Periodic pencils of flat connections and their p-curvature

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1. Let X be a smooth variety$/k=\bar{k}$
$X\times V \longrightarrow X$ a trivial vector bundle on X.

Def. A *polynomial family* of flat connections on X with fiber V is a family of flat connections
$\nabla(\vec{s})=d-B(s_1,\dots,s_n)$, where $B\in\Omega^1(X)\otimes \mathrm{End}V$
depends polynomially on $s_1,\dots,s_n$; i.e.
$dB-[B,B]=0$. Such a family is called a *pencil* if
$B=s_1B_1+\dots+s_nB_n$, i.e. the dependence of B on s_i is linear homogeneous.
So the flatness conditions on B_i are

$dB_i = 0$, $[B_i, B_j] = 0$

(if $\dim X = 1$, these conditions are empty).
since these are 2-forms on X.

Def. A family $\nabla(\vec{s})$ is said to be periodic if there exist shift operators $A_j(\vec{s}) \in GL(V)(k(\vec{s})[X])$ such that

$$\nabla(\vec{s}+\vec{e_j}) \circ A_j(\vec{s}) = A_j(\vec{s}) \circ \nabla(\vec{s}).$$

It turns out that such pencils have many remarkable properties, and at the same time there are many interesting examples.

2. Let $k = \mathbb{C}$. Fix $x_0 \in X$ and let $\rho_{\vec{s}} : \pi_1(X, x_0) \longrightarrow GL(V)$ be the monodromy representation of $\nabla(\vec{s})$.

Definition. A family $\nabla(\vec{s})$ has periodic monodromy if

$\forall j$, $\rho_{\vec{s}} \cong \rho_{\vec{s}+\vec{e}_j}$ for a large set of $\vec{s}$ (not contained in a countable union of proper analytic subsets).

Theorem 1. A pencil $\nabla(\vec{s})$ of $\mathbb{C}^n$ has periodic monodromy if and only if its monodromy representation is defined over a finite Galois extension of $\mathbb{C}(\vec{q})$ and is Galois stable, where $\vec{q} = (q_1, \dots, q_n)$,

$$q_j = e^{2\pi i s_j}.$$

Sketch of proof. We have a holomorphic map

$$\rho : \mathbb{C}^n \longrightarrow \mathrm{Hom}(\pi_1(X, x_0), GL(V)) / GL(V).$$

This map is $\mathbb{Z}^n$-periodic, so we get a holomorphic map

$$\bar{\rho} : (\mathbb{C}^\times)^n = \mathbb{C}^n / \mathbb{Z}^n \to \mathrm{Hom}(\pi_1(X, x_0), GL(V)) / GL(V).$$

Since ∇ is a pencil, ρ is exponentially bounded, so $\bar{\rho}$ has polynomial growth. Thus $\bar{\rho}$ is a regular algebraic map. This map has an étale slice $\tilde{\rho} : \widetilde{(\mathbb{C}^\times)^n} \longrightarrow Hom(\pi_1(X, x_0), GL(V))$ where $\widetilde{(\mathbb{C}^\times)^n}$ is a finite cover of an open subset of $(\mathbb{C}^\times)^n$. $\blacksquare$

Theorem 2. A family $\nabla(\vec{s})$ with regular singularities is periodic if and only if it has periodic monodromy.

Sketch of proof. $\Rightarrow$ is obvious, so only need to prove $\Leftarrow$. If ∇ has RS and periodic monodromy, by the Riemann–Hilbert correspondence for sufficiently generic $\vec{s}$ there is an isomorphism $A_j(\vec{s}): \nabla(\vec{s}+\vec{e_j}) \cong \nabla(\vec{s})$.

Pick a basis $f_1, f_2, \ldots$ of $k[X]$. There exists d such that for a large set of $\vec{s}$,

$$A_j(\vec{s},x) = \sum_{i=1}^{d} A_{ji}(\vec{s}) f_i(x).$$

Then the condition that A_j is an isomorphism $\nabla(\vec{s}) \cong \nabla(\vec{s}+\vec{e_j})$ is a finite system of linear equations on A_{ji}, which has an invertible solution for a large set of $\vec{s}$. Then by elimination of quantifiers it has a solution over $k(\vec{s})$. ∎

3. Theorem 2 allows us to give many examples of periodic pencils.

Example 0: The hypergeometric pencil.

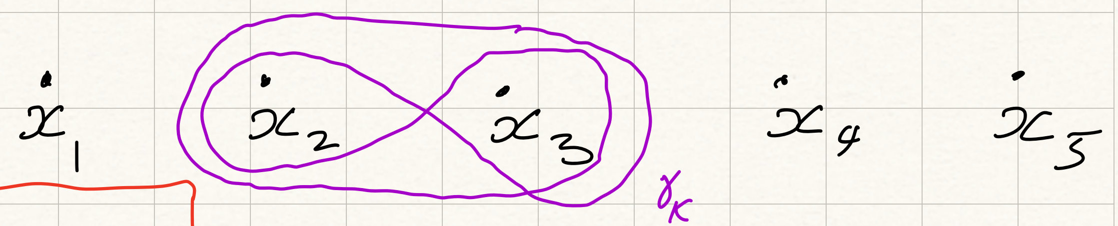
Let $I = (I_1, \ldots, I_r)$ and consider the pencil corresponding to the differential equations

$$\frac{\partial I_j}{\partial x_i} = s_i \frac{I_i - I_j}{x_i - x_j}.$$

with parameters $(s_1, \ldots, s_r)$. The basic solutions are given by (spanning the space V_s given by $\sum s_k I_k = 0$)

$$\boxed{I_{j,\gamma_K} = \int_{\gamma_K} \prod_{i=1} (t - x_i)^{-s_i} \frac{dt}{t - x_j}}$$

Where γ_K is the Pochhammer loop around x_K, x_{K+1}:



and $\boxed{I_{j,0} = 1}$. Then the shift operator is defined on V_s by

$$\boxed{A_k(s, x) = \sum_{i \neq k} \frac{(s_k + 1)(E_{ii} - E_{ik}) + s_i (E_{kk} - E_{ki})}{x_i - x_k}}$$

and

$$A_k(s, x) \mathbb{1} = \mathbb{1}$$

(where $\mathbb{1} = (1, \ldots, 1)^T$).

Example 1. KZ connections. Let $\mathfrak{g}$ be a simple f.d. Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ Cartan, $\lambda_1, \dots, \lambda_r \in \mathfrak{h}^*$ weights, $\beta \in Q_+$, M_λ be the Verma module over $\mathfrak{g}$ with highest weight λ, and consider the space

$$V = (M_{\lambda_1} \otimes \dots \otimes M_{\lambda_r})[\lambda_1 + \dots + \lambda_r - \beta].$$

(this space depends only on β, not on $\lambda_1, \dots, \lambda_r$). The KZ connection is the connection on $X = \mathbb{C}^r \setminus$ diagonals

$$\boxed{\nabla_{KZ} = d - \hbar \sum_{i=1}^{r} \left(\sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \right) dz_i,}$$

where $\Omega \in (S^2\mathfrak{g})^{\mathfrak{g}}$ is the Casimir tensor. This connection is flat, and it is a pencil with parameters $\vec{s} = (\hbar, \hbar\lambda_1, \dots, \hbar\lambda_r)$ $(1 + r \cdot \mathrm{rank}(\mathfrak{g})$ parameters$)$.

Prop 3. ∇_{KZ} is periodic (up to rescaling $\vec{s}$ by an integer)

Proof. Since ∇_{KZ} has regular singularities, it suffices to show that it has periodic monodromy, which follows from the Drinfeld-Kohno theorem:
The monodromy given by R-matrices for the quantum group $U_q(\mathfrak{g})$, which depend on $q = e^{\pi i \hbar}$ (in the simply laced case) and q^{λ_j}.
Ex 0 is the simplest nontrivial special case.
Generalizations: KZ for Kac-Moody algebras, Lie superalgebras, trigonometric KZ, elliptic KZ, KZ in Deligne category, etc.

Example 2. Casimir connections.
The Casimir connection is a connection on $\mathfrak{h}_{reg} \subset \mathfrak{h}$ with

fiber $V = M_\lambda[\lambda - \beta]$, given by

$$\nabla_{Cas} = d - \hbar \sum_{\alpha \in R_+} \frac{e_\alpha f_\alpha + f_\alpha e_\alpha}{2} \frac{d\alpha}{\alpha}$$

where e_α, f_α are root elements of $\mathfrak{g}$. It is flat and forms a pencil with parameters $\vec{s} = (\hbar, \hbar\lambda)$.

Prop. 4. ∇_{Cas} is a periodic pencil (up to rescaling $\vec{s}$).

Proof. Since ∇_{Cas} has RS, it suffices to show that it has periodic monodromy. But this follows from the theorem of A. Appel and V. Toledano-Laredo that the monodromy of ∇_{Cas} is given by the quantum Weyl group of $U_q(\mathfrak{g})$, which depends on $q = e^{\pi i \hbar}$ and q^λ. ☒

Generalizations: Casimir connections for Kac-Moody algebras, trigonometric Casimir connections.

Example 3. Dunkl connections:

W - finite Weyl group with reflection representation $\mathfrak{h}$, V a representation of W. The Dunkl connection is the connection on $\mathfrak{h}_{reg}$ with fiber V

$$\nabla_{Dunkl} = d - \hbar \sum_{s \in Reflections(W)} s \cdot \frac{d\alpha_s}{\alpha_s}$$

α_s = root corr. to s

This is a flat pencil with parameter $\hbar$.

Prop. 5. ∇_{Dunkl} is a periodic pencil.

Proof. It has RS, so only need to check periodic monodromy. But this follows from the fact that the monodromy of the Dunkl connection is given by the Hecke algebra H_q, $q = e^{2\pi i \hbar}$. (Ginzburg, Guay, Opdam, Rouquier).

Generalizations: Dunkl connections for complex reflection groups, trigonometric Dunkl connections

Example 4. Let $\tilde{Y} \to Y$ be a conical symplectic resolution of singularities with finitely many torus fixed points. Consider the equivariant quantum connection $\nabla_{\tilde{Y}}$ with base $H^2(\tilde{Y}, \mathbb{C}^*) \setminus$ divisor and fiber

$H^*(\tilde{Y}, \mathbb{C})$. This is a flat family depending on equivariant parameters $s_1, \dots, s_n$ of the torus $T = (\mathbb{C}^\times)^n$ acting on $\tilde{Y}_{\dagger f}$, and it is known to be a pencil (in a certain basis called the stable basis).

Prop. 6. ∇_Y is a periodic pencil.

Proof. In the work of Okounkov, Pandharipande, Maulik, Braverman, ... it is shown that $\exists$ shift operators coming from geometry. (Stable envelopes).

4. Singularities of periodic pencils.

One of the important properties of periodic pencils is that their singularities occur on hyperplanes defined over $\mathbb{Q}$ (at least up to shift). This echoes the fact that in representation theory singularities tend to occur on such hyperplanes, as many examples of periodic pencils arise from representation theory (as we saw above). For example, let $\mathcal{B}_j \subset \mathbb{C}^n$ be the set of points where $\nabla(\vec{s}) \not\cong \nabla(\vec{s}+\vec{e}_j)$, and $\overline{\mathcal{B}_j}$ be its Zariski closure. Then $\overline{\mathcal{B}_j}$ is contained in the pole divisor of A_j.

Theorem 7. Every codimension 1 irreducible component of $\overline{\mathcal{B}_j}$ is a hyperplane defined over $\mathbb{Q}$ up to shift. Moreover, if $\nabla(\vec{s})$ is generically irreducible then poles of $\bar{A}_j \in PGL(V)$ (projection of A_j) occur on such hyperplanes.

Similar results occur for other types of singularities of $\nabla(\vec{s})$, such as non-semisimplicity loci, jumping loci of endomorphism algebras, etc.

5. To prove Theorem 7, we will use the following theorem, which is interesting in its own right.

Theorem 8. Every periodic pencil has regular singularities.

This can be proved using

p-curvature as we will explain below.

Now the proof of Theorem 7 can be obtained from the following theorem of James Ax.

Theorem 9. (J. Ax, 1971). Let $Y \subset \mathbb{C}^n$ be an irreducible algebraic hypersurface such that $\exp(Y) \subset (\mathbb{C}^*)^n$ is also contained in an algebraic hypersurface. Then Y is a hyperplane defined over $\mathbb{Q}$, up to shift.

Proof of theorem 7: By Th. 8 and the RH correspondence,

$$\nabla(\vec{s}+\vec{e_j}) \not\cong \nabla(\vec{s}) \Longleftrightarrow$$

$$\rho_{\vec{s}+\vec{e_j}} \not\cong \rho_{\vec{s}}$$

But since $\rho_{\vec{s}}$ by Theorem 1 is defined over some finite extension of $\mathbb{C}(e^{2\pi i \vec{s}})$, we see that $\exp(\mathcal{B}_j)$ is contained in an algebraic hypersurface $Z \subset (\mathbb{C}^\times)^n$. So by Theorem 9, $\forall$ codim 1 comp. of $\overline{\mathcal{B}_j}$ is a hyperplane defined over $\mathbb{Q}$ up to shift. $\blacksquare$

⑥ Now consider flat connections in characteristic $p>0$. In characteristic 0, if a connection is flat (= has zero curvature) then it has a full set of formal flat sections near every point. This is, however, false in characteristic p: e.g. the equation $\frac{dy}{dt} = y$ has no $\neq 0$ formal solutions near 0. (the solution $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$ is defined only in char 0).

That's because connections in char. p, besides usual curvature, have another kind of curvature defined by Grothendieck, called the **p-curvature**. Namely, if D_i are covariant derivatives corresponding to coordinates $x_1, \dots, x_r$ on X then the p-curvature is the collection of (commuting) operators

$$\boxed{C_i = D_i^p.}$$

Claim. These operators belong to $k[X]\otimes \mathrm{End}\,V$ (i.e. there are no derivatives).

Example: for the above equation $\nabla = \partial - 1$ so $C = \nabla^p = (\partial-1)^p = \partial^p - 1 = -1$.

Now, if you have a formal flat section f then $\nabla_i f = 0$, so $\nabla_i^p f = 0$, hence $C_i f = 0$.

So to have a fundamental set of formal solutions, we must have $C_i = 0$ for all i.

More generally, for generic $x \in X$ the dimension of the space of formal solutions over $k[[x_1^p, \ldots, x_r^p]]$ is the dimension of the common kernel) of $C_i(x)$ (and it is a free module with an algebraic basis).

Theorem 10. If $\nabla(\vec{s}) = d - \sum_{j=1}^{n} s_j B_j$ a periodic pencil then the p-curvature operators C_i of $\nabla(\vec{s})$ are isospectral to

$$\sum_{j=1}^{n} (s_j - s_j^p)\, B_{ij}^{(1)}, \qquad B_j \overset{\mathrm{def}}{=} \sum_{i=1}^{r} B_{ij}\, dx_i,$$

where $B_{ij}^{(1)}$ is the Frobenius twist of B_{ij} (basically it means that all matrix coefficients are raised to power p).

This is a striking fact

because in general we can say very little about the p-curvature for arbitrarily large p.

Proof. (for simplicity $\dim X = 1$, $n=1$, so $\nabla(s) = \partial_x - sB$).

Let $C(s)$ be the p-curvature of ∇. Since $\nabla(s+1) \cong \nabla(s)$, we have that $C(s+1)$ is conjugate to $C(s)$.

Now let $b_i(s) = \operatorname{Tr} \Lambda^i C(s)$ (coefficients of the characteristic polynomial). Then

$$b_i(s+1) = b_i(s). \qquad (*)$$

Also $C(s) = \nabla(s)^p$ is a polynomial in s of degree p, so

$b_i(s)$ is a polynomial of s of degree pi. Equation $(*)$ implies that $b_i(s) = \beta_i(s - s^p)$ where β_i is a polynomial of degree i. But also $C(0) = 0$, so $C(s)$ is divisible by s, hence $b_i(s)$ is divisible by s^i. Thus $\beta_i(u) = \beta_i^0 \cdot u^i$.

It remains to compute β_i^0. To this end look at the leading term of $b_i(s)$.

We have $C(s) = -s^p B^p + O(s^{p-1})$, $s \to \infty$

so $\beta_i^0 = \operatorname{tr} \Lambda^i (B^p) = \operatorname{tr} \Lambda^i (B^{(1)})$.

Thus $C(s)$ is isospectral

to $(s-s^p)B^{(i)}$, as claimed ∎

Corollary 11. The p-curvature of a periodic pencil is nilpotent when $s_i \in \mathbb{F}_p$.

6. Theorem 10 gives us information about the p-curvature of connections in Ex 1–4 (KZ, Casimir, Dunkl, Quantum conn), since they can be reduced to characteristic p for almost all p.

Def. (N. Katz) A flat connection defined over $\overline{\mathbb{Q}}$ is globally nilpotent if its reduction to characteristic p has nilpotent p-curvature for almost all p.

Theorem 12. A periodic pencil over $\overline{\mathbb{Q}}$ evaluated at rational values of s_j is globally nilpotent.

This follows from Corollary 11.

Theorem 13 (N. Katz 1970). Any globally nilpotent connection has regular singularities.

This gives a proof of Thm 8 that periodic pencils have regular singularities — it follows from Theorem 12 and Theorem 13.

Remark. A theorem of N. Katz

says that if a connection $\nabla/\bar{\mathbb{Q}}$ is geometric (i.e. semisimple and irreducible constituents are direct summands of Gauss-Manin connections)

then ∇ is globally nilpotent. The converse (for semisimple connections) is the André-Bombieri-Dwork conjecture. This generalizes the Grothendieck-Katz conjecture that if ∇ has zero p-curvature upon reduction to almost all primes then it has finite monodromy (= algebraic fundamental solution).

We don't know if

all periodic pencils are geometric; many of them are (such as KZ, Casimir) but for some it is not known despite many efforts (e.g. Dunkl connections for exceptional groups). So these E_6, E_7, E_8 are explicit examples for which the ABD conjecture is open.

7. Def. A regular flat connection ∇ defined over $\overline{\mathbb{Q}}$ is *quasi-geometric* if its monodromy representation is also defined over $\overline{\mathbb{Q}}$.

The pencils in Examples 1–4 evaluated at $\vec{s} \in \mathbb{Q}^n$ are quasigeometric. Also every geometric connection is quasigeometric (since monodromy can be implemented by moving around cycles in the fiber of the family). The following theorem generalizes the monodromy theorem in Hodge theory.

Theorem 14. Every quasigeometric connection has quasiunipotent monodromy along divisors in a compactification $X \subset \overline{X}$.

Proof Repeats Brieskorn's proof of the monodromy theorem, written down by Deligne.

Let λ be an eigenvalue of the residue of ∇ at

a codim $=1$ component D of $\overline{X} \setminus X$ in some trivialization.
Then $e^{2\pi i \lambda}$ is an eigenvalue of the monodromy of ∇ around D. If ∇ is quasi-geometric then both are algebraic, so by a theorem of Gelfond and Schneider (1934) $\lambda \in \mathbb{Q}$, and $e^{2\pi i \lambda}$ is a root of 1. $\blacksquare$

Conjecture 15. If $\mathcal{C}$ is a braided fusion category$/\mathbb{C}$ then the regular connections on configuration spaces corresponding to its braid group representations (and mapping class group representations if $\mathcal{C}$ is modular) are quasi-geometric.

Theorem 16. If $\mathcal{C} = \mathrm{Rep}\, V$, where V is a strongly rational vertex algebra defined over $\overline{\mathbb{Q}}$ then Conjecture 15 holds for $\mathcal{C}$.

Proof. In this case the corresponding connection is the KZ connection of V, which by definition is defined over $\overline{\mathbb{Q}}$. The monodromy of this connection is defined over $\overline{\mathbb{Q}}$ by Ocneanu rigidity.