LMS Prospects in Mathematics
Algebra

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Algebra is the study of abstract algebraic structures. E.g.

- Groups
- Rings
- Vector spaces and modules
- Lie algebras
- Hopf algebras
- Hecke Algebras

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- Fusion Systems
- Cluster Algebras
- Diagram algebras

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Typically, there are two types of problems:

(I) Existence/Classification.
Decide if there exist algebraic structures (of a given sort) satisfying a given property or properties and if so describe them all up to isomorphism.

(II) Representations.
Describe the homomorphisms of a given algebraic structure into linear structures of the same type.

Some examples of classification.

- Abelian groups of order 2^3: C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$ [special case of the structure theorem for finitely generated abelian groups]
- Groups of order 2^3: The abelian ones, D_8, and Q_8.
- Groups of order 2^{11}: Open.

Two useful classification theorems.

- Classification of simple Lie Algebras over $\mathbb{C}$ (Cartan-Killing, late 19th century).
- Classification of finite simple groups (late 20th century). The proof runs to over 10,00 pages, spread over about 500 journal articles, by more than 100 mathematicians.

Representations

Let G be a group and let V be a vector space over a field K.

$\mathrm{GL}(V)$:=group of all invertible linear transformations $V \to V$.

A *representation* of a group G over K is a group homomorphism

$$\rho : G \to \mathrm{GL}(V).$$

If V has dimension n, then via choice of basis, $\mathrm{GL}(V)$ is identified with the general linear group $\mathrm{GL}(n, K)$, that is, the group of invertible $n \times n$-matrices with entries in K. Consequently, ρ is identified with a homomorphism from G into $\mathrm{GL}(n, K)$.

Goal of representation theory: Describe all representations of G over K and the maps between them. More formally: Describe $\mathrm{Mod}-KG$, the category of KG-modules.

A sampling of UK Algebra research.

- Finite Group Theory (structure of finite simple groups and related algebraic groups, permutation groups, p-groups, fusion systems).Bath, Birkbeck, Birmingham, Bristol, Leicester, Imperial, St. Andrews, Warwick.
- Representation theory of finite groups. Birmingham, City, Lancaster, Bristol, Manchester.
- Infinite/geometric group theory (finitely generated groups acting on spaces, word problems on groups). Glasgow, Oxford, Royal Holloway, Southampton.
- Representations of algebraic groups and Lie algebras. Aberdeen, Bath, Birmingham, Leeds, Newcastle, York, Warwick.
- Combinatorial Representation Theory (Coxeter groups, Hecke algebras, diagram algebras), categorification, higher representation theory. Cambridge, City, Kent, East Anglia, Queen Mary, York.
- Geometric Representation Theory (Representations from actions on varieties).Bath, Cambridge, Edinburgh, King's, Glasgow, Oxford.
- Representation of finite dimensional algebras, Cluster algebras. Bath, Bristol, King's, Leeds.
- 2-Representations, categorification, homotopical and homological methods. Bath, City, East Anglia, Leeds, Lancaster, York.

A problem in group representation theory.

Let G be a finite group, V a $\mathbb{C}$-vector space and $\rho : G \to \mathrm{GL}(V)$ a representation.

A subspace $W \subseteq V$ is G-invariant if $\rho(g)(w) \in W$ for all $w \in W$, $g \in G$. If 0 and V are the only G-invariant subspaces of V, then we say that ρ is *irreducible*.

Frobenius (1896): If ρ is irreducible, then $\dim V$ is a divisor of $|G|$.

Fix a prime number p dividing $|G|$. Set

$n_p(G) =$ number of irreducible representations such that $p \nmid \dim V$.

McKay Conjecture (1972)

Let P be a Sylow p-subgroup of G and let $H = N_G(P)$. Then $n_p(G) = n_p(H)$.

There has been major progress on the McKay conjecture and related problems in the last 15 years, using deep results from finite group theory, combinatorial and geometric representation theory.

We still lack a structural understanding of the equation the conjecture predicts.