The polynomial Szemerédi theorem and beyond

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Mean ergodic theorem

Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure-preserving dynamical system and $f \in L^2(\mu)$. Then

$$\frac{1}{N}\sum_{n=1}^{N} T^n f \to \int_X f \quad \text{in} \quad L^2(\mu).$$

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Failure of mean ergodic theorem for polynomial iterates

Consider $P(n) = 2n$, $X = \mathbb{Z}/2\mathbb{Z}$, $Tx = x + 1 \mod 2$.

Then $T^2 = Id$, and so $T^{2n}f = f$ for every n. Hence

$$\frac{1}{N}\sum_{n=1}^{N} T^{2n}f = f.$$

Now take $X = \mathbb{Z}/3\mathbb{Z}$, $Tx = x + 1 \mod 3$. Then T^2 is ergodic, and so

$$\frac{1}{N}\sum_{n=1}^{N} T^{2n}f \to \int_X f := \frac{f(0)+f(1)+f(2)}{3}$$

in $L^2(\mu)$.

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Note that

$$n^2 = \begin{cases} 0 \mod 3 & \text{if } n = 0 \mod 3 \\ 1 \mod 3 & \text{if } n = 1, 2 \mod 3. \end{cases}$$

Hence

$$\frac{1}{N}\sum_{n=1}^{N} T^{n^2} f(x) \to \frac{f(x) + 2f(x+1)}{3}$$

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For instance, an irrational translation on $\mathbb{T}$ is totally ergodic.

Then

$$\frac{1}{N}\sum_{n=1}^{N} T^{P(n)}f \to \int_X f \quad \text{in} \quad L^2(\mu)$$

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Multiple ergodic averages with polynomial iterates

We will study averages

$$\frac{1}{N}\sum_{n=1}^{N} T^{P_1(n)}f_1 \cdots T^{P_k(n)}f_k$$

where

1. $(X, \mathcal{X}, \mu, T)$ is an invertible measure preserving dynamical system;
2. $P_1, \ldots, P_k \in \mathbb{Z}[n]$ are distinct polynomials with $P_i(0) = 0$ (we call such polynomials *integral*);
3. $f_1, \ldots, f_k \in L^\infty(\mu)$.

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Furstenberg's proof of Szemerédi theorem

Theorem (Furstenberg 1977)

Let $k \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T)$ be a system and $A \in \mathcal{X}$ be a set with positive measure. Then

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > 0.$$

As a corollary, there exists $n > 0$ such that

$$\mu(A \cap T^{-n}A \cap \cdots \cap T^{-(k-1)n}A) > 0.$$

In particular, there exists $x \in A$ such that

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Theorem (Szemerédi 1975)

Let $k \geq 3$. Then each dense subset of $\mathbb{N}$ contains a k-term arithmetic progression

$$m,\ m+n,\ \ldots,\ m+(k-1)n$$

with $n \neq 0$.

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Theorem (Bergelson & Leibman 1996)

Let $P_1, \ldots, P_k$ be distinct integral polynomials and $(X, \mathcal{X}, \mu, T)$ be a system. Suppose that $A \in \mathcal{X}$ has positive measure. Then

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Corollary

Let $P_1, \ldots, P_k$ be distinct integral polynomials. Then each dense subset of $\mathbb{N}$ contains a polynomial progression of the form

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Convergence of multiple ergodic averages

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$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(A\cap T^{-P_1(n)}A\cap\cdots\cap T^{-P_k(n)}A) > 0.$$

Does the limit exist?

Theorem (Host & Kra 2005; Leibman 2005)

Let $(X, \mathcal{X}, \mu, T)$ be a system, $P_1, \ldots, P_k$ be distinct integral polynomials and $f_1, \ldots, f_k \in L^\infty(\mu)$. Then

$$\frac{1}{N}\sum_{n=1}^{N} T^{P_1(n)}f_1 \cdots T^{P_k(n)}f_k$$

converges in $L^2(\mu)$.

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Weak convergence

Strong convergence implies weak convergence, hence we have the following corollary.

Corollary

Let $(X, \mathcal{X}, \mu, T)$ be a system, $P_1, \ldots, P_k$ be distinct integral polynomials and $f_0, \ldots, f_k \in L^\infty(\mu)$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X f_0 \cdot T^{P_1(n)} f_1 \cdots T^{P_k(n)} f_k$$

exists.

If $f_0 = \ldots = f_k = 1_A$ for $A \in \mathcal{X}$, then we get that

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Factors

A *factor* of a system $(X, \mathcal{X}, \mu, T)$ is a T-invariant sub-σ-algebra $\mathcal{Y}$.

Equivalently, a factor of $(X, \mathcal{X}, \mu, T)$ is a system $(Y, \mathcal{Y}, \nu, S)$ together with a *factor map* $\pi : X \to Y$ satisfying

1. $\nu = \mu \circ \pi^{-1}$;
2. $\pi \circ T = S \circ \pi$.

For instance, the translation

$$S : \mathbb{T} \to \mathbb{T}$$
$$x \mapsto x + a$$

is a factor of the system

$$T : \mathbb{T}^2 \to \mathbb{T}^2$$
$$(x,\ y) \mapsto (x + a,\ y + x),$$

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The idea behind Host-Kra proof

Theorem (Host & Kra 2005)

Let $(X, \mathcal{X}, \mu, T)$ be a system, $P_1, \ldots, P_k$ be distinct integral polynomials and $f_1, \ldots, f_k \in L^\infty(\mu)$. Then the average

$$\frac{1}{N}\sum_{n=1}^{N} T^{P_1(n)} f_1 \cdots T^{P_k(n)} f_k$$

converges in $L^2(\mu)$.

For distinct integral polynomials $P_1, \ldots, P_k$, we want to find a well-structured factor $\mathcal{Y}$ of $(X, \mathcal{X}, \mu, T)$ such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{P_1(n)} f_1 \cdots T^{P_k(n)} f_k$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{P_1(n)} \mathbb{E}(f_1|\mathcal{Y}) \cdots T^{P_k(n)} \mathbb{E}(f_k|\mathcal{Y})$$

The tower of Host-Kra factors

For an ergodic system $(X, \mathcal{X}, \mu, T)$, Host and Kra constructed a tower of factors

$$\mathcal{Z}_0 \subseteq \mathcal{Z}_1 \subseteq \mathcal{Z}_2 \subseteq \cdots \subseteq \mathcal{X}$$

with the following property.

Theorem (Host & Kra 2005)

Let $(X, \mathcal{X}, \mu, T)$ be ergodic, $f_1, \ldots, f_k \in L^\infty(\mu)$ and $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a family of distinct integral polynomials. Then there exists $s \in \mathbb{N}$, such that the $L^2(\mu)$ limits agree

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$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{P_1(n)} \mathbb{E}(f_1|\mathcal{Z}_s) \cdots T^{P_k(n)} \mathbb{E}(f_k|\mathcal{Z}_s).$$

We say that the factor $\mathcal{Z}_s$ is characteristic for the family $\mathcal{P}$ and the system $(X, \mathcal{X}, \mu, T)$.

Importantly, s depends only on $P_1, \ldots, P_k$ but not on X or $f_1, \ldots, f_k$.

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The structure of Host-Kra factors

The factor $\mathcal{Z}_0$ of $(X, \mathcal{X}, \mu, T)$ is the σ-algebra of T-invariant sets.

The factor $\mathcal{Z}_1$ of an ergodic system is the Kronecker factor, i.e. the factor generated by eigenfunctions of T.
Equivalently, it is the maximal factor of X s.t. $(X, \mathcal{Z}_1, \mu, T)$ is isomorphic to a rotation on a compact abelian group.

More generally, the factor $\mathcal{Z}_s$ is an inverse limit of s-step nilsystems, i.e. systems of the form $(G/\Gamma, \mathcal{Y}, \nu, S)$, where:

1. G/Γ is an s-step nilmanifold, i.e. G is an s-step nilpotent Lie group and Γ is a cocompact lattice;
2. $\mathcal{Y}$ is the Borel σ-algebra;
3. ν is the Haar measure;
4. $Sx = gx$ is a left multiplication map for some $g \in G$ (called *nilrotation*).

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2. $\mathcal{Y}$ is the Borel σ-algebra;
3. ν is the Haar measure;
4. $Sx = gx$ is a left multiplication map for some $g \in G$ (called *nilrotation*).

Examples of nilsystems

A translation $Tx = x + a$ on $G/\Gamma = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ defines a 1-step nilsystem.

An example of a 2-step nilsystem is

$$T : \mathbb{T}^2 \to \mathbb{T}^2$$
$$(x,\ y) \mapsto (x+a,\ y+x).$$

This system can be realized as G/Γ for

$$G = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then T corresponds to the left multiplication by $g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$, which is:

$$g \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & y+x \\ 0 & 1 & x+a \\ 0 & 0 & 1 \end{pmatrix}.$$

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The proof of the existence of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{P_1(n)} f_1 \cdots T^{P_k(n)} f_k$$

would go as follows:

1. Some Host-Kra factor $\mathcal{Z}_s$ is characteristic for the average, so we can assume that all the functions are $\mathcal{Z}_s$-measurable.
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Host-Kra factors

Recall the equality

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{P_1(n)}f_1\cdots T^{P_k(n)}f_k$$
$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{P_1(n)}\mathbb{E}(f_1|\mathcal{Z}_s)\cdots T^{P_k(n)}\mathbb{E}(f_k|\mathcal{Z}_s).$$

Recall that $\mathcal{Z}_s$ is a factor of $\mathcal{Z}_{s+1}$ for each $s \in \mathbb{N}$.

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Question

For the rest of the talk, we assume that T is *totally ergodic*, i.e. $T, T^2, T^3, \ldots$ are all ergodic.

Question

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a family of distinct integral polynomials. What is the smallest s such that the factor $\mathcal{Z}_s$ is characteristic for $\mathcal{P}$ for all totally ergodic systems?

Thus, we look for the smallest s such that

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for all totally ergodic systems $(X, \mathcal{X}, \mu, T)$ and $f_1, \ldots, f_k \in L^\infty(\mu)$.

This smallest s is called the *complexity* of $\mathcal{P}$.

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Linearly independent polynomials

Theorem (Frantzikinakis & Kra 2005)

The factor $\mathcal{Z}_0$ is characteristic for linearly independent families such as $\{n, n^2, \ldots, n^k\}$. Thus,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdots T^{n^k} f_k$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n\, \mathbb{E}(f_1|\mathcal{Z}_0) \cdots T^{n^k}\, \mathbb{E}(f_k|\mathcal{Z}_0)$$

whenever T is totally ergodic.

$\mathcal{Z}_0$ is the σ-algebra $\mathcal{I}$ of T-invariant sets, hence

$$T^{n^i}\, \mathbb{E}(f_i|\mathcal{Z}_0) = T^{n^i}\, \mathbb{E}(f_i|\mathcal{I}) = \mathbb{E}(f_i|\mathcal{I}) = \int_X f_i,$$

and so

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Summary of known results

1. Linearly independent families have complexity 0 (Frantzikinakis & Kra 2005).
2. The linear family $\{n, 2n \ldots, kn\}$ has complexity $k-1$ (Host & Kra 2005; Ziegler 2007).
3. For any nonconstant integral polynomial P, the family $\{P(n), 2P(n), \ldots, kP(n)\}$ has complexity $k-1$ (Frantzikinakis 2008).
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Idea

The complexity of $\mathcal{P} = \{P_1, \ldots, P_k\}$ is related to algebraic relations

$$Q_0(m) + Q_1(m + P_1(n)) + \ldots + Q_k(m + P_k(n)) = 0$$

satisfied by the polynomial progression

$$m,\ m + P_1(n),\ \ldots, m + P_k(n).$$

Algebraic relations

An *algebraic relation of degree s* satisfied by $\{P_1, \ldots, P_k\}$ is a relation of the form

$$Q_0(m) + Q_1(m + P_1(n)) + \ldots + Q_k(m + P_k(n)) = 0,$$

where $\max_i \deg Q_i = s$.

For instance, the family $\{n, n^2, n + n^2\}$ satisfies one linear relation (up to scaling):

$$m - (m + n) - (m + n^2) + (m + n + n^2) = 0.$$

The family $\{n, 2n, n^2\}$ satisfies any linear combination of the relations:

$$m - 2(m + n) + (m + 2n) = 0 \quad \text{and}$$
$$(m^2 + 2m) - 2(m + n)^2 + (m + 2n)^2 - 2(m + n^2) = 0.$$

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Complexity conjecture

Conjecture

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a family of distinct integral polynomials. The factor $\mathcal{Z}_s$ is characteristic for $\mathcal{P}$ iff $\mathcal{P}$ satisfies no relation of degree $s+1$ or higher.

E.g., the conjecture is saying:

1. $\mathcal{Z}_1$ is characteristic for $\{n, n^2, n+n^2\}$ because of the relation
$$m - (m+n) - (m+n^2) + (m+n+n^2) = 0.$$
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Why do algebraic relations matter?

It suffices to find the characteristic factor for the weak convergence, i.e. the smallest s for which

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\int_X f_0\cdot T^{P_1(n)}f_1\cdots T^{P_k(n)}f_k$$
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Since T is measure-preserving, we can rewrite the integral as

$$\int_X f_0\cdot T^{P_1(n)}f_1\cdots T^{P_k(n)}f_k=\int_X T^m f_0\cdot T^{m+P_1(n)}f_1\cdots T^{m+P_k(n)}f_k.$$

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Consider the family $\mathcal{P} = \{n,\ 2n,\ 3n\}$ and note the algebraic relation

$$m^2 - 3(m+n)^2 + 3(m+2n)^2 - (m+3n)^2 = 0.$$

Take nonzero $a \in \mathbb{T}$, let $e(y) = e^{2\pi i y}$ and consider the expressions

$$e(am^2),\ e(-3a(m+n)^2),\ e(3a(m+2n)^2),\ e(-a(m+3n)^2)$$

Morally, we can express these exponentials as functions

$$T^m f_1,\ T^{m+n} f_2,\ T^{m+2n} f_3,\ T^{m+3n} f_4,$$

where $f_i(x,y) = e(ac_i y)$ for some $c_i \in \mathbb{Z}$ and

$$T(x,\ y) = (x+a,\ y+2x+a) \quad \text{on} \quad \mathbb{T}^2.$$

Thus,

$$T^m f_1 \cdot T^{m+n} f_2 \cdot T^{m+2n} f_3 \cdot T^{m+3n} f_4 = 1,$$

but

$$\mathbb{E}(f_i|\mathcal{Z}_1)(x,y) = \int_{\mathbb{T}} f_i(x,y')dy' = \int_{\mathbb{T}} e(ac_i y')dy' = 0,$$

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Take nonzero $a \in \mathbb{T}$, let $e(y) = e^{2\pi iy}$ and consider the expressions

$$e(am^2),\ e(-3a(m+n)^2),\ e(3a(m+2n)^2),\ e(-a(m+3n)^2)$$

Morally, we can express these exponentials as functions

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Easy direction in the complexity conjecture

Conjecture

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a family of distinct integral polynomials. The factor $\mathcal{Z}_s$ is characteristic for $\mathcal{P}$ iff $\mathcal{P}$ satisfies no relation of degree $s+1$ or higher.

Extending the previous example, we can show that if $\mathcal{P}$ satisfies an algebraic relation of degree $s+1$, then there is a system $(X, \mathcal{X}, \mu, T)$ and functions $f_0, \ldots, f_k \in L^\infty(\mu)$ such that

$$T^m f_0 \cdot T^{m+P_1(n)} f_1 \cdots T^{m+P_k(n)} f_k = 1$$

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Homogeneity

A *homogeneous* relation of degree s is a relation of the form

$$a_0 m^s + a_1(m + P_1(n))^s + \ldots + a_k(m + P_k(n))^s = 0.$$

A family $\{P_1, \ldots, P_k\}$ is *homogeneous* if all the relations that it satisfies are sums of homogeneous relations.

For instance, the family $\{n, n^2, n + n^2\}$ satisfies only the relation

$$m - (m + n) - (m + n^2) + (m + n + n^2) = 0$$

(up to scaling), hence it is homogeneous.

The family $\{n, 2n, n^2\}$ satisfies the nonhomogeneous relation

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Complexity conjecture holds for homogeneous families

Conjecture

The factor $\mathcal{Z}_s$ is characteristic for $\mathcal{P} = \{P_1, \ldots, P_k\}$ iff $\mathcal{P}$ satisfies no relation of degree $s+1$ or higher.

Theorem (K. 2021)

The conjecture holds for all homogeneous families $\mathcal{P}$.

Examples:

1. $\mathcal{Z}_1$ (Kronecker factor) is characteristic for $\mathcal{P} = \{n,\ n^2,\ n+n^2\}$ because it satisfies one homogeneous relation up to scaling:

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   (this example was previously proved by Frantzikinakis)

2. $\mathcal{Z}_1$ (Kronecker factor) is characteristic for $\mathcal{P} = \{n,\ 2n,\ n^3,\ 2n^3\}$ because it only satisfies linear combinations of two homogeneous relations

$$m - 2(m+n) - (m+2n) = 0 \quad \text{and} \quad m - 2(m+n^3) - (m+2n^3) = 0.$$

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More examples of homogeneous families

1. Linear families $\{a_1 n, \ldots, a_k n\}$;
2. Families $\{a_1 P(n), \ldots, a_k P(n)\}$;
3. Linearly independent families such as $\{n,\ n^2, \ldots, n^k\}$
4. Families satisfying only linear relations

$$a_0 m + a_1(m + P_1(n)) + \ldots + a_k(m + P_k(n)) = 0$$

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Families satisfying only linear relations

Corollary (K. 2021)

Suppose that a family $\mathcal{P}$ satisfies only linear relations

$$a_0 m + a_1(m + P_1(n)) + \ldots + a_k(m + P_k(n)) = 0$$

and no higher order relations. Then the Kronecker factor $\mathcal{Z}_1$ is characteristic for $\mathcal{P}$, i.e.

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{P_1(n)} f_1 \cdots T^{P_k(n)} f_k$$
$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{P_1(n)} \mathbb{E}(f_1|\mathcal{Z}_1) \cdots T^{P_k(n)} \mathbb{E}(f_k|\mathcal{Z}_1) \quad \text{in} \quad L^2(\mu)$$

for every totally ergodic system $(X, \mathcal{X}, \mu, T)$ and $f_1, \ldots, f_k \in L^\infty(\mu)$.

Proof: reduction to nilsystems

1. We use Host-Kra's result to replace

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X T^m f_0 \cdot T^{m+P_1(n)} f_1 \cdots T^{m+P_k(n)} f_k$$

by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X T^m \mathbb{E}(f_0|\mathcal{Z}_s) \cdot T^{m+P_1(n)} \mathbb{E}(f_1|\mathcal{Z}_s) \cdots T^{m+P_k(n)} \mathbb{E}(f_k|\mathcal{Z}_s)$$

for some $s \in \mathbb{N}$.

2. We approximate the system by a totally ergodic nilsystem G/Γ.
3. For an ergodic element $a \in G$, we find the closures of

$$\left(a^m x,\ a^{m+P_1(n)} x,\ \ldots,\ a^{m+P_k(n)} x\right)_{n,m\in\mathbb{N}}$$

inside G^{k+1}/Γ^{k+1} for $x \in G/\Gamma$.

Closure of polynomial sequences

Let G/Γ be a nilmanifold and $g(n) = g_0 g_1^n g_2^{n^2} \cdots g_s^{n^s}$ be a *polynomial sequence* on G which is *irrational*.

Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be a homogeneous family and

$$g^P(m, n) = (g(m),\ g(m + P_1(n)),\ \ldots,\ g(m + P_k(n))).$$

My main result gives an explicit description of the closure of g^P inside the product nilmanifold $(G/\Gamma)^{k+1}$.

This result works if and only if $\mathcal{P}$ is homogeneous.

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Application: equality of limits

Corollary (K. 2021)

Let $(X, \mathcal{X}, \mu, T)$ *be a totally ergodic system and* $f_1, \ldots, f_4 \in L^\infty(\mu)$*. Then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot T^{n^3} f_3 \cdot T^{2n^3} f_4$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \mathop{\mathbb{E}}_{r\in[N]} T^r f_3 \cdot T^{2r} f_4$$

Application: Multiple recurrence

Khintchine's recurrence theorem says that for every $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} : \mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon\}$$

is syndetic (i.e. it has bounded gaps).

It was proved by Bergelson, Host, Kra & Ruzsa that for ergodic systems, the set

$$\{n \in \mathbb{N} : \mu(A \cap T^{-n}A \cap \cdots \cap T^{-kn}A) > \mu(A)^{k+1} - \varepsilon\}$$

is syndetic for $k \leq 3$, but this can fail for higher k.

A similar multiple recurrence property has been proved e.g. for:

1. $\{n, n^2, \ldots, n^k\}$ (Frantzikinakis & Kra 2005);
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Corollary (K. 2021)

Let $P_1, \ldots, P_k$ be integral polynomials satisfying only linear relations (and some technical assumptions on the coefficients). Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $A \in \mathcal{X}$ have positive measure.
For every $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} : \mu(A \cap T^{-P_1(n)}A \cap \cdots \cap T^{-P_k(n)}A) > \mu(A)^{k+1} - \varepsilon\}$$

is syndetic.

Open questions

1. Does the complexity conjecture hold for nonhomogeneous families?
2. Is it true that the complexity of a family of k polynomials is at most $k-1$?
3. If

$$Q_0(m) + Q_1(m + P_1(n)) + \ldots + Q_k(m + P_k(n)) = 0,$$

then is it true that $\max \deg Q_i \leq k-1$?

4. What about pointwise convergence?
5. What if we look at polynomial iterates of several commuting transformations, so averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{P_1(n)} f_1 \cdots T_k^{P_k(n)} f_k?$$

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2. Is it true that the complexity of a family of k polynomials is at most $k-1$?
3. If

$$Q_0(m) + Q_1(m + P_1(n)) + \ldots + Q_k(m + P_k(n)) = 0,$$

then is it true that $\max \deg Q_i \leq k-1$?

4. What about pointwise convergence?
5. What if we look at polynomial iterates of several commuting transformations, so averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T_1^{P_1(n)} f_1 \cdots T_k^{P_k(n)} f_k?$$

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