Pathwise well-posedness of stochastic nonlinear Schrödinger equation with multiplicative noises

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Problem

Stochastic nonlinear Schrödinger equation posed on d-dimensional torus $\mathbb{T}^d$:

$$\begin{aligned} i\partial_t u + \Delta u &= |u|^{2k}u + \boxed{u\Phi\zeta} \quad \text{multiplicative noise} \qquad \textbf{(SNLS)} \\ u|_{t=0} &= u_0 \in H^s(\mathbb{T}^d) \end{aligned}$$

- power nonlinearity $\mathcal{N}(u) = |u|^{2k}u$, $k \in \mathbb{N}_0$
- ζ = space-white, time-fractional/white noise on $\mathbb{R} \times \mathbb{T}^d$
- Φ = Hilbert-Schmidt operator from $L^2(\mathbb{T}^d)$ to $H^s(\mathbb{T}^d)$: $\widehat{\Phi f}(n) = \phi(n)\widehat{f}(n)$ s.t.

$$\phi(n)(1+|n|^2)^{s/2} \in \ell^2(\mathbb{Z}^d)$$

- ■ Cylindrical process $W(t,x) = \sum_{n\in\mathbb{Z}^d} \beta_n(t)\phi(n)e^{in\cdot x}$
- ■ $\{\beta_n\}$ = independent complex fractional Brownian motions with $\beta_{-n} = \overline{\beta_n}$ & Hurst index $\gamma \in (0,1) \Rightarrow \beta_n(t) \in \mathcal{C}_t^{\gamma-\varepsilon}$; $\gamma = 1/2$ corresp. to Brownian motion

$$\Longrightarrow \quad W(t,x) \in \mathcal{C}_t^{\gamma-\varepsilon}H_x^s$$

Goal: Get pathwise local well-posedness

Duhamel formulation: ($S_t = e^{it\Delta}$ linear Schrödinger propagator)

$$u_t = S_t u_0 - i\int_0^t S_{t-r}\mathcal{N}(u)(r)dr - i\underbrace{\int_0^t S_{t-r}u_r dW_r} \qquad \textbf{(mild)}$$

$\Longrightarrow$ Need to define the stochastic convolution

$\gamma = 1/2$ White-in-time, Ito calculus

- de Bouard-Debussche '99, '03
- Brzeźniak-Millet '14
- Brzeźniak-Hornung-Weis '19
- Cheung-Mosincat '19

■ The stochastic convolution

$$\boxed{\int_0^t S_{t-r}u_r dW_r} \qquad \textbf{(Sto.Conv.)}$$

= Ito integral, NOT pathwise

■ What if $\gamma \neq 1/2$? No martingale structure, THEN?

Main task: Define **(Sto.Conv.)** in a **pathwise** manner for all $\gamma \in (0,1)$

> **Theorem** (Oh-Zheng '22) ≈ 1st pathwise LWP result
>
> For $\gamma \in [1/2, 1)$, and $s > d/2$. Then, (**SNLS**) is locally well posed in $H^s(\mathbb{T}^d)$.

First difficulty: deficiency of temporal regularities

- $f : \mathbb{R} \to \mathbb{R}$ continuous, $g : \mathbb{R} \to \mathbb{R}$ of bounded variation

$$\Longrightarrow \quad \mathcal{I}_{t,r} := \int_r^t f dg \quad \text{Riemann-Stieltjes integral}$$

- Extension to $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$ $\Longrightarrow$
  - ■ $\alpha + \beta > 1$: **Young** integral (Young '36)
  - ■ $\alpha + \beta \le 1$: **rough** integral (**rough path theory**: Lyons '98, Gubinelli '04)
- Back to the infinite-dimensional setting (i.e. $H^s(\mathbb{T}^d)$-valued paths)

$$\int_0^t \underbrace{S_{t-r}}_{\text{bad}}\ \underbrace{u_r}_{\gamma-\varepsilon}\ \underbrace{dW_r}_{\gamma-\varepsilon} \Longrightarrow \text{deficiency in temporal regularities even in Young case}$$

Reduction via interaction representation: Apply $\boxed{v_t = S_{-t}u_t}$ to (**mild**) $\Longrightarrow$

Equivalent formulation:

$$v_t = v_0 + (-i)\int_0^t S_{-r}\mathcal{N}(S_r v_r)dr + (-i)\int_0^t S_{-r}(S_r v_r)dW_r$$

- Why interaction representation? $\Leftarrow v \in \mathcal{C}_t^\gamma H_x^s$ leads to $S_t v_t \in \mathcal{C}_t^\gamma H_x^{s-2\gamma}$
  "interplay between temporal and spatial regularities"
- $H^s(\mathbb{T}^d)$ Hilbert-algebra $\Rightarrow$ power-nonlinearity $\mathcal{N}$ = easy term

From now on, focus on

$$v_t = v_0 + \int_0^t S_{-r}\, dW_r\, S_r\, v_r \qquad \textbf{(mild2)}$$

Next: Young & rough cases $\Leftarrow$ controlled rough paths + PDE ideas

Notations: $(\delta f)_{t_1,t_2} = f_{t_1} - f_{t_2}$, $\quad (\delta g)_{t_1,t_2,t_3} = g_{t_1,t_3} - g_{t_1,t_2} - g_{t_2,t_3}$ for $t_1 > t_2 > t_3$

Sewing lemma (Gubinelli '04, Gubinelli-Tindel '10) *Formally*, for a 2-parameter process in $H^s(\mathbb{T}^d)$ with $\delta g \sim 1+\varepsilon$ (temporal) regularity $\|(\delta g)_{t_1,t_2,t_3}\|_{H_x^s} \lesssim |t_1 - t_3|^{1+\varepsilon}$, we can *invert* $g = \Lambda\delta g$ with Λ = sewing map.

Young case: $\gamma \in (1/2, 1)$

Main task: Need to define

$$\boxed{\mathcal{I}_{t,r} = \int_r^t S_{-t_1}dW_{t_1}S_{t_1}v_{t_1}}$$

- $\mathcal{I}_{t,r} = \int_r^t S_{-t_1}dW_{t_1}S_{t_1}v_r + R_{t,r} =: \mathbf{X}_{t,r}v_r + R_{t,r}$
  - ■ introduced integral (random) operators $\mathbf{X}_{t,r}$ (**Q1**: definition & properties?)
  - ■ How to define the remainder $R_{t,r}$?
- Apply the finite-difference operator δ to both sides (see Notations):

(i) LHS additive over disjoint interval $\Longrightarrow (\delta\mathcal{I})_{t_1,t_3} = \mathcal{I}_{t_1,t_3} - \mathcal{I}_{t_1,t_2} - \mathcal{I}_{t_2,t_3} = 0$

(ii) For RHS: Simple algebraic computation $\Longrightarrow (\delta\mathbf{X}v)_{t_1,t_2,t_3} = -\mathbf{X}_{t_1,t_2}(\delta v)_{t_2,t_3}$

$$\text{(i)} + \text{(ii)} \Longrightarrow (\delta R)_{t_1,t_2,t_3} = \underbrace{\mathbf{X}_{t_1,t_2}}_{\gamma-\varepsilon?}\underbrace{(\delta v)_{t_2,t_3}}_{\gamma-\varepsilon} \to \text{sum} = 2\gamma - 2\varepsilon > 1$$

- Sewing lemma $\Longrightarrow$ We can define

$$R = \Lambda\delta R = \Lambda\mathbf{X}\delta v = -\Lambda\delta\mathbf{X}v$$

Once **Q1** answered, we can define

$$\boxed{\int_r^t S_{-t_1}dW_{t_1}S_{t_1}v_{t_1} = \big[(\mathrm{Id} - \Lambda\delta)(\mathbf{X}v)\big]_{t,r}}$$

Answer to Q1 :

$$\boxed{\|\mathbf{X}_{t,r}\|_{H^s \to H^s} \lesssim |t-r|^{\gamma-\varepsilon}} \qquad \textbf{(OP)}$$

On Fourier side: $\mathcal{F}_x(\mathbf{X}_{t,r}f)(n) = \sum_{\substack{n_1,n_2\in\mathbb{Z}^d \\ n_1+n_2=n}} \underbrace{\int_r^t e^{it_1(|n|^2-|n_2|^2)}\phi(n_1)d\beta_{n_1}(t_1)}_{\text{Wiener integral}}\widehat{f}(n_2)$

with split $\mathbf{X} = \mathbf{X}^{\mathfrak{g}} + \underbrace{\mathbf{X}^{\mathfrak{b}}}_{|n| \sim |n_2| \gg |n_1|}$ (n_1 much smaller, $\phi(n_1)$ can not absorb n)

Key ingredients in the proof of (OP)

- **good part:** Random matrix estimates
  - ■ $\gamma = 1/2$: Deng-Nahmod-Yue '22, Bringmann '22, Oh-Wang-Zine '22
  - ■ $\gamma > 1/2$: fractional Brownian motion analogue Oh-Zheng '22
- **bad part**: Schur's test + Kolmogorov continuity argument

Rough case: $\gamma = 1/2$, Brownian-in-time

- Like (scalar) rough integral, immediate issue = deficiency of temporal regularity
- worse mapping property of the integral operator $\mathbf{X}$ (to be proved similarly):

$$\mathbf{X}_{t,r}^{\mathfrak{g}} : H^s(\mathbb{T}^d) \to H^s(\mathbb{T}^d) \qquad \text{and} \qquad \mathbf{X}_{t,r}^{\mathfrak{b}} : H^s(\mathbb{T}^d) \to H^{s-\varepsilon}(\mathbb{T}^d)$$

$\Longrightarrow$ Need to reformulate (**mild2**):

(I) **Finite difference viewpoint from controlled rough path:**

$$\begin{aligned} (\delta v)_{t,r} &= \mathbf{X}_{t,r}(v_\bullet) && (1) \\ &= \mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet) + \mathbf{X}_{t,r}^{\mathfrak{b}}(v_\bullet) \\ &= \mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet) + \mathbf{X}_{t,r}^{\mathfrak{b}}(v_0) + \mathbf{X}_{t,r}^{\mathfrak{b}}\big[(\delta v)_{\bullet,0}\big] && (2) \end{aligned}$$

(II) **Partial iterations of Duhamel formulation:**

$$\begin{aligned} (\delta v)_{t,r} &= \mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet) + \mathbf{X}_{t,r}^{\mathfrak{b}}(v_0) + \mathbf{X}_{t,r}^{\mathfrak{b}}\big[\mathbf{X}_{\bullet,0}(v_\bullet)\big] \quad \Leftarrow \text{plug (1) into (2)} \\ &= \mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet) + \mathbf{X}_{t,r}^{\mathfrak{b}}(v_0) + \mathbf{X}_{t,r}^{\mathfrak{b}}\big[\mathbf{X}_{\bullet,0}(v_0)\big] + \mathbf{X}_{t,r}^{\mathfrak{b}}\big[\mathbf{X}_{\bullet,0}((\delta v)_{\bullet,0})\big] \\ &=: \mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet) + \sum_{j=1}^{2}\mathbf{X}_{t,r}^{\mathfrak{b}}\big(\mathbf{X}_{\bullet,0}^{j-1}v_0\big) + \mathbf{X}_{t,r}^{\mathfrak{b}}\big(\mathbf{X}_{\bullet,0}^{2}(v_\bullet)\big) \\ &= \sum_{j=1}^{J}\mathbf{X}_{t,r}^{\mathfrak{b}}\big(\mathbf{X}_{\bullet,0}^{j-1}v_0\big) + \mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet) + \underbrace{\mathbf{X}_{t,r}^{\mathfrak{b}}\big[\mathbf{X}_{\bullet,0}^{J}(v_\bullet)\big]} \end{aligned}$$

for W temporal smooth: vanishes as $J \to \infty$

where $\mathbf{X}_{t,r}^{j}(v_\bullet) = \mathbf{X}_{t,r}\big[\mathbf{X}_{\bullet,0}^{j-1}(v_\bullet)\big]$

$$\text{(I)} + \text{(II)} \quad \Longrightarrow \quad \boxed{(\delta v)_{t,r} = \sum_{j=1}^{\infty}\mathbf{X}_{t,r}^{\mathfrak{b}}\big(\mathbf{X}_{\bullet,0}^{j-1}v_0\big) + \mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet)} \qquad \textbf{(mild3)}$$

■ **infinite sum** = influence of bad part from each iteration, only depends on v_0

■ mapping property of $\mathbf{X}^{\mathfrak{b}}$ on generic element in $H^s(\mathbb{T}^d)$ = bad
BUT $\mathbf{X}_{t,r}^{\mathfrak{b}}\big(\mathbf{X}_{\bullet,0}^{j-1}v_0\big) \in H^s(\mathbb{T}^d)$ defined a priori via stochastic analysis

Main task now = Define $\mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet)$; Then solve (**mild3**) via standard fixed-point

- **Impose a controlled structure:** In view of (**mild3**)

$$\boxed{(\delta v)_{t,r} = \sum_{j=1}^{\infty}\mathbf{X}_{t,r}^{\mathfrak{b}}\big(\mathbf{X}_{\bullet,0}^{j-1}v_0\big) + \mathbf{X}_{t,r}^{\mathfrak{g}}v_r + R_{t,r}} \qquad \boxed{R_{t,r} \sim 1-}$$

- **Insert the controlled structure** into $\mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet) = \mathbf{X}_{t,r}^{\mathfrak{g}}(v_r) + \mathbf{X}_{t,r}^{\mathfrak{g}}((\delta v)_{\bullet,r}) \Longrightarrow$

$$\mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet) = \boxed{\mathbf{X}_{t,r}^{\mathfrak{g}}(v_r) + \mathbf{X}_{t,r}^{\mathfrak{g}}\Big(\sum_{j=1}^{\infty}\mathbf{X}_{t,r}^{\mathfrak{b}}\big(\mathbf{X}_{\bullet,0}^{j-1}v_0\big)\Big) + \underbrace{\mathbf{X}_{t,r}^{\mathfrak{g}}\big(\mathbf{X}_{\bullet,r}^{\mathfrak{g}}\big)}_{\mathbb{X}_{t,r}^{\mathfrak{g}}}v_r} + \underbrace{Q_{t,r}}_{\text{def.?}}$$

- Apply δ as before $\Longrightarrow (\delta Q)_{t_1,t_2,t_3} = \underbrace{\mathbf{X}_{t_1,t_2}^{\mathfrak{g}}}_{(1/2)-}\underbrace{R_{t_2,t_3}}_{1-} + \underbrace{\mathbb{X}_{t_1,t_2}^{\mathfrak{g}}}_{1-}\underbrace{(\delta v)_{t_2,t_3}}_{(1/2)-} \sim 1+$

- Sewing lemma $\Longrightarrow Q = \Lambda\delta Q \Longrightarrow \mathbf{X}_{t,r}^{\mathfrak{g}}(v_\bullet) = \Big[(\mathrm{Id} - \Lambda\delta)\boxed{\text{previous box}}\Big]_{t,r}$
  - ■ infinite-sum term motivated by spatial-regularity loss
  - ■ 2nd-order operator $\mathbb{X}^{\mathfrak{g}}$ motivated by deficiency in temporal and spatial regularity

What else to do? Rougher noise ($\gamma < 1/2$; $s \le d/2$), $u^k\Phi\zeta$ for $k \ge 2$, ...

still reading? Thanks! **END**