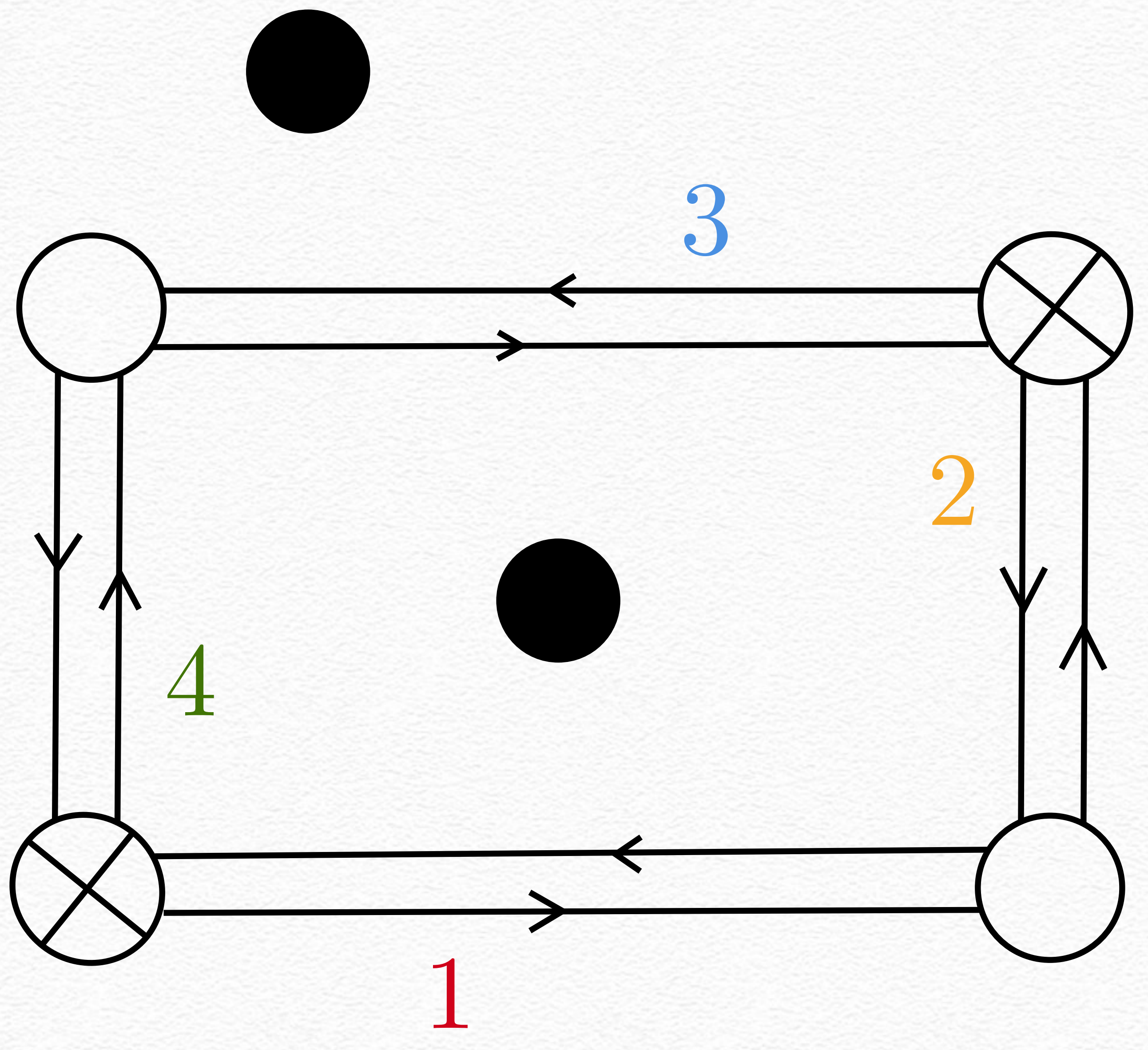
Expressed as sum over branched holomorphic covering maps of a $\mathbb{P}^1$ by the worldsheet over exactly three branchpoints (in target space).

[cf. Gross-Taylor '92 for 2d Yang-Mills]

Such branched covers of $\mathbb{P}^1$ special: admitted only by **Arithmetic RS** - Belyi's Theorem. [Belyi '80]

Exactly the worldsheets we saw in our dictionary between Feynman Diagrams and worldsheets.

Belyi Maps



$K-$ Vertices ($\otimes$):

$$\sigma_K = (14)(23)$$

Faces ($\bullet$):

$$\sigma_f = (13)(24)$$

$M-$ Vertices ($\bigcirc$):

$$\sigma_M = (12)(34)$$

Two step process to rewrite Matrix model correlators

$$\langle \prod_i Tr(K^{l_i}) \prod_j Tr(M^{n_j}) \rangle_{KM}$$

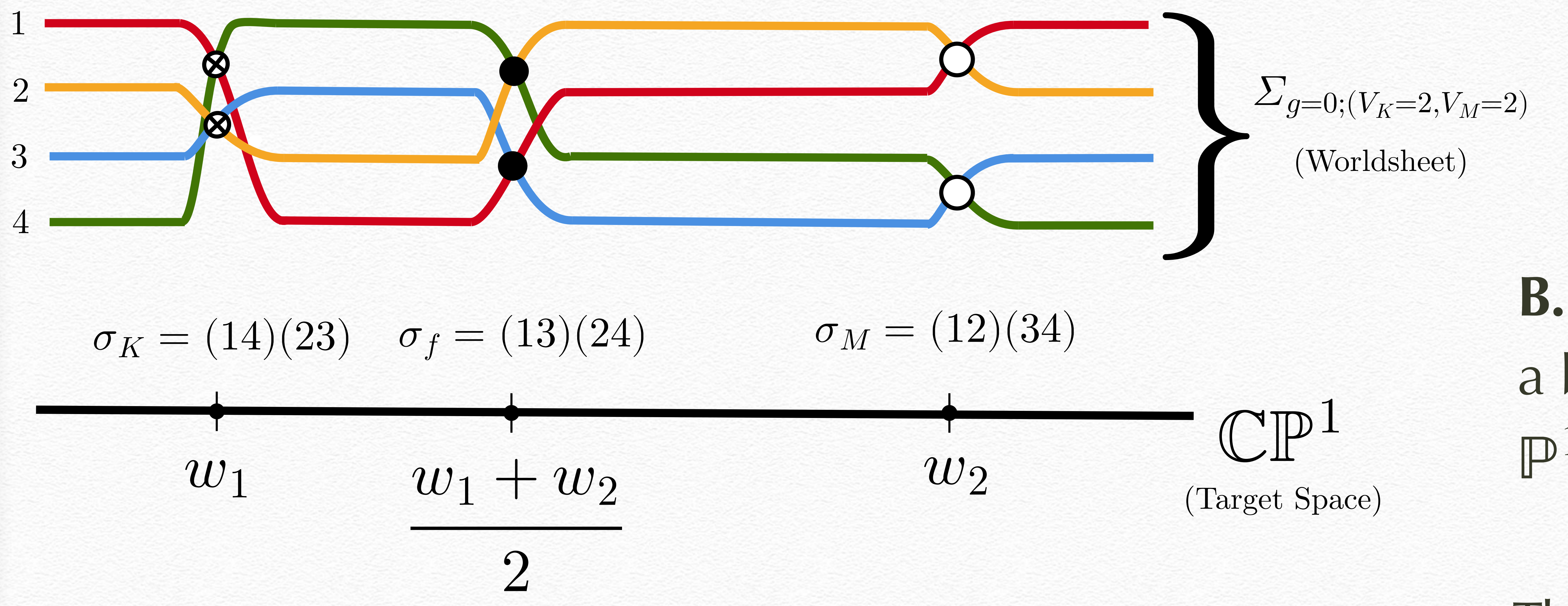
A. Each contribution (wick contraction) can be written in terms of three permutations - cycle structure.

Two of them are determined by the cycle structure of the K, M vertices $(l_1)(l_2)\ldots(l_k)$ and $(n_1)(n_2)\ldots(n_m)$

The third cycle depends on the specific wick contraction (diagram). Cycle structure in terms of faces.

[di Francesco-Itzykson '92; de Mello Koch-Ramgoolam'10; R.G., R.G-Pius'10-'11]

Belyi Maps



Two step process to rewrite Matrix model correlators

$$\langle \prod_i Tr(K^{l_i}) \prod_j Tr(M^{n_j}) \rangle_{KM}$$

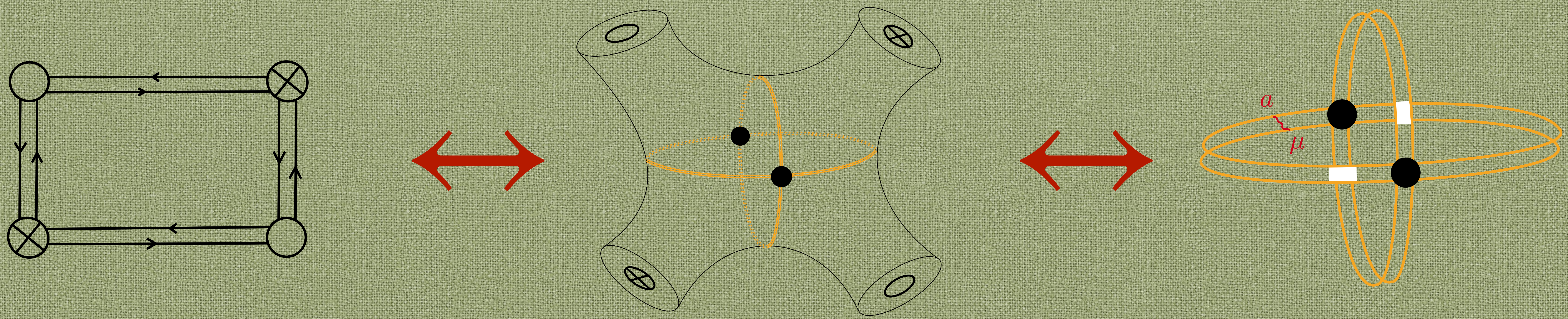
B. Each diagram can be viewed as a branched covering of the target space $\mathbb{P}^1$ by the worldsheet.

The degree of the cover is the number of edges $\sum l_i = \sum n_j$. The three permutations $\leftrightarrow$ three branch points.

Thus sum over Feynman diagrams counts the number of such branched Belyi maps. (Cf. Grothendieck)

Closed string dual is thus an A-model topological string theory with a target topology $\mathbb{P}^1$.

Open-Closed-Open Triality

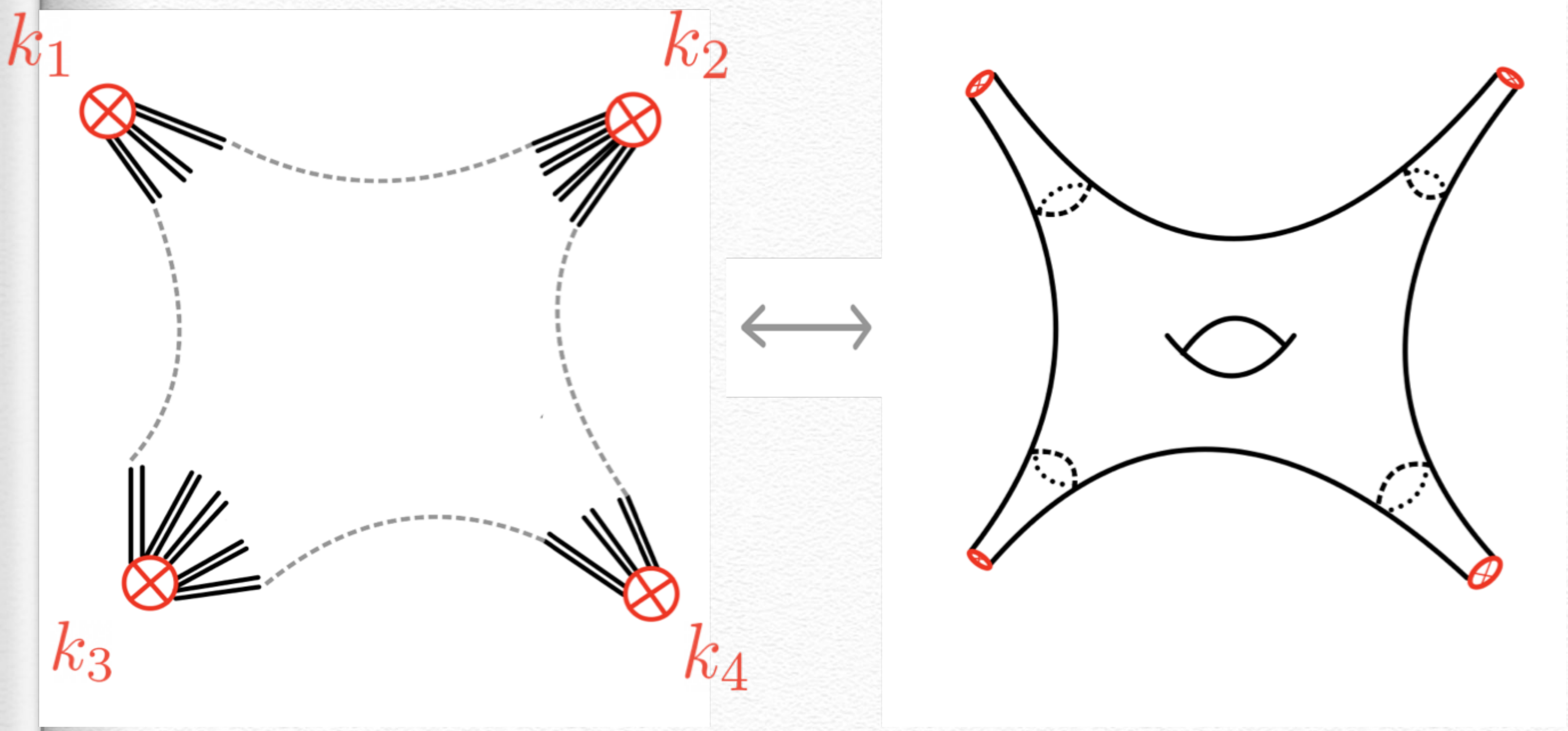


V-type vs. F-type: Refinement of Open-Closed Duality

[R. G'10; R. G-Mazenc '22]

Vertex Type

e.g. $\mathcal{N} = 4$ SYM, Double-Scaled Matrix Models

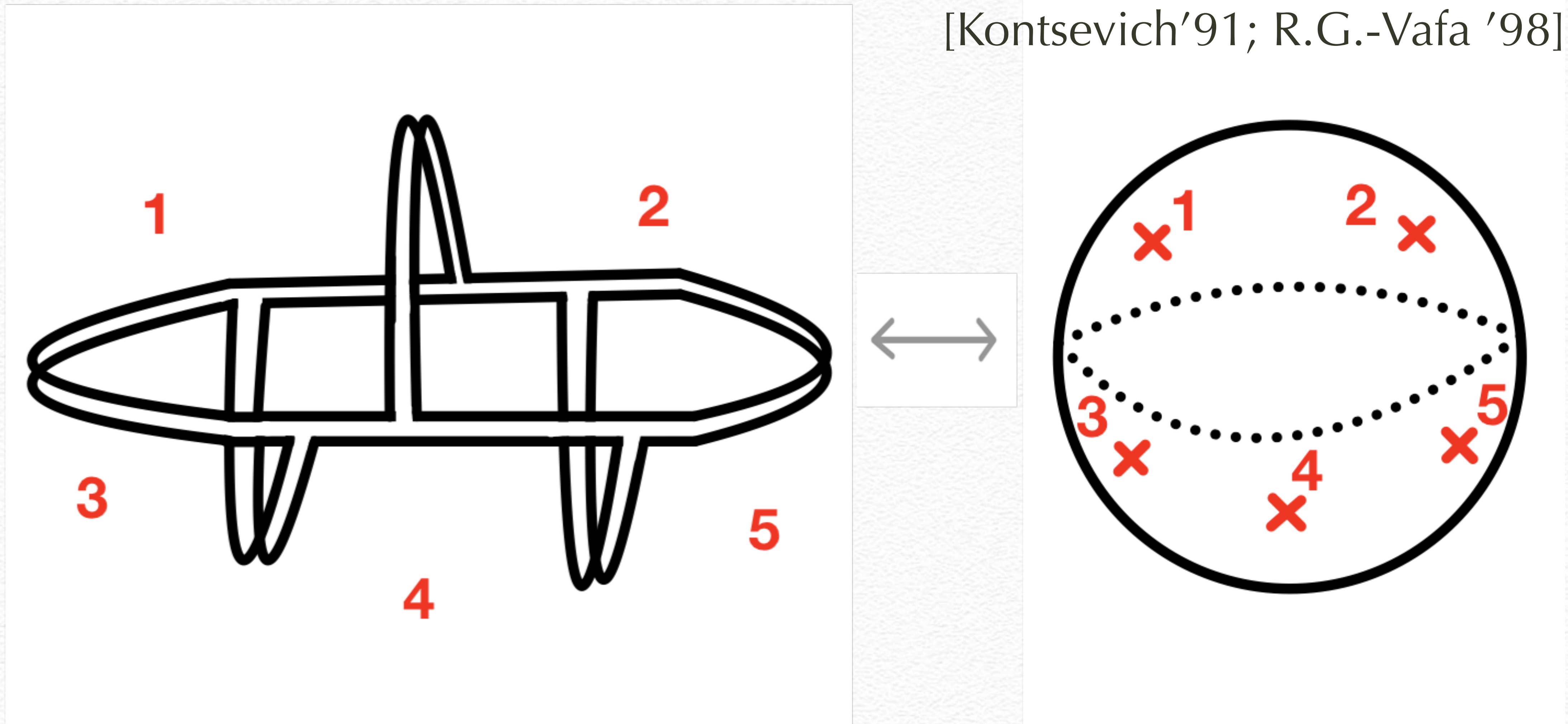


- (Ext.) Diagram Vertices ↔ Closed String insertions
- (Skeleton of) Feynman Diag = Dual of Strebel Graph

Face Type

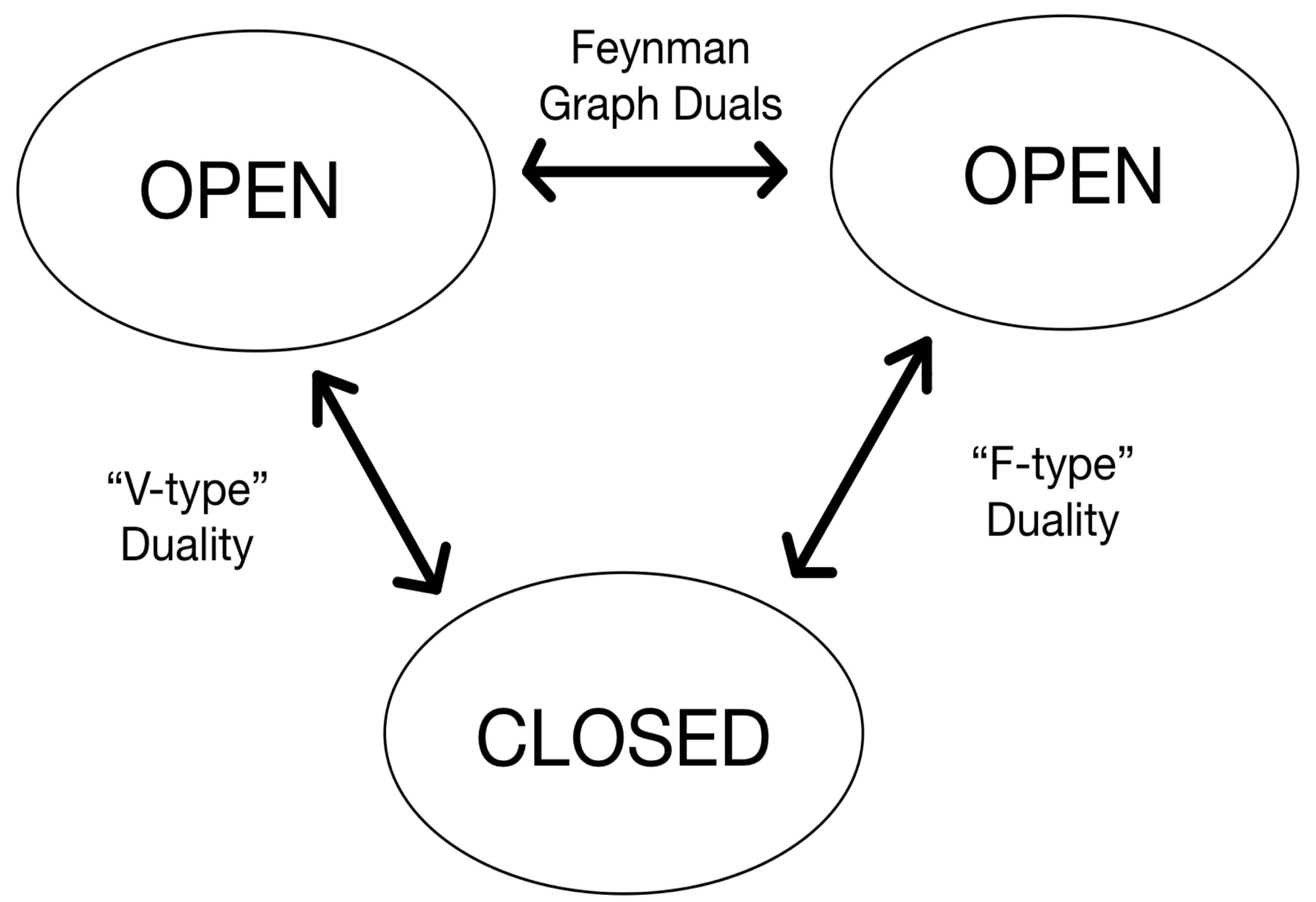
E.g. Chern-Simons duality, Kontsevich Model

[Kontsevich'91; R.G.-Vafa '98]



- Diagram Faces (branes) ↔ Closed String insertions
- Feynman Diagram = Strebel Graph

V-type & F-type: Triality

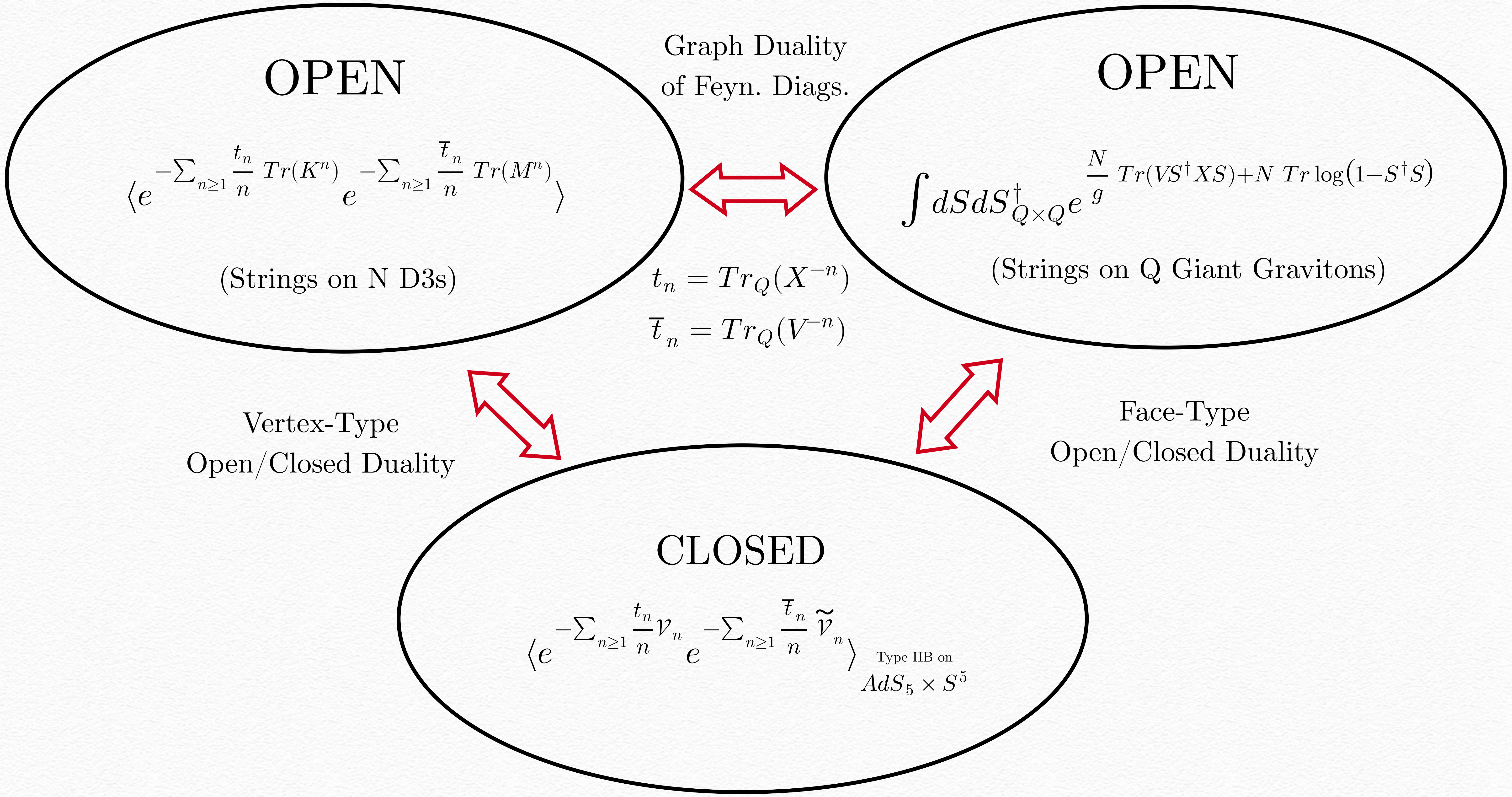


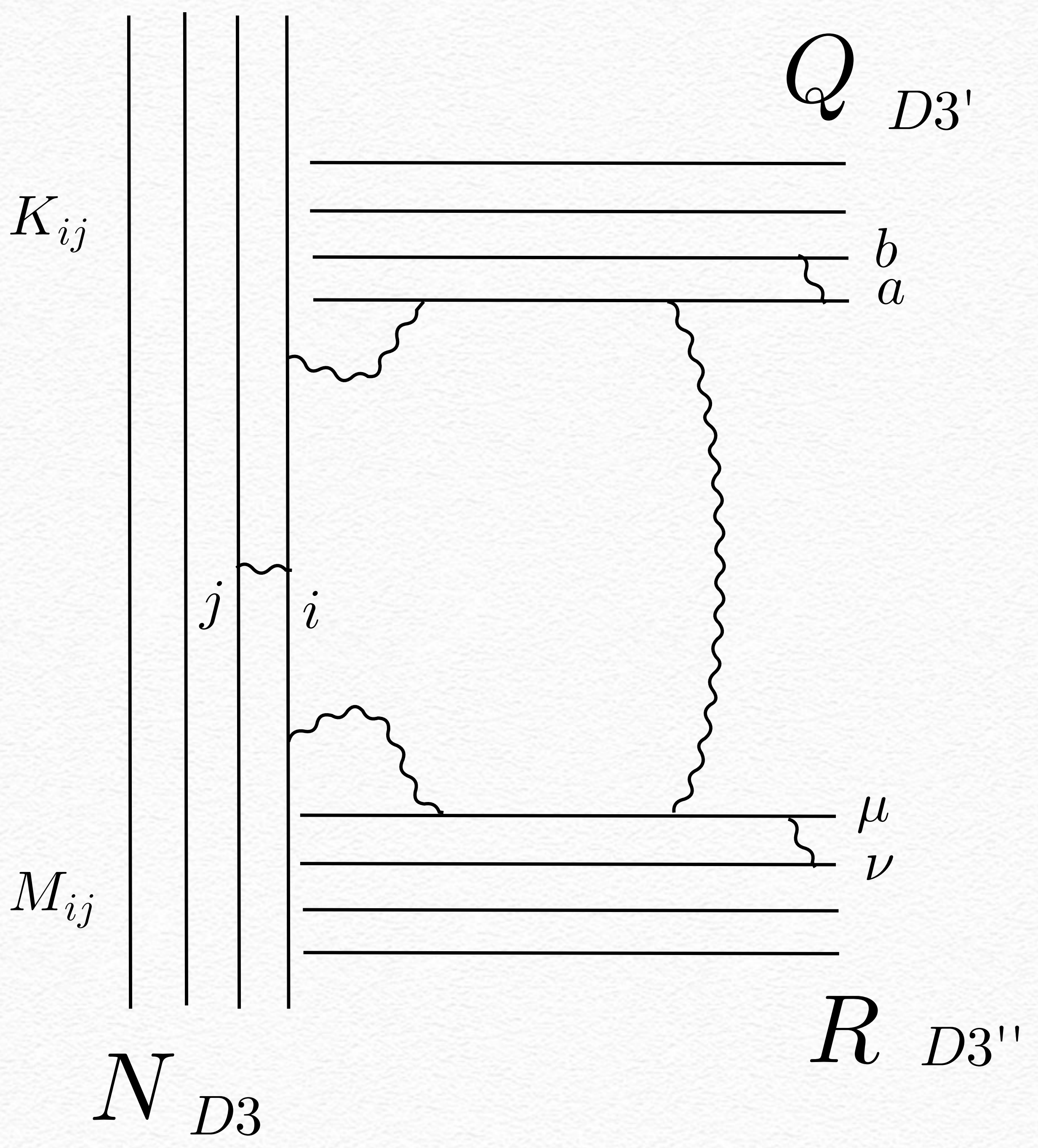
Two complementary open string descriptions arising from open strings on different D-branes.

Early example: Minimal string theory with ZZ-branes and FZZT branes.

[Maldacena-Moore-Seiberg-Shih '04]

V-type & F-type: Triality



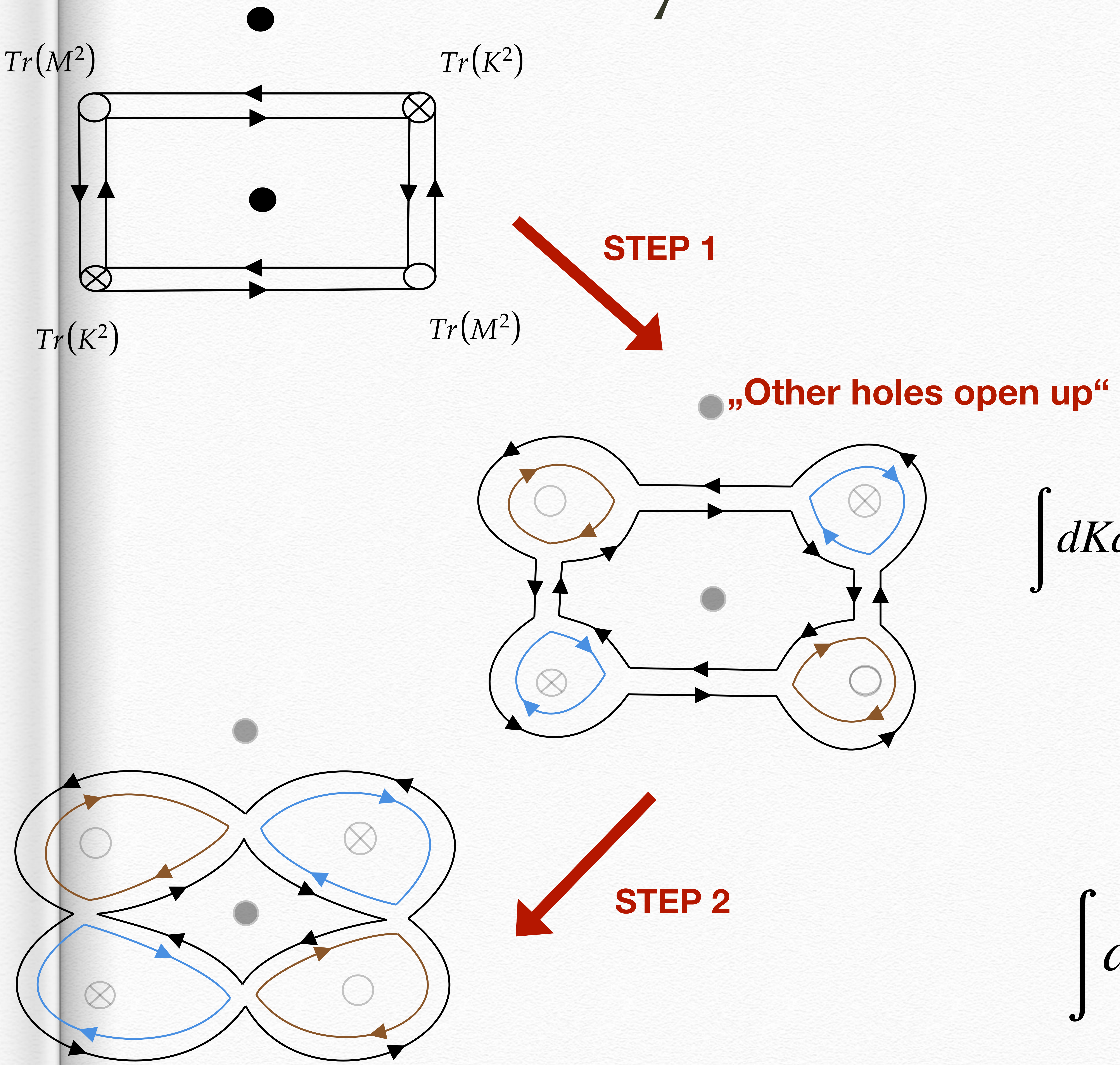


Can realise this in the simple matrix integrals describing this subsector of $\mathcal{N} = 4$ SYM

Explicit integrating in and out of fields - implements dynamical graph duality.

Dynamical Graph Duality

[R. G'10; R. G-Mazenc '22]



$$\int dKdM_{N\times N}e^{-\frac{N}{\lambda}\,Tr(KM)+\sum_n t_n Tr(K^n)+\sum_n \tilde{t}_n Tr(M^n)}$$

STEP 1:
Integrate in $\psi^\dagger_{ia},\chi^\dagger_{i\mu}$

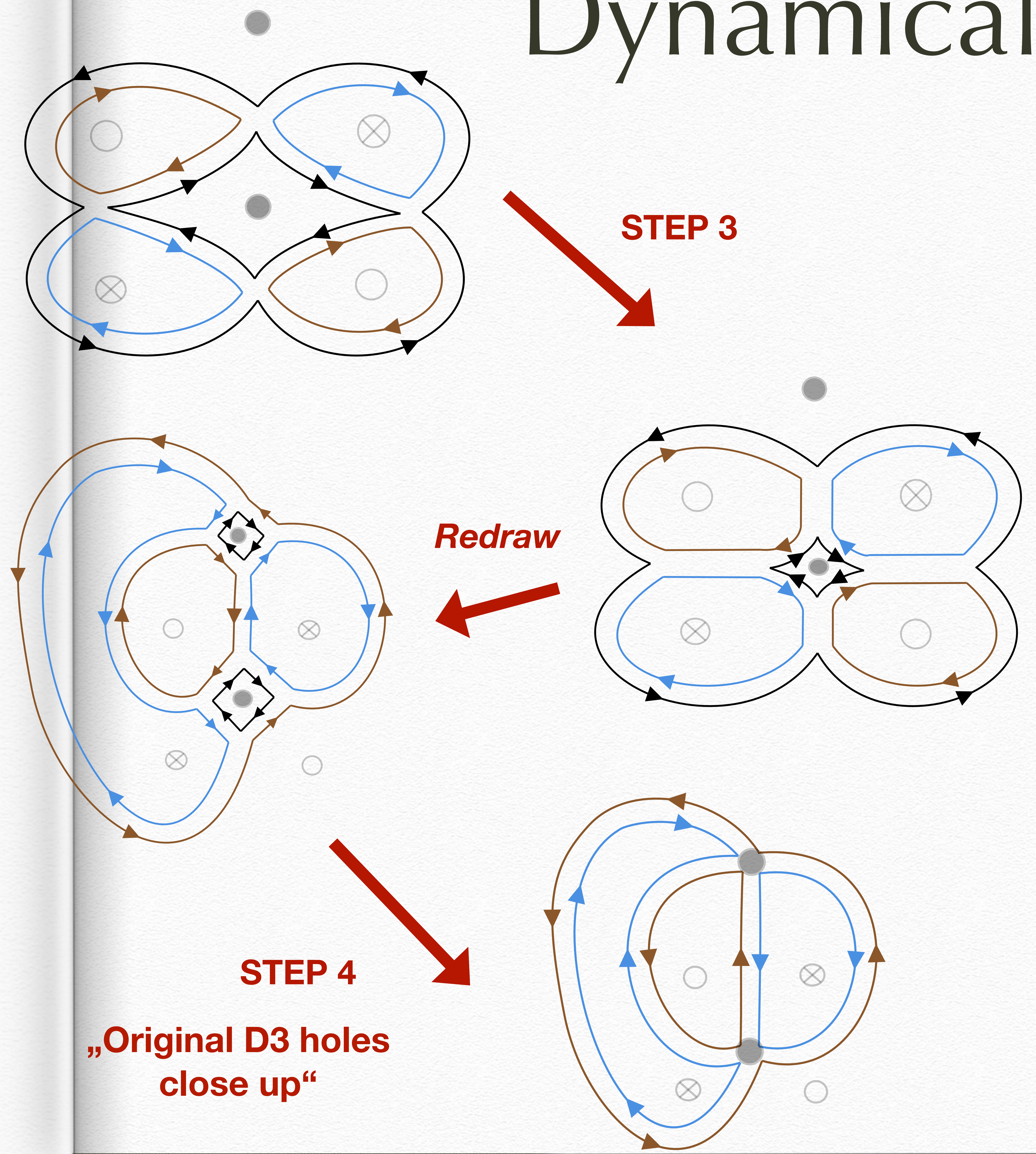
$$\int dKdMd\psi d\psi^\dagger d\chi d\chi^\dagger e^{-\frac{N}{\lambda}Tr_N(KM)+\psi^\dagger_{ia}(X_{ab}\delta_{ij}-\delta_{ab}M_{ij})\psi_{jb}+\chi^\dagger_{i\mu}(V_{\mu\nu}\delta_{ij}-K_{ij}\delta_{\mu\nu})\chi_{j\nu}}$$

STEP 2:
Integrate out K_{ij}, M_{ij}

$$\int d\psi d\psi^\dagger d\chi d\chi^\dagger e^{\psi^\dagger_{ia}X_{ab}\psi_{ib}+\chi^\dagger_{i\mu}V_{\mu\nu}\chi_{i\nu}+\frac{\lambda}{N}\psi^\dagger_{ia}\psi_{ja}\chi^\dagger_{j\mu}\chi_{i\mu}}$$

Dynamical Graph Duality

[R. G'10; R. G-Mazenc '22]



$$\int d\psi d\psi^\dagger d\chi d\chi^\dagger e^{\psi^\dagger_{ia}X_{ab}\psi_{ib}+\chi^\dagger_{i\mu}V_{\mu\nu}\chi_{i\nu}+\frac{\lambda}{N}\psi^\dagger_{ia}\psi_{ja}\chi^\dagger_{j\mu}\chi_{i\mu}}$$

STEP 3:

Integrate in $S^\dagger_{a\mu}$

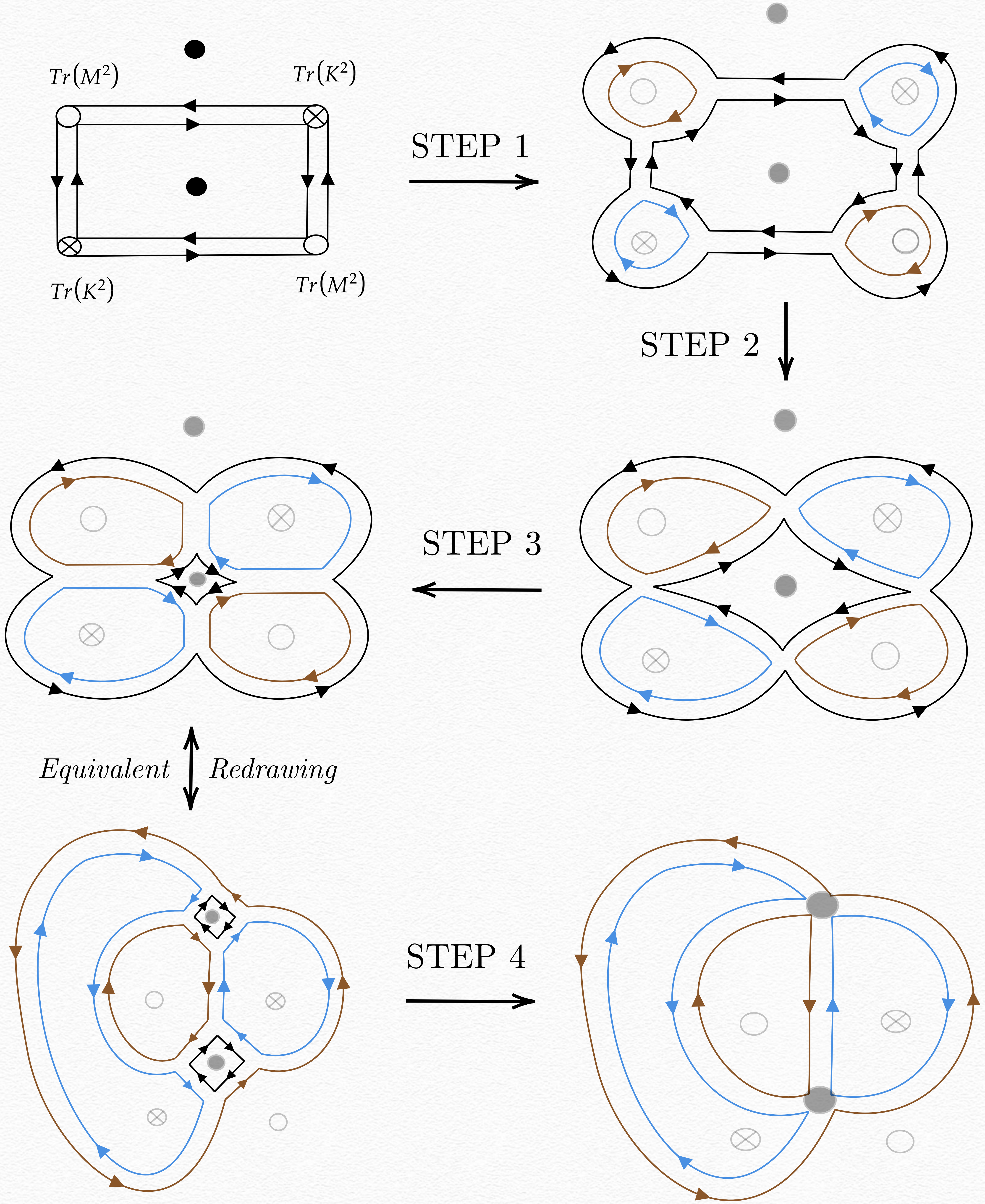
$$\int d\psi d\psi^\dagger d\chi d\chi^\dagger dS dS^\dagger e^{-\frac{N}{\lambda}S^\dagger_{a\mu}S_{\mu a}+S^\dagger_{a\mu}\psi_{aj}\chi^\dagger_{j\mu}+S_{a\mu}\psi^\dagger_{aj}\chi_{j\mu}+\psi^\dagger_{ia}X_{ab}\psi_{bi}+\chi^\dagger_{i\mu}V_{\mu\nu}\chi_{\nu i}}$$

STEP 4:

Integrate out $\psi^\dagger_{ia},\chi^\dagger_{i\mu}$

$$\int dS dS^\dagger_{Q\times R} e^{-\frac{N}{\lambda}Tr(VS^\dagger XS)+N\sum_{k\geq 1}\frac{1}{k}Tr\left((S^\dagger S)^k\right)}$$

Play the Movie!



Thanks for your attention