# ``Dualities'' via non-invertible defects


Paul Fendley

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Talk based on a pair of papers with Luisa Eck, **2302.14081** and **2304.08484**

and on more general work with David Aasen and Roger Mong: **2008.08598**

# I will discuss how fusion categories provide a valuable tool for lattice statistical mechanics

For many, the story of topological quantum computation began with

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

**A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS[1]**

BY VAUGHAN F. R. JONES[2]

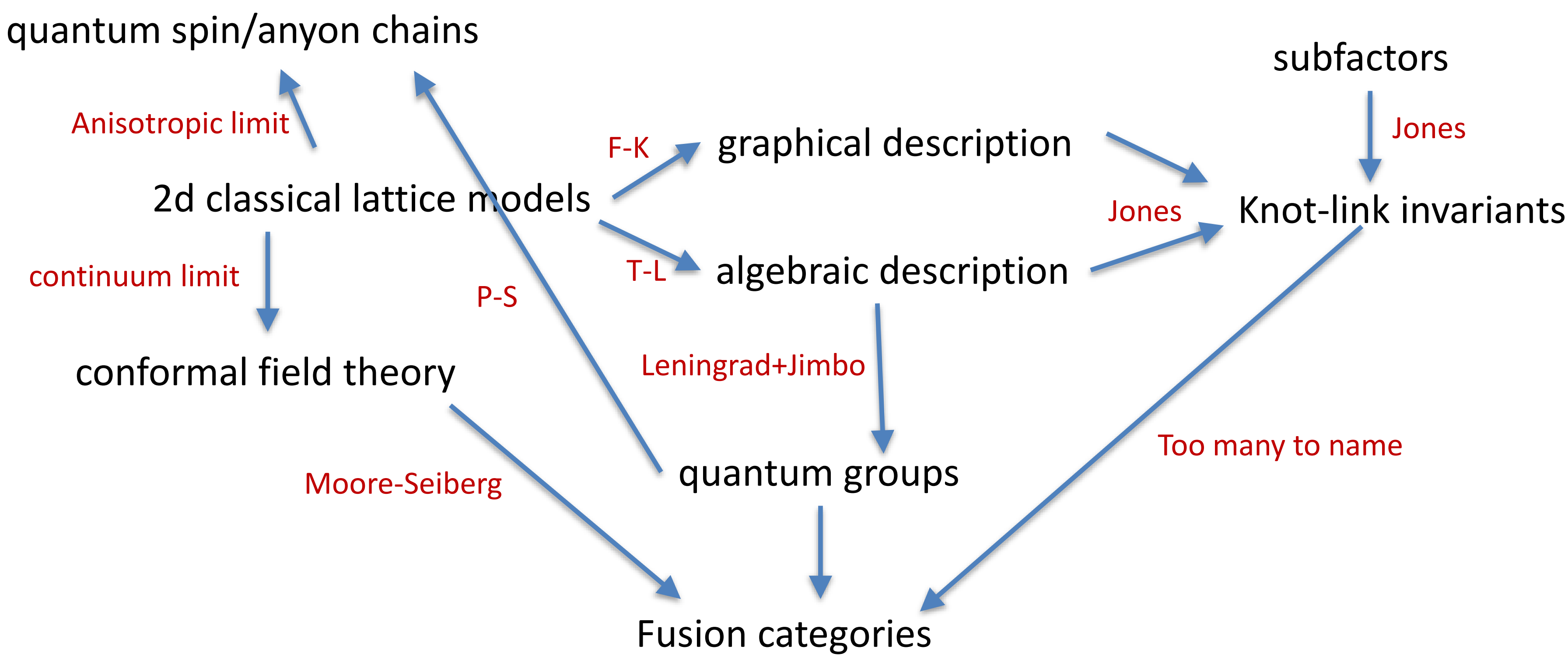
But when you read the paper, you see

For real $t$, D. Evans pointed out that an explicit representation of $A_n$ on $\mathbf{C}^{2n+2}$ was discovered by H. Temperley and E. Lieb [**23**], who used it to show the equivalence of the Potts and ice-type models of statistical mechanics. A readable account of this can be found in R. Baxter's book [**2**]. This represen-

Parallel with great discoveries in knot theory, closely related results came from integrable statistical mechanics, including quantum-group algebras and conformal field theory. Such interplay allowed the development of fusion categories.

# The prehistory of the Delaney diagram

according to a statistical mechanic




And the rest is history…

# Today

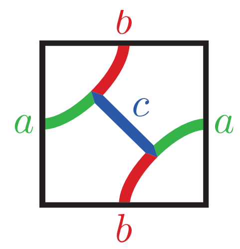
2d classical lattice models,
quantum spin/anyon chains ← algebraic description

↑ Fusion categories

Ask not what I can do for fusion categories,
ask what fusion categories can do for me.

# Fusion categories allow one to construct topological defects both on the lattice and in the continuum

The partition function in the presence of topological defects is independent of local deformations of the defects. Just need to keep track of how they branch/fuse, and how they wrap around cycles. Schematically:




Working in a Hilbert-space formalism, the defect-line creation operators necessarily commute with the Hamiltonian/transfer matrix.

Topological defects generate non-invertible/categorical/ generalized ``dualities'' and symmetries.

# Canonical example is pre-prehistory: Kramers-Wannier duality

AUGUST 1, 1941 PHYSICAL REVIEW VOLUME 60

## Statistics of the Two-Dimensional Ferromagnet. Part I

H. A. KRAMERS, *University of Leiden, Leiden, Holland*

AND

G. H. WANNIER, *University of Texas, Austin, Texas*[1]

(Received April 7, 1941)

In an effort to make statistical methods available for the treatment of cooperational phenomena, the Ising model of ferromagnetism is treated by rigorous Boltzmann statistics. A method is developed which yields the partition function as the largest eigenvalue of some finite matrix, as long as the manifold is only one dimensionally infinite. The method is carried out fully for the linear chain of spins which has no ferromagnetic properties. Then a sequence of finite matrices is found whose largest eigenvalue approaches the partition function of the two-dimensional square net as the matrix order gets large. It is shown that these matrices possess a symmetry property which permits location of the Curie temperature if it exists and is unique. It lies at

$$J/kT_c = 0.8814$$

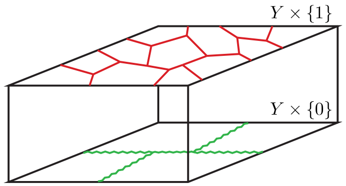
if we denote by $J$ the coupling energy between neighboring spins. The symmetry relation also excludes certain forms of singularities at $T_c$, as, e.g., a jump in the specific heat. However, the information thus gathered by rigorous analytic methods remains incomplete.

The key to generalizing K-W duality is to write lattice models in terms of fusion categories. Enables one to derive exact results without integrability.

# Two ways to proceed
# (both in a talk and in research)

One (``math style”) is to set up the general structure, show all you can show, and then maybe work out some examples.

In a sentence: We use 3d Turaev-Viro-Barrett-Westbury to construct 2d classical lattice models, quantum spin/anyon chains and their topological defects in one fell swoop.




Aasen, Fendley and Mong

The ``physics style” is to work out some examples in detail and then try to say something more general.

In a sentence: We use topological defects to relate a quartet of spin chains and give exact results for their physical properties.

Eck and Fendley

Bonus: one model I focus on turns out to be rather interesting in its own right, and probably experimentally realizable

# One of the nice things about studying statistical mechanics is the contact between formalism and reality.

Rydberg-atom arrays provide a way of realizing all sorts of interesting phases in lattice models. The exceptional ability to tune interactions allows one to carefully examine these phases and the transitions between them.

In the Rydberg blockade, each site is effectively a two-state system that can be viewed as empty or as occupied by a hard-core boson. The blockade means that neighbouring (or more) sites cannot both be occupied.

Such “quantum simulators” have already revealed much interesting physics in strongly interacting systems. One exciting example is quantum scars.

expt: Lukin group
theory: Turner, Michailidis, Abanin, Serbyn, Papic

# Key example today:
# The integrable Rydberg-blockade square ladder

We find a one-parameter family of integrable Hamiltonians.

Getting there requires formal work, which turns out very interesting in its own right.

This ladder is related to the XXZ chain by a non-invertible "duality". It also turns out to possess an unusual non-invertible symmetry.

We also map out a three-parameter phase diagram using the integrable line, a non-invertible symmetry, perturbation theory and numerics. (won't get to)

# Outline

1. Writing the XXZ chain in terms of an algebra

2. A quartet of Hamiltonians obeying this algebra, all integrable
   - Three-state antiferromagnet
   - Ising zigzag ladder
   - Rydberg-blockade ladder

3. Non-invertible mappings and symmetries

4. The physics of the integrable Rydberg-blockade ladder

# 1. ``Equivalences'' via algebras

Proc. Roy. Soc. Lond. A. **322**, 251–280 (1971)
Printed in Great Britain

**Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem**

By H. N. V. Temperley
*Department of Applied Mathematics, University College, Swansea, Wales, U.K.*
and E. H. Lieb†
*Department of Mathematics, Massachussetts Institute of Technology, Cambridge, Mass., U.S.A.*

Aka Tutte polynomial

Aka Potts model partition function

A transfer-matrix approach is introduced to calculate the 'Whitney polynomial' of a planar lattice, which is a generalization of the 'percolation' and 'colouring' problems. This new approach turns out to be equivalent to calculating eigenvalues and traces of Heisenberg type operators on an auxiliary lattice which are very closely related to problems of 'ice' or 'hydrogen-bond' type that have been solved analytically by Lieb (1967*a* to *d*). Solutions for

aka six-vertex model/XXZ chain

# ``Equivalence/duality'' here means a linear identity between partition functions

Temperley and Lieb wrote transfer matrices of different models in terms of operators obeying the same algebra. Their work yielded many "equivalences", i.e. linear identities between partition functions of e.g. Potts, XXZ, loop models.

The Temperley-Lieb algebra also underlies the Jones polynomial.

By now, many generalizations are known, including (at least) one for each simple Lie algebra $G$ and each positive integer $k$, known as $G_k$. These yield both more general knot and link invariants, and interesting (integrable) lattice models.

One of these is quite elegant and useful, but has escaped much attention…

# The XXZ chain, algebraically

The Hilbert space is a chain of $L$ two-state systems (on half-integer-labelled sites).
Hamiltonian (with periodic boundary conditions) written in terms of Pauli matrices:

$$H_{\mathrm{XXZ}} = \frac{1}{2}\sum_{j=1}^{L}\Big(X_{j-\frac{1}{2}}X_{j+\frac{1}{2}} + Y_{j-\frac{1}{2}}Y_{j+\frac{1}{2}} + \Delta(1 + Z_{j-\frac{1}{2}}Z_{j+\frac{1}{2}})\Big)$$

Write $H = \sum_{j=1}^{L}(S_j + \Delta P_j)$ , with generators $S_j,\ P_j$ defined as

$$S_j = \frac{1}{2}\Big(X_{j-\frac{1}{2}}X_{j+\frac{1}{2}} + Y_{j-\frac{1}{2}}Y_{j+\frac{1}{2}}\Big)\ , \qquad P_j = \frac{1}{2}\Big(1 + Z_{j-\frac{1}{2}}Z_{j+\frac{1}{2}}\Big)\ .$$

These generators commute when far apart, and less obviously satisfy

$$\big(S_j\big)^2 = 1 - P_j\ , \qquad (P_j)^2 = P_j\ , \qquad S_jP_j = P_jS_j = 0\ ,$$
$$S_jS_{j\pm1}S_j \;=\; P_jS_{j\pm1}P_j = 0$$

A solution of the Yang-Baxter equation follows solely from this algebra.

Any such model is integrable for any $\Delta$

Maassarani

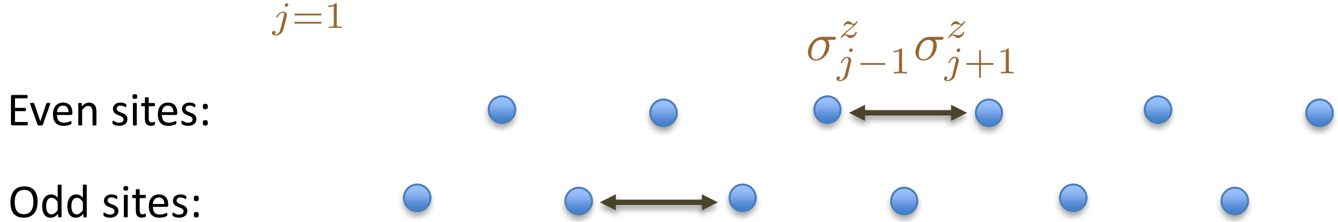
# 2. A quartet of ``equivalent'' Hamiltonians

Martina, Protogenov and Verbus; Finch; Braylovskaya, Finch and Frahm;
Gils et al; Aasen, Fendley and Mong; Lootens, Verstraete et al

$$H = \sum_{j=1}^{L} \left(S_j + \Delta P_j\right)$$

a. Zig-zag Ising ladder

$$H_{\text{zig}} = \frac{1}{2} \sum_{j=1}^{L} \Big(\sigma^z_{j-1}\left(1 + \sigma^x_j\right)\sigma^z_{j+1} + \Delta\left(1 - \sigma^x_j\right)\Big)$$




Thus two transverse-field Ising chains coupled by interaction around each triangle.

Symmetry is only $\mathbb{Z}_2 \times \mathbb{Z}_2$, not $U(1)$: need to be careful with "equivalence"!

## b. Three-state Potts antiferromagnetic chain

Impose ``zero temperature'' constraint that adjacent states are different,
i.e. with each state labelled $A,\ B$ or $C,$ then $AA,\ BB$ or $CC$ are forbidden,
while e.g. $ABCBCABACACAB$ is allowed.

$$H = \sum_{j=1}^{L} \left( S_j + \Delta P_j \right)$$

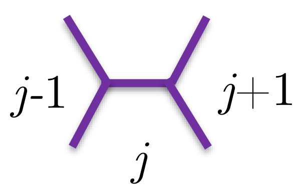
$S_j$ changes state at site $j$ if allowed, e.g. $ABA \leftrightarrow ACA$

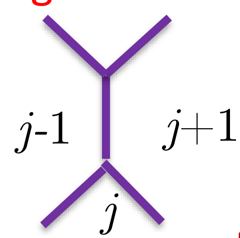
$\Delta P_j$ gives energy $\Delta$ if changing state at site $j$ not allowed, e.g. $ABC$

Generalises old work on classical model at $\Delta = -1/2.$

Baxter; Saleur; Cardy, Jacobsen & Sokal; Delfino

Writing in terms of domain walls gives a graphical presentation via the
domain walls of 3-state Potts satisfying the chromatic algebra

$S_j =$ 


$P_j =$ 


Fendley and Krushkal

# To go further, exploit the fact that the S-P algebra follows naturally from the projectors of the fusion category $SU(2)_4$

A fusion category is comprised of ``objects'' that obey ``fusion rules''.

The $SU(2)_k$ fusion rules are a truncated version of $SU(2)$. For $SU(2)_4$ , stop at spin 2.

Fusion with spin 1:

$$\frac{1}{2} \otimes 1 = \frac{1}{2} \oplus \frac{3}{2}$$

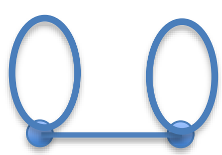
$$\frac{3}{2} \otimes 1 = \frac{1}{2} \oplus \frac{3}{2}$$

$$0 \otimes 1 = 1$$

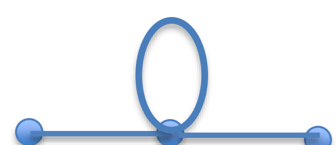
$$1 \otimes 1 = 0 \oplus 1 \oplus 2$$

$$2 \otimes 1 = 1$$

Encode these rules in pictures, where line indicates fusion with spin 1



$\frac{1}{2}$ $\frac{3}{2}$

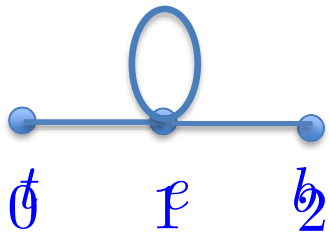


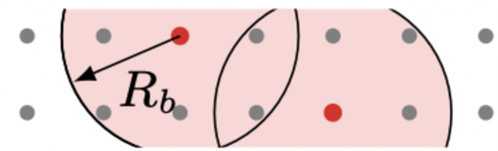
$0$ $1$ $2$

The fusion category gives other Hamiltonians whose generators obey same algebra!

# c. Integrable Rydberg-blockade ladder

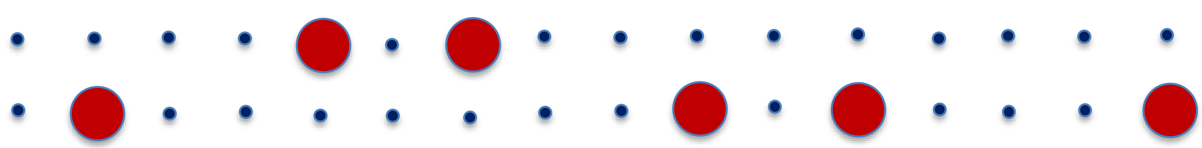
Using category, can construct ``height'' (RSOS/IRF) models with nice properties.
For $SU(2)_4$ with degrees of freedom 0,1,2 obeying $1 \otimes 1 = 0 + 1 + 2$ ,
adjacent heights on a chain must be adjacent on the diagram




Allowed configuration is $121101011212 1112$ -> *ebeteteebebeeeb*

Interpret as a ladder: 0 is particle on top rung, 2 on bottom, 1 is empty rung

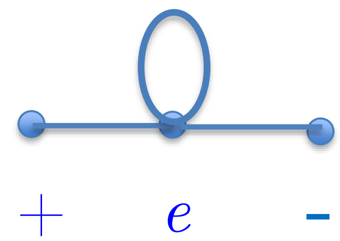


1 2 1 1 0 1 0 1 1 2 1 2 1 1 1 2

Rules mean at most one particle per square. The Rydberg blockade!

# Integrable Rydberg-blockade ladder

$$|\pm\rangle = \frac{1}{\sqrt{2}}\big(|t\rangle \pm |b\rangle\big)$$




Using the category, find $S_j,\ P_j$ so that

$$H_{IRL} = \sum_j \left[ p_j + p_j^\dagger + s_{j-1}s_{j+1} + \Delta\Big(n_j^- + \big(n_{j-1}^{(e)} - n_{j+1}^{(e)}\big)^2\Big)\right]$$

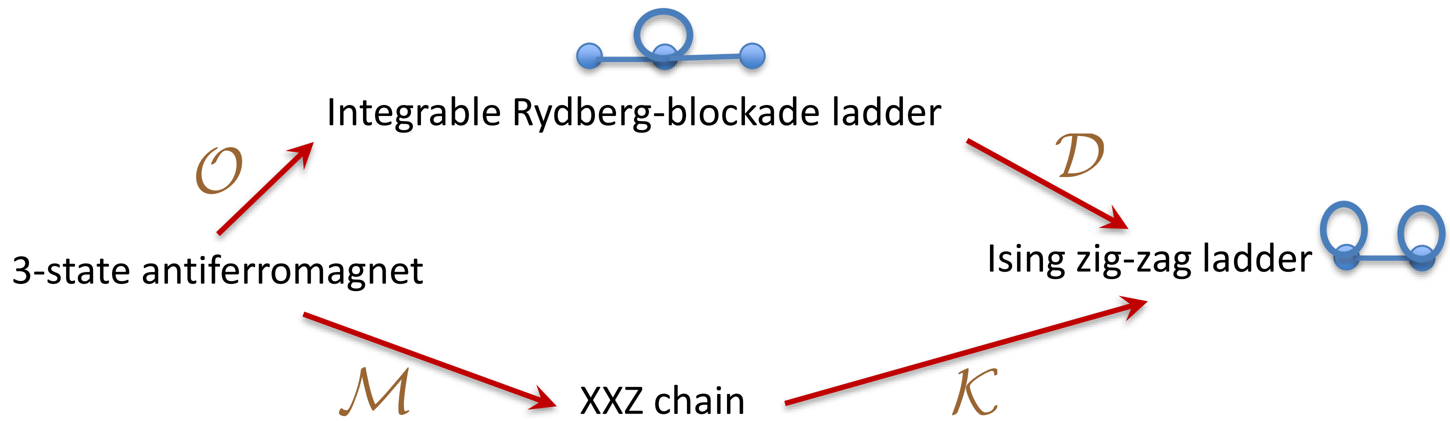
where $p_j$ annihilates a plus state, $s_j$ swaps $+ \leftrightarrow -$, and $n_j^{(a)} = 0,\ 1$ depending on which of $a = e, +, -$ is on rung $j$.

Be careful with ``equivalent'':

No $U(1)$, but only $\mathbb{Z}_2 \times \mathbb{Z}_2$ from flipping $t \leftrightarrow b$ on odd rungs or on even rungs.

# 3. Non-invertible mappings and symmetries

To use (known) XXZ physics with the Rydberg-blockade ladder, utilise exact but non-invertible maps. Fusion categories and orbifolds give all four maps explicitly.




Find $\mathcal{D}\mathcal{O} = \mathcal{M}\mathcal{K}$ and the Hamiltonians ``commute'' e.g. $\mathcal{K}\, H_{\mathrm{XXZ}} = H_{\mathrm{zig}}\, \mathcal{K}$

Non-invertible because e.g. $\mathcal{K}\,\mathcal{K}^{\dagger} = 1 + \mathcal{F}$ where $\mathcal{F}$ flips all spins.

Thus $1 + \mathcal{F}$ has zero eigenvalues, and maps only work in certain sectors.
Can use twisted boundary conditions to get others.

# Describe non-invertible mappings/symmetries using topological defects

Aasen, Fendley and Mong


A topological defect provides an interface between the two ``equivalent'' theories. Partition function is independent of deformations of path.

Topological defect-line creation operators necessarily commute with the Hamiltonian/transfer matrix. They generate ”dualities” when they map between different models, and “symmetries” when the models are the same.

The quotes are because these operators are typically non-unitary. Sometimes they are not even invertible.

Do not need to impose integrability for such symmetries/dualities to exist.

# Fusion categories provide a key tool

Types of topological defects are labelled by objects in the Drinfeld centre.

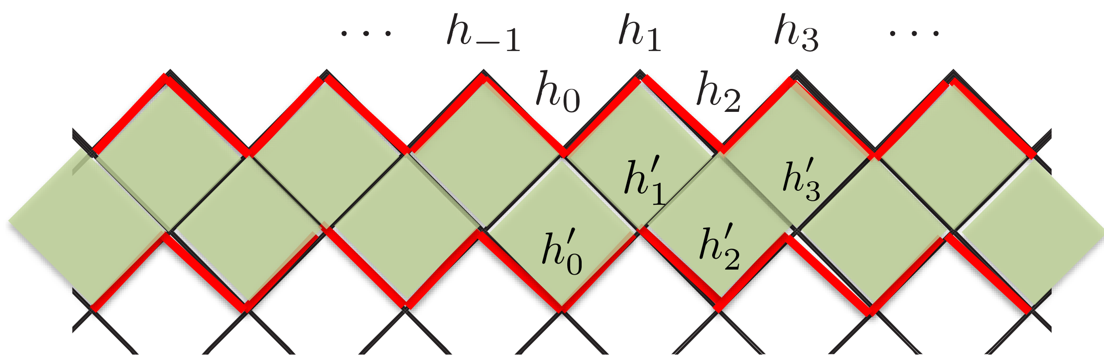
The construction via T-V-B-W ensures they have nice properties.

In particular, they are topological! Deform them without changing the physics.

Also guarantee that the defects themselves satisfy fusion rules and F moves.

# Transfer matrix/Hamiltonian

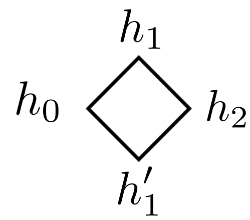
Easiest to work in a (Euclidean) space time (i.e. classical) picture. Degrees of freedom are ``heights'' living on the sites of some lattice, e.g.




Basis elements of Hilbert space are allowed height configurations along zig-zag line.

Transfer matrix $T$ then adds rows with appropriate Boltzmann weights.

$$Z = \sum_{\text{heights}} \prod_{\text{faces}}$$

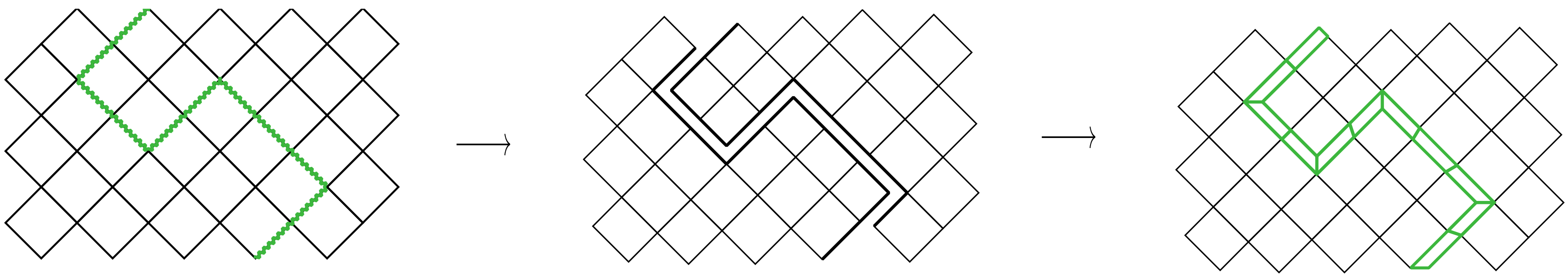



$$= \mathrm{tr}\, T^R$$

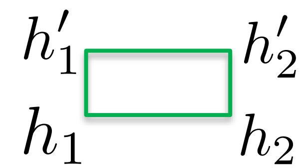
for $R$ rows and periodicity in vertical direction

Hamiltonian is then found in a limit where $T \sim 1 + H + \ldots$

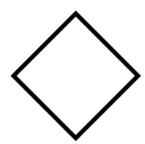
# Inserting a defect



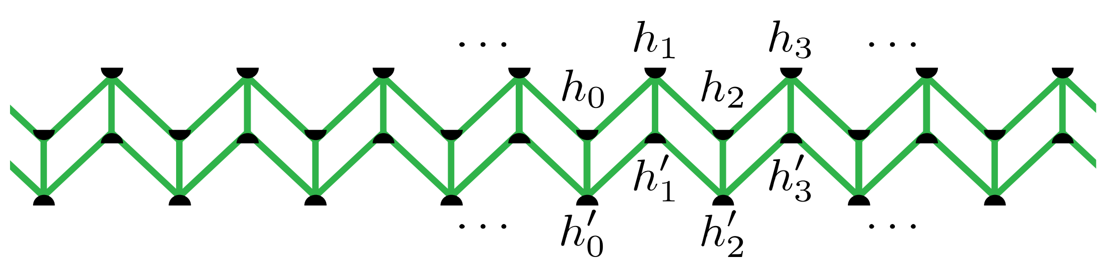
The defects have a weight depending on the adjacent heights:




In the presence of the defect, the partition function is modified to

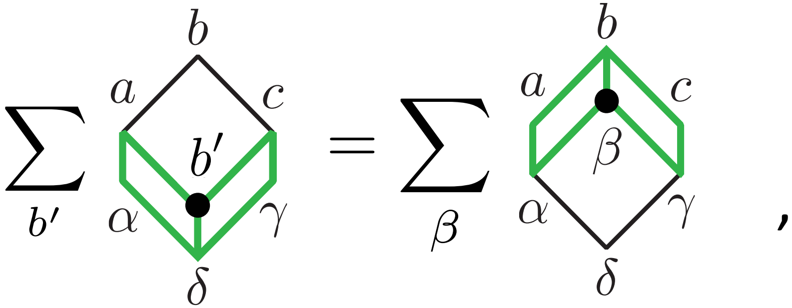
$$\mathcal{Z} = \sum_{\text{heights}} \prod_{\text{faces}}$$  $$\times \prod_{\text{along defect}}$$ 

Defect-line creation operator: $\mathcal{D} =$

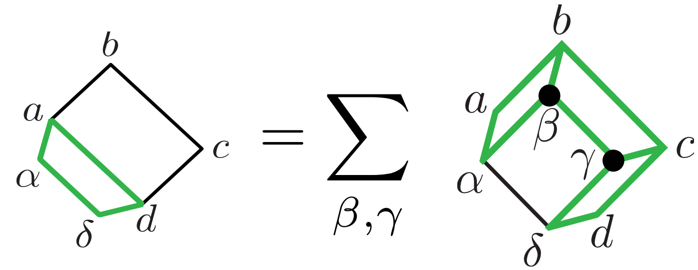


# To make defects topological, they must satisfy defect commutation relations

For $\mathcal{Z}$ to be invariant under deformations of the defect's path:

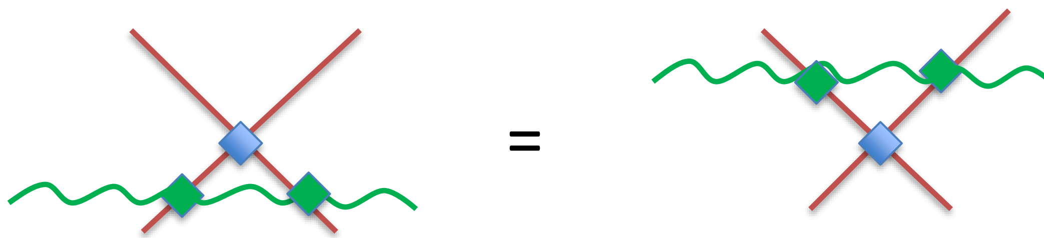



,



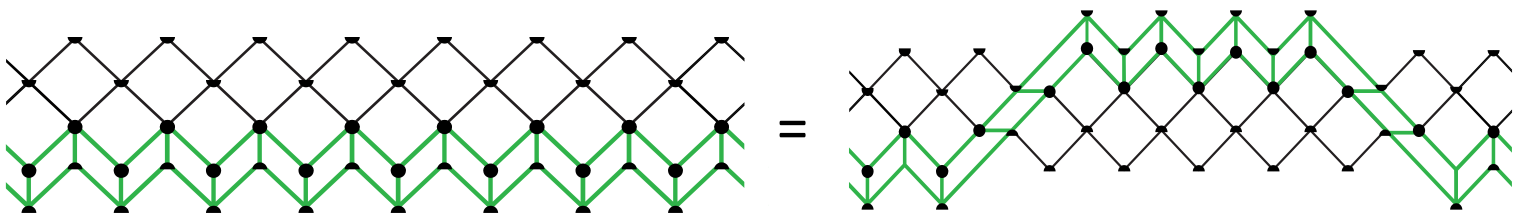

In tensor networks, these have been dubbed ``pulling-through'' conditions:

Verstraete et al




Then

$\mathcal{D}T =$

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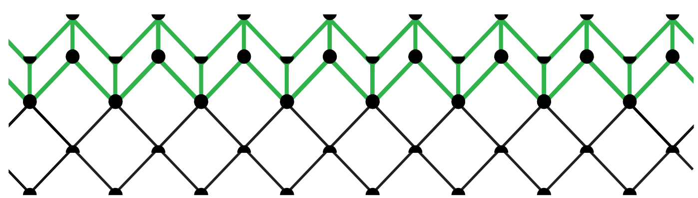
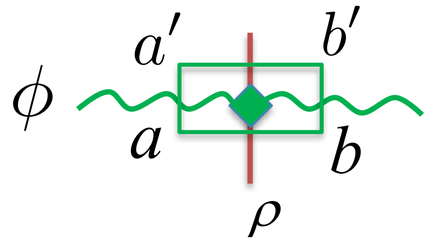


$= T\mathcal{D}$

Each object in the category yields defect weights satisfying the commutation relations. Akin to braiding, but can define without going off of surface (objects are in the Drinfeld centre).

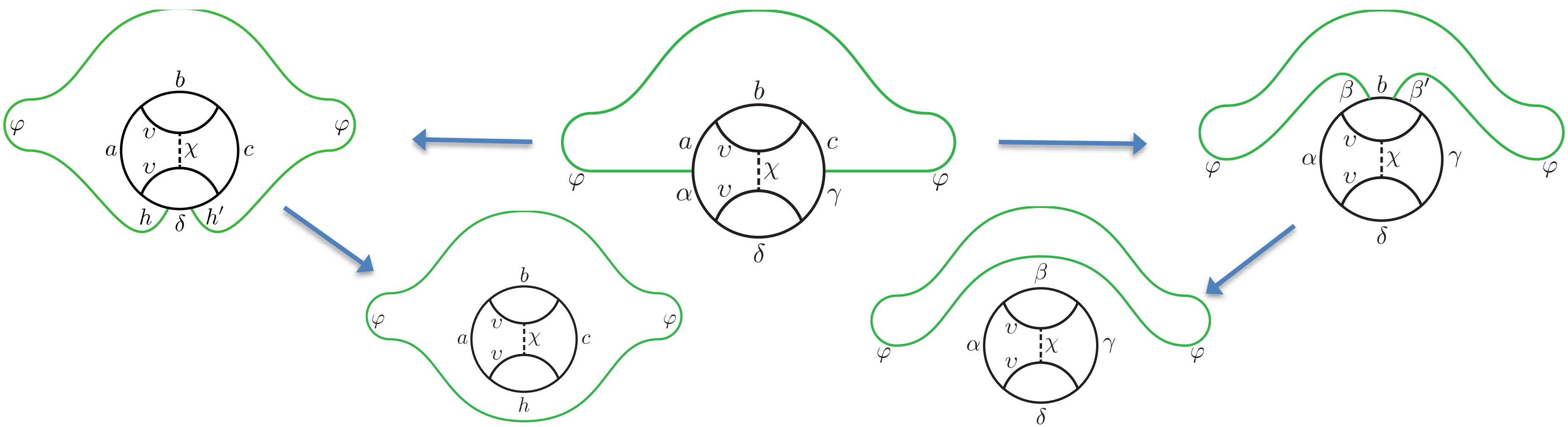
Label of type of defect




$$= \begin{bmatrix} \phi & a' & a \\ \rho & b & b' \end{bmatrix}$$

Category generalization of 6j symbols, i.e. $F$ symbols

Proof of defect commutation relations is essentially:

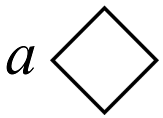



# Do not need to impose integrability.

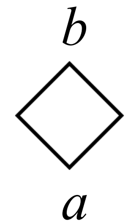
However, many (almost all?) critical integrable models can be built using categories.

# Two types of topological defects in Ising

For Ising, fluctuating degrees of freedom on only half the sites:

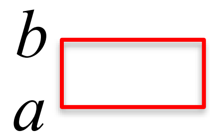



$= e^{K_x \delta_{ab}}$




$= e^{K_y \delta_{ab}}$

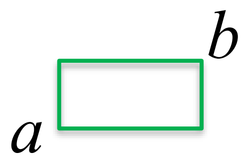
spin-flip defect:




$=$


$= 1 - \delta_{ab}$

duality defect:

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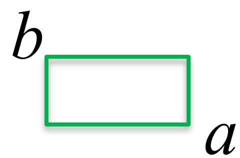
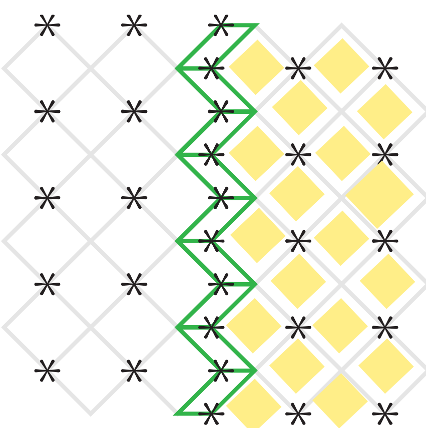
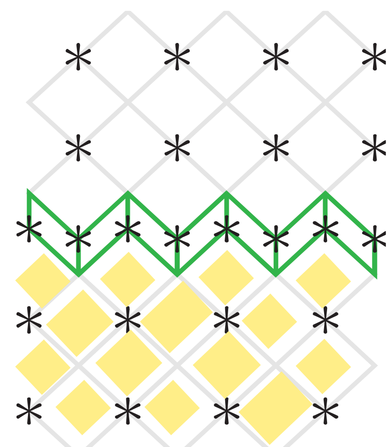


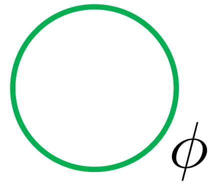

$= \frac{1}{2^{1/4}} (-1)^{\frac{1}{4}(a-1)(b-1)}$



Couplings on one side of duality defect are dual values of those on the other!

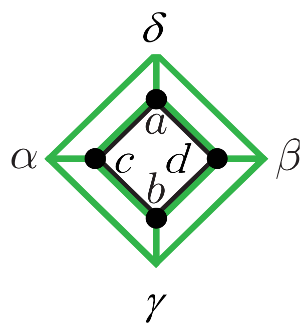


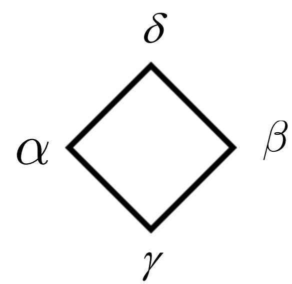
# Identities between partition functions with different defect configuations

In a category, 

$= d_\phi$

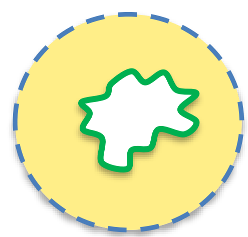
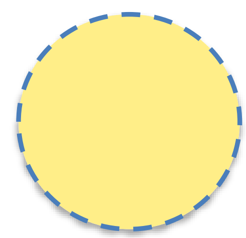
By either the abstract manipulations or using the explicit defect weights, find

$\sum_{a,b,c,d}$ 

$= d_\phi$ 


Use commutation relations to move defect around. Thus for any path, partition functions on a disc are related as

$$\mathcal{Z} = d_\phi Z$$


$= d_\phi$


# Micro to macro

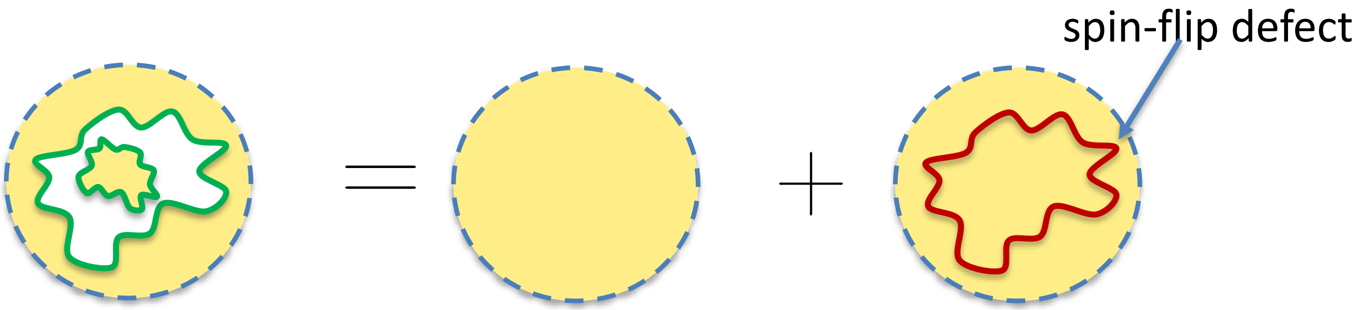
Micro to macro: defects fuse in the same way as the corresponding objects

$$\alpha \otimes \beta = \oplus_\gamma N_{\alpha\beta}^{\gamma}\ \gamma \quad \Longrightarrow \quad \mathcal{D}_\alpha \mathcal{D}_\beta = \sum_\gamma N_{\alpha\beta}^{\gamma} \mathcal{D}_\gamma$$

$$\sigma \otimes \sigma = 1 + \psi$$

In Ising category and CFT:

$$\mathcal{D}_\sigma \mathcal{D}_\sigma = 1 + \mathcal{D}_\psi$$

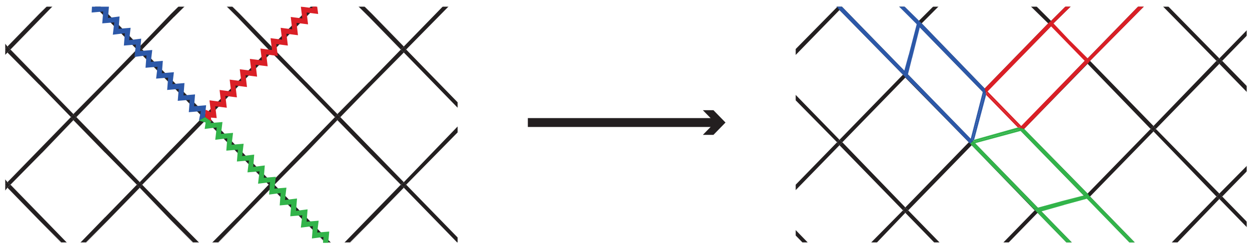



Kramers-Wannier duality of ising is not invertible:

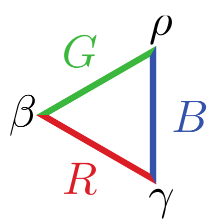
$$(\mathcal{D}_\psi)^2 = 1 \quad \Rightarrow \quad (\mathcal{D}_\sigma)^4 = 2(\mathcal{D}_\sigma)^2$$

# Branching and fusing

Another huge payoff is that the category setup makes it straightforward to define junctions of these topological defects

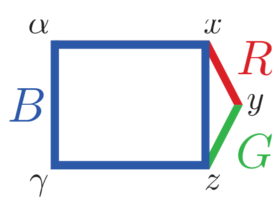


Defining

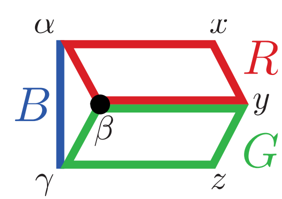



$$= (d_R d_G d_B)^{\frac{1}{4}} \begin{bmatrix} R & G & B \\ \rho & \gamma & \beta \end{bmatrix}$$

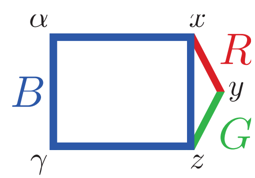
gives the junction commutation relation



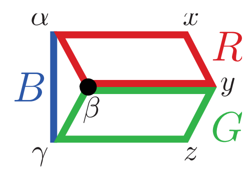

$$= \sum_{\beta}$$

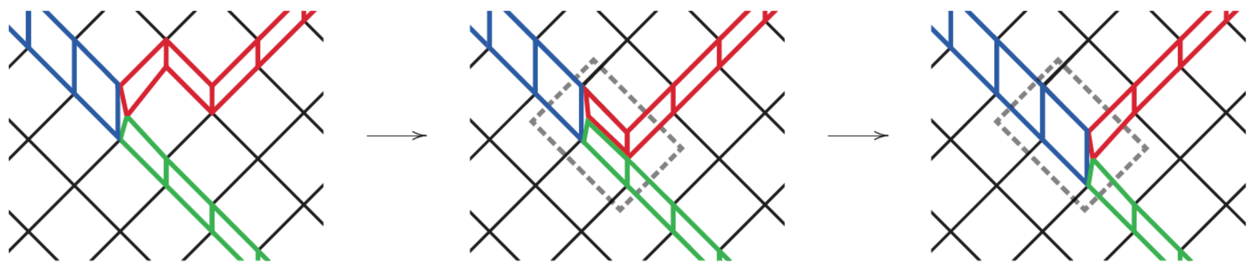


# Lattice to continuum: Defects themselves satisfy F moves

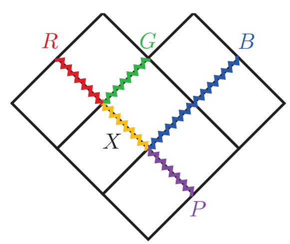



$$= \sum_{\beta}$$

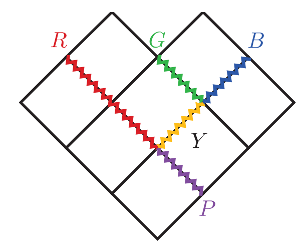


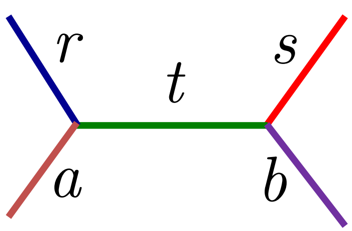
A little more work then gives



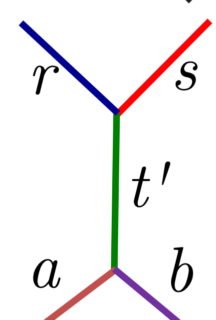

$$= \sum_{Y} F\begin{bmatrix} G & B \\ R & P \end{bmatrix}_{XY}$$




Ignore lattice!

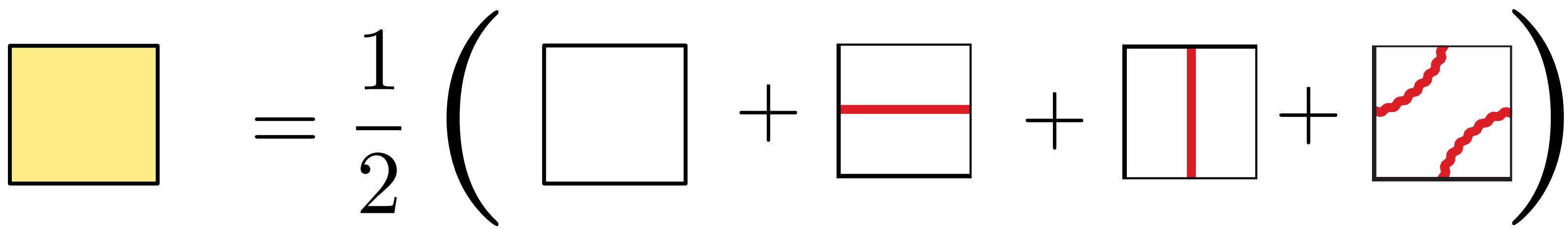


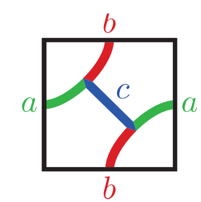

$$= \sum_{t' \in \mathcal{C}} F_{tt'} \begin{bmatrix} r & s \\ a & b \end{bmatrix}$$

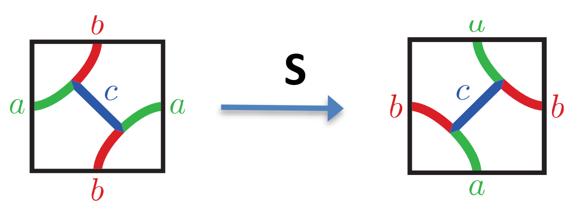


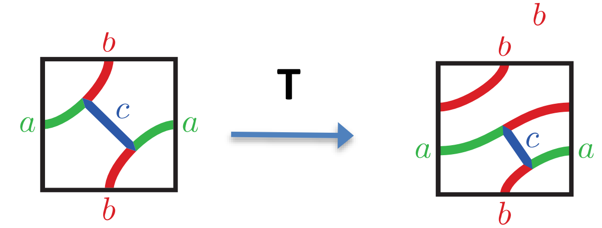
# Duality and modular transformations on the torus

Use these F-moves to give an easy graphical proof of the Ising relation:




A general basis for the toroidal partition functions is $Z^{b}_{ac} \equiv$ 

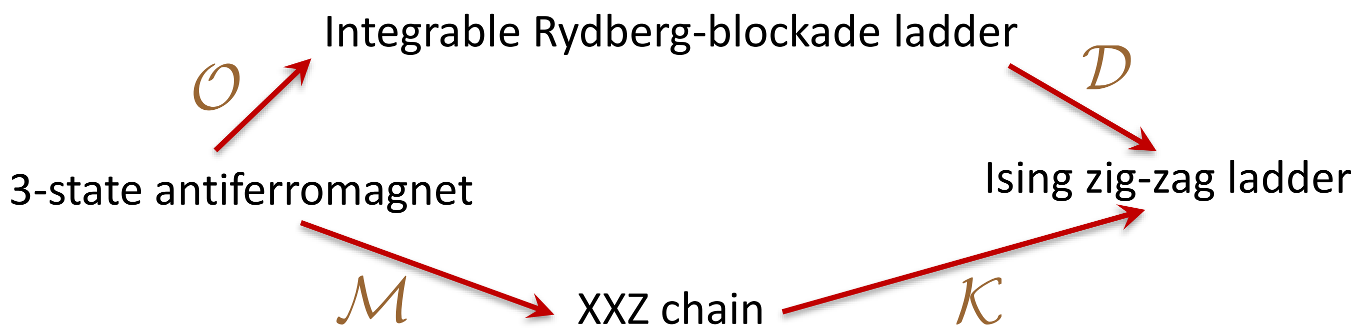




Can use F-moves find modular transformations exactly as a linear transformation:

$$\mathbf{S}(Z^{b}_{ac}) = \sum_{\gamma} \left[F^{ba}_{ab}\right]_{c\gamma} Z^{a}_{b\gamma} \qquad \mathbf{T}(Z^{b}_{ac}) = \sum_{\gamma} \left[F^{ab}_{ba}\right]_{c\gamma} Z^{b}_{\gamma a}$$

# Non-invertible ``self-duality'' of the Rydberg ladder




The map $\mathcal{D}$ is associated with the spin-½ object of $SU(2)_4$ . Since

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 0 \oplus 1 \qquad \Longrightarrow \qquad \mathcal{D}^\dagger \mathcal{D} = 1 + \widetilde{\mathcal{D}}$$

where $\widetilde{\mathcal{D}}$ is associated with the spin-$1$ object. By construction, $\widetilde{\mathcal{D}} H_{\rm IRL} = H_{\rm IRL} \widetilde{\mathcal{D}}$

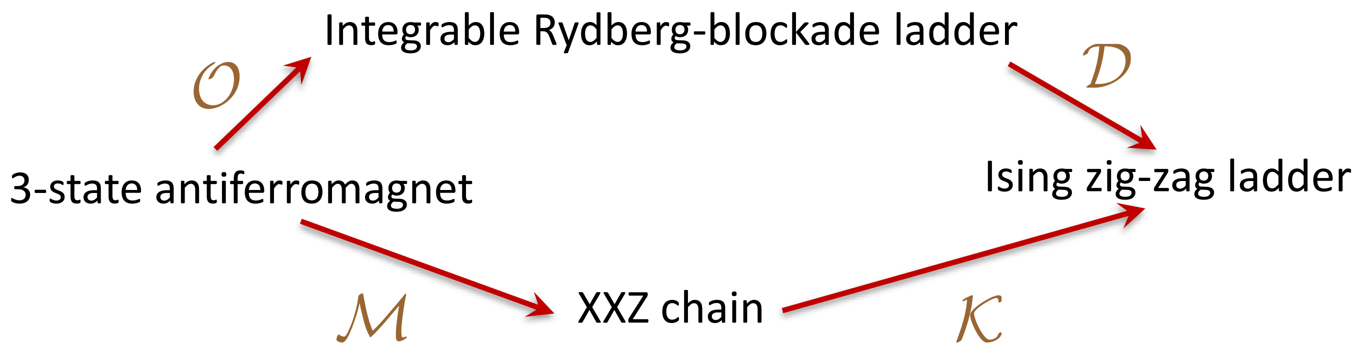
This symmetry generator is not invertible because

$$1 \otimes 1 = 0 \oplus 1 \oplus 2 \qquad \Longrightarrow \qquad \widetilde{\mathcal{D}}^2 = 1 + \mathcal{F} + \widetilde{\mathcal{D}}$$

Because it maps a Hilbert space to itself, we call this symmetry a self-duality.

Turns out $\mathcal{D} = \mathcal{O} R \mathcal{O}^\dagger$, where $R$ is $\mathbb{Z}_3$ transformation of 3-state antiferromagnet.

# Non-invertible symmetries of the Rydberg ladder




Another non-invertible symmetry of the Rydberg-blockade ladder is a remnant of the $U(1)$ symmetry of the XXZ chain. However, the map $\mathcal{K}$ annihilates the generator $Q$.

Instead, can map $Q^2$ via $\widetilde{Q} \equiv \mathcal{O}\mathcal{M}^\dagger Q^2 \mathcal{M}\mathcal{O}^\dagger$. Result commutes with $H_{\mathrm{IRL}}$

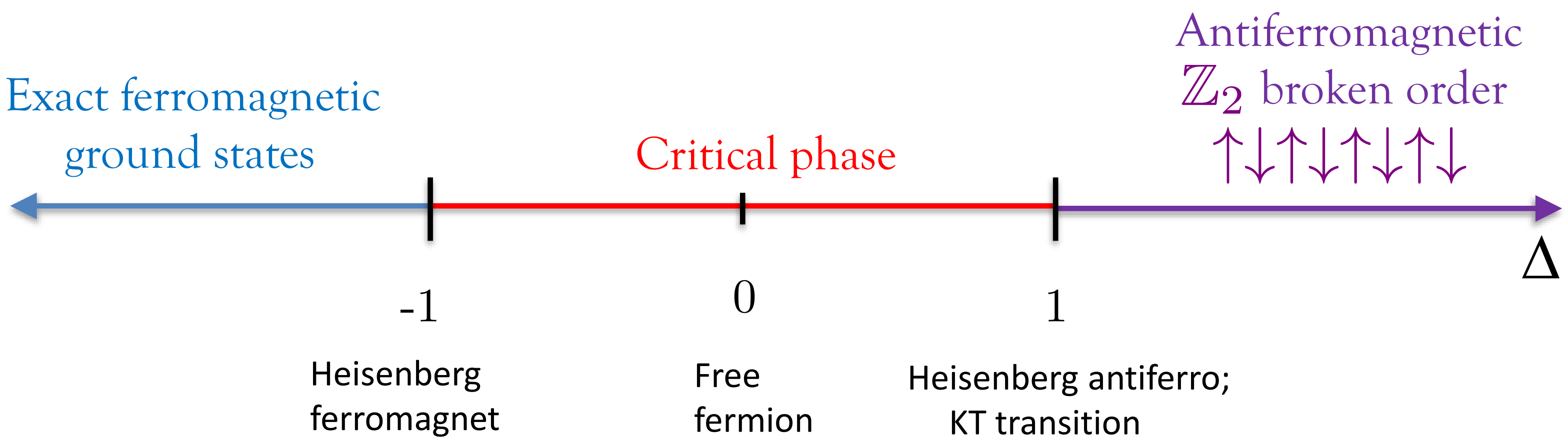
It acts non-trivially on only half of the Hilbert space. Explicitly,,

$$\mathcal{Q} = \tfrac{1}{2}\left(1+\mathcal{F}\right)\sum_{j=1}^{L}\sum_{k=0}^{L-1}\prod_{l=0}^{k}(-1)^{1-n^-_{j+l}}$$

This symmetry survives perturbation!

# 4. The physics of the integrable line

## Phase diagram of XXZ



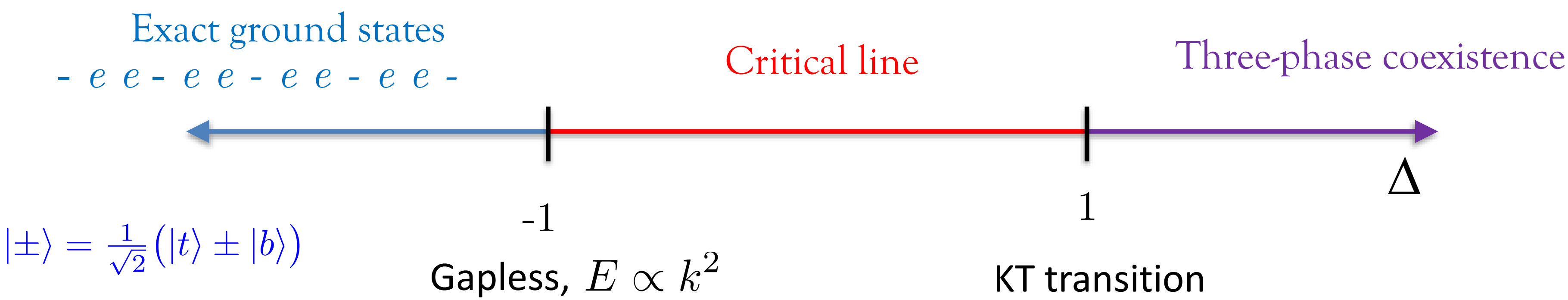

Critical ground state survives all mappings: all models are critical for $-1 \leq \Delta \leq 1$

The effective field theory of XXZ is a free massless boson of radius $R$, where

$$\Delta = -\cos \frac{\pi}{2R^2} \qquad R \geq \frac{1}{\sqrt{2}}$$

``Equivalent'' models are all $c = 1$ CFTs, but not the same ones!

# Phase diagram of integrable Rydberg-blockade ladder




We thus have located an exact critical line in the Rydberg-blockade ladder!

Its effective CFT is a $c = 1$ orbifold with $\Delta = -\cos\frac{9\pi}{2R^2}$

As $\Delta \to \infty$, three ground states are

$$e\ e\ e\ e\ e\ e\ e\ e\ e\ e\ e\ e$$
$$e + e + e + e + e + e +$$
$$+ e + e + e + e + e + e$$

No conventional symmetry relates them, but self-duality does.

Three ground states for all $\Delta > 1$ !

# Off the integrable line

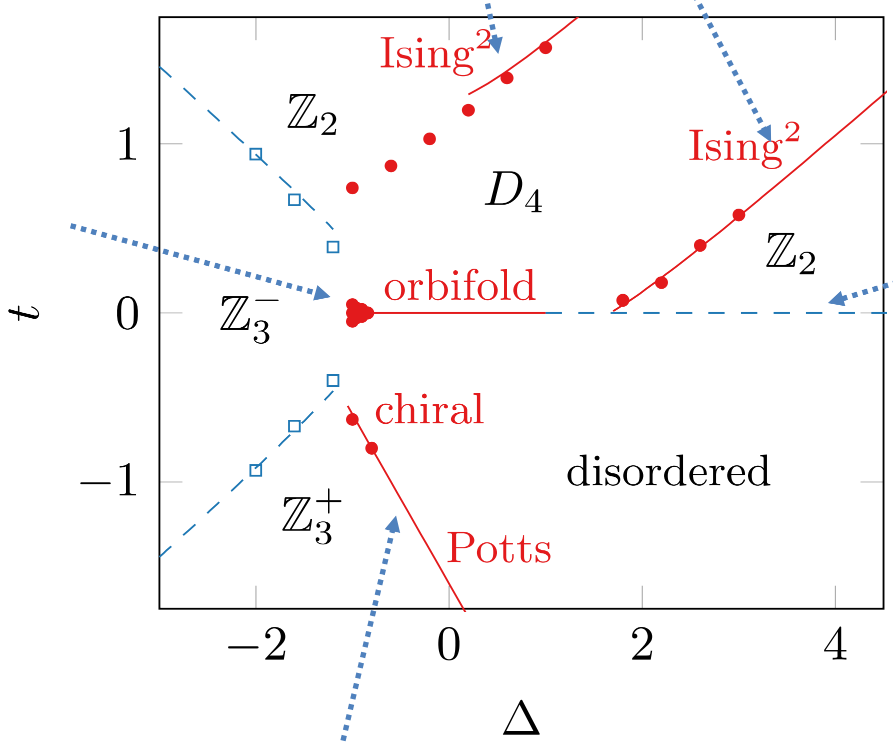
$$H = \sum_{j=1}^{L} \left( (1-w)\left(p_j + p_j^\dagger\right) + (1+2w)s_{j-1}s_{j+1} + (\Delta - 2t)n_j^- + (\Delta + t)\left(n_{j-1}^e - n_{j+1}^e\right)^2 \right)$$

Ising^2 transition from $t+\Delta$ large limit

First-order transitions: blue/dashed
second-order: red/solid

Lines from perturbations or exact; dots from ED

Small critical bubble!




First-order transition at phase coexistence plane, including $t=w=0$ self-dual line.

Find both chiral and non-chiral Potts transitions

# Conclusions

- Models defined from fusion categories build in exact topological defects
- Provides concrete realisation of some non-invertible symmetries and ``dualities''
- Exact results and phase diagram for the Rydberg-blockade ladder
- Make contact with experiment!?!
- This algebraic approach to XXZ useful for other things?