Operator Preconditioning (OPC)

Ralf Hiptmair

Seminar for Applied Mathematics, ETH Zürich

One-World Numerical Analysis (OWNA) Seminar Series
October 4, 2021 on ZOOM

> **Alle schwärmen von OPC: Aber welche Wirkung hat es?**
>
> OPC soll gesund, jung und schön machen – fast zu schön, um wahr zu sein, oder? Wir schauen uns den Stoff, der aus den Trauben

(www.womenshealth.de)

Preconditioning

Preconditioning

Finite element method (FEM)

Preconditioning

Finite element method (FEM)
Boundary element method (BEM)

Preconditioning

Finite element method (FEM)
Boundary element method (BEM)

►

Large sparse/compressed
linear systems of equations
$\mathbf{Ax} = \mathbf{b}$

Preconditioning

Finite element method (FEM)
Boundary element method (BEM)

► Large sparse/compressed linear systems of equations

$$\mathbf{Ax} = \mathbf{b}$$

Preconditioning

Finite element method (FEM)
Boundary element method (BEM)

► Large sparse/compressed linear systems of equations

$$\mathbf{Ax} = \mathbf{b}$$

Preconditioning

Finite element method (FEM)

Boundary element method (BEM)

► Large sparse/compressed linear systems of equations $\mathbf{Ax} = \mathbf{b}$

Iterative solution by preconditioned Krylov-subspace methods

Preconditioning

Finite element method (FEM)

Boundary element method (BEM)

► Large sparse/compressed linear systems of equations

$\mathbf{A}\mathbf{x} = \mathbf{b}$

Iterative solution by preconditioned Krylov-subspace methods

Preconditioner for $\mathbf{A} \in \mathbb{R}^{n,n}$: Linear operator $\mathbf{C} : \mathbb{R}^n \to \mathbb{R}^n$,

Preconditioning

Finite element method (FEM)

Boundary element method (BEM)

► Large sparse/compressed linear systems of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$

Iterative solution by preconditioned Krylov-subspace methods

Preconditioner for $\mathbf{A} \in \mathbb{R}^{n,n}$: Linear operator $\mathbf{C} : \mathbb{R}^n \to \mathbb{R}^n$,

$\mathbf{x} \mapsto \mathbf{C}(\mathbf{x})$ "cheap"

Preconditioning

Finite element method (FEM)

Boundary element method (BEM)

► Large sparse/compressed linear systems of equations $\mathbf{Ax} = \mathbf{b}$

Iterative solution by preconditioned Krylov-subspace methods

Preconditioner for $\mathbf{A} \in \mathbb{R}^{n,n}$: Linear operator $\mathbf{C} : \mathbb{R}^n \to \mathbb{R}^n$,

$\mathbf{x} \mapsto \mathbf{C}(\mathbf{x})$ "cheap" & "$\kappa(\mathbf{CA})$ small"

Preconditioning

Finite element method (FEM)

Boundary element method (BEM)

► Large sparse/compressed linear systems of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$

Iterative solution by preconditioned Krylov-subspace methods

Preconditioner for $\mathbf{A} \in \mathbb{R}^{n,n}$: Linear operator $\mathbf{C} : \mathbb{R}^n \to \mathbb{R}^n$,

$\mathbf{x} \mapsto \mathbf{C}(\mathbf{x})$ "cheap" **&** "$\kappa(\mathbf{CA})$ small"

Best-known: $\mathbf{A}$ s.p.d. and conjugate gradient (CG) method

$$\left\|\mathbf{e}^{(k)}\right\| \leq \frac{2\gamma^k}{1+\gamma^{2k}} \left\|\mathbf{e}^{(0)}\right\| \;, \quad \gamma := \frac{\sqrt{\kappa(\mathbf{CA})}-1}{\sqrt{\kappa(\mathbf{CA})}+1} \;.$$

$\kappa(\mathbf{CA}) \hat{=}$ spectral condition number

Preconditioning

Finite element method (FEM)

Boundary element method (BEM)

Large sparse/compressed linear systems of equations $\mathbf{Ax} = \mathbf{b}$

Iterative solution by preconditioned Krylov-subspace methods

Preconditioner for $\mathbf{A} \in \mathbb{R}^{n,n}$: Linear operator $\mathbf{C} : \mathbb{R}^n \to \mathbb{R}^n$,

$\mathbf{x} \mapsto \mathbf{C}(\mathbf{x})$ "cheap" & "$\kappa(\mathbf{CA})$ small"

Krylov methods:

- CG
- CGS, CGN
- MINRES, GMRES, ORTHODIR
- BiCG, BICGStab

Galerkin Discretization

Galerkin Discretization

Linear variational problem: $(\ell \in V')$

$$u \in V: \quad \mathrm{a}(u, v) = \ell(v) \quad \forall v \in V\ . \tag{VP}$$

Galerkin Discretization

Linear variational problem: $(\ell \in V')$

$$u \in V: \quad \mathsf{a}(u, v) = \ell(v) \quad \forall v \in V\ . \tag{VP}$$

$\mathsf{a} : V \times V \to \mathbb{C} \hat{=}$ bounded (sesqui-)linear form

Galerkin Discretization

Linear variational problem: $(\ell \in V')$

$$u \in V: \quad \mathrm{a}(u, v) = \ell(v) \quad \forall v \in V\ . \tag{VP}$$

$\mathrm{a} : V \times V \to \mathbb{C} \triangleq$ bounded (sesqui-)linear form

$$\mathrm{a}(\cdot,\cdot) \quad \leftrightarrow \quad \text{linear operator} \quad A : V \to V'\ (A \in L(V, V')),\ \|A\| = \sup_{u,v} \frac{|\mathrm{a}(u, v)|}{\|u\|_V \|v\|_V}\ .$$

Galerkin Discretization

Linear variational problem: $(\ell \in V')$

$$u \in V: \quad \mathrm{a}(u, v) = \ell(v) \quad \forall v \in V\ . \tag{VP}$$

$\mathrm{a} : V \times V \to \mathbb{C} \triangleq$ bounded (sesqui-)linear form

$$\mathrm{a}(\cdot,\cdot) \quad \leftrightarrow \quad \text{linear operator} \quad A : V \to V' \ (A \in L(V, V')),\ \|A\| = \sup_{u,v} \frac{|\mathrm{a}(u,v)|}{\|u\|_V \|v\|_V}\ .$$

$$\sup_{v \in V} \frac{|\mathrm{a}(u,v)|}{\|v\|_V} \geq \gamma \|u\|_V \ \forall u \in V$$

Galerkin Discretization

Linear variational problem: $(\ell \in V')$

$$u \in V\colon \quad \mathrm{a}(u, v) = \ell(v) \quad \forall v \in V\ . \tag{VP}$$

$\mathrm{a} : V \times V \to \mathbb{C} \triangleq$ bounded (sesqui-)linear form

$\mathrm{a}(\cdot,\cdot) \quad \leftrightarrow \quad$ linear operator $\quad A : V \to V'$ $(A \in L(V, V'))$, $\|A\| = \sup\limits_{u,v} \dfrac{|\mathrm{a}(u,v)|}{\|u\|_V \|v\|_V}$.

$$\sup_{v \in V} \frac{|\mathrm{a}(u,v)|}{\|v\|_V} \geq \gamma \|u\|_V \ \forall u \in V \qquad \Leftrightarrow \qquad \|A^{-1}\| \leq \gamma^{-1}\ .$$

Galerkin Discretization

Linear variational problem: $(\ell \in V')$

$$u \in V: \quad \mathrm{a}(u, v) = \ell(v) \quad \forall v \in V \,. \tag{VP}$$

$\mathrm{a} : V \times V \to \mathbb{C} \mathrel{\hat{=}}$ bounded (sesqui-)linear form

$\mathrm{a}(\cdot,\cdot) \quad \leftrightarrow \quad$ linear operator $\quad A : V \to V' \; (A \in L(V, V')), \; \|A\| = \sup\limits_{u,v} \dfrac{|\mathrm{a}(u,v)|}{\|u\|_V \|v\|_V} \,.$

$$\sup_{v \in V} \frac{|\mathrm{a}(u,v)|}{\|v\|_V} \geq \gamma \|u\|_V \; \forall u \in V \quad \Leftrightarrow \quad \|A^{-1}\| \leq \gamma^{-1} \,.$$

Replace $\quad V \quad \longrightarrow \quad V_h := \mathrm{Span}\,\{b_1, \ldots, b_N\},\; N := \dim V_h < \infty,$

(VP) $\quad \longrightarrow \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad (\mathbf{A})_{ij} := \mathrm{a}(b_j, b_i), \quad \mathbf{A} \in \mathbb{C}^{N,N}$

Galerkin Discretization

Linear variational problem: $(\ell \in V')$

$$u \in V: \quad \mathrm{a}(u,v) = \ell(v) \quad \forall v \in V \,. \tag{VP}$$

$\mathrm{a} : V \times V \to \mathbb{C} \mathrel{\hat=}$ bounded (sesqui-)linear form

$$\mathrm{a}(\cdot,\cdot) \quad \leftrightarrow \quad \text{linear operator} \quad A : V \to V' \ (A \in L(V,V')), \ \|A\| = \sup_{u,v} \frac{|\mathrm{a}(u,v)|}{\|u\|_V \|v\|_V} \,.$$

$$\sup_{v \in V} \frac{|\mathrm{a}(u,v)|}{\|v\|_V} \geq \gamma \|u\|_V \ \forall u \in V \qquad \Leftrightarrow \qquad \|A^{-1}\| \leq \gamma^{-1} \,.$$

Replace $\quad V \longrightarrow V_h := \mathrm{Span}\,\{b_1,\dots,b_N\},\ N := \dim V_h < \infty,$

$\quad$ (VP) $\longrightarrow \mathbf{A}\mathbf{x} = \mathbf{b}, \quad (\mathbf{A})_{ij} := \mathrm{a}(b_j, b_i), \quad \mathbf{A} \in \mathbb{C}^{N,N}$

OPC: Main Theorem

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.

Bi-/Sesqui-linear forms $\mathsf{a} : V \times V \to \mathbb{K}$, $\mathsf{b} : W \times W \to \mathbb{K}$.

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable

❷ $\dim V_h = \dim W_h =: N$

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable

❷ $\dim V_h = \dim W_h =: N$

h-uniform discrete inf-sup conditions, e.g.,

$$\exists c_A > 0: \quad \sup_{v_h \in V_h} \frac{|a(u_h, v_h)|}{\|v_h\|_V} \geq c_A \|u_h\|_V \quad \forall u_h \in V_h, \ \forall h .$$

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable

❷ $\dim V_h = \dim W_h =: N$

- Stable discrete duality pairing : sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \ .$$

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

  ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable

  ❷ $\dim V_h = \dim W_h =: N$

- Stable discrete duality pairing : sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h .$$

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

  ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable

  ❷ $\dim V_h = \dim W_h =: N$

- Stable discrete duality pairing : sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \; .$$

Spectral condition number $\kappa(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}\mathbf{A}) \leq \|A_h\| \, \|A_h^{-1}\| \; \|B_h\| \, \|B_h^{-1}\| \frac{\|d\|^2}{c_D^2}$

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

  ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable

  ❷ $\dim V_h = \dim W_h =: N$

- Stable discrete duality pairing : sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h\ .$$

Spectral condition number $\kappa(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}\mathbf{A}) \leq \|A_h\| \|A_h^{-1}\| \|B_h\| \|B_h^{-1}\| \dfrac{\|d\|^2}{c_D^2}$

any Galerkin matrices

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

❶ $A_h := A_{|V_h} : V_h \mapsto V'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable

❷ $\dim V_h = \dim W_h =: N$

- Stable discrete duality pairing : sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \,.$$

Spectral condition number $\kappa(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}\mathbf{A}) \leq \|A_h\| \, \|A_h^{-1}\| \; \|B_h\| \, \|B_h^{-1}\| \dfrac{\|d\|^2}{c_D^2}$

▶ preconditioner $\mathbf{C} = \mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}$

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

❶ $A_h := A_{|V_h} : V_h \mapsto V'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable

❷ $\dim V_h = \dim W_h =: N$

- Stable discrete duality pairing : sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \;.$$

Spectral condition number $\kappa(D_h^{-1} B_h D_h^{-*} A_h) \leq \|A_h\| \, \|A_h^{-1}\| \; \|B_h\| \, \|B_h^{-1}\| \frac{\|d\|^2}{c_D^2}$

▶ preconditioner $\mathbf{C} = \mathbf{D}^{-1} \mathbf{B} \mathbf{D}^{-H}$

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable

❷ $\dim V_h = \dim W_h =: N$

- Stable discrete duality pairing : sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \,.$$

$$\kappa(\mathbf{A}^H \mathbf{C}^H \mathbf{C} \mathbf{A}) \leq \left(\|A_h\| \, \|A_h^{-1}\| \; \|B_h\| \, \|B_h^{-1}\| \frac{\|d\|^2}{c_D^2} \right)^2$$

▶ preconditioner $\mathbf{C} = \mathbf{D}^{-1} \mathbf{B} \mathbf{D}^{-H}$

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.

O. STEINBACH AND W. WENDLAND, *The construction of some efficient preconditioners in the boundary element method*, Adv. Comput. Math, 9 (1998), pp. 191–216.

S. CHRISTIANSEN AND J.-C. NÉDÉLEC, *Des préconditionneurs pour la résolution numérique des équations intégrales de frontiére de l'acoustique*, C.R. Acad. Sci. Paris, Ser. I Math, 330 (2000), pp. 617–622.

R. HIPTMAIR, *Operator preconditioning*, Computers and Mathematics with Applications, 52 (2006), pp. 699–706.

P. ESCAPIL-INCHAUSPÉ AND C. JEREZ-HANCKES, *Bi-parametric operator preconditioning*, arXiv:2011.05028 [math.NA], 2020.

Spectral condition number ... c_D^2

preconditioner $\mathbf{C} = \mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}$

OPC: Main Theorem

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable

❷ $\dim V_h = \dim W_h =: N$

- Stable discrete duality pairing : sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \,.$$

Spectral condition number $\kappa(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}\mathbf{A}) \leq \|A_h\| \, \|A_h^{-1}\| \; \|B_h\| \, \|B_h^{-1}\| \frac{\|d\|^2}{c_D^2}$

▶ preconditioner $\mathbf{C} = \mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}$

The Plan

1. (Simple) Abstract Framework
2. Finite Element Applications: Equivalent Operator Preconditioning
3. Boundary Element Applications: Calderón Preconditioning
4. Calderón Preconditioning for Screen Problems

What Next ?

1. (Simple) Abstract Framework
2. Finite Element Applications: Equivalent Operator Preconditioning
3. Boundary Element Applications: Calderón Preconditioning
4. Calderón Preconditioning for Screen Problems

Abstract Framework

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that
  ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable
  ❷ $\dim V_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \, .$$

Abstract Framework

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that
  ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable
  ❷ $\dim V_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0\colon \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \,.$$

▶ $V \triangleq$ *Hilbert space* H, inner product $(\cdot,\cdot)_H$, norm $\|\cdot\|_H$

Abstract Framework

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that
  ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable
  ❷ $\dim V_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \, .$$

- $V \triangleq$ *Hilbert space* H, inner product $(\cdot, \cdot)_H$, norm $\|\cdot\|_H$
- $V_h \triangleq$ Galerkin trial/test space $H_h \subset H$, $N := \dim H_h < \infty$

Abstract Framework

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that
  ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable
  ❷ $\dim V_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \, .$$

- $V \triangleq$ *Hilbert space* H, inner product $(\cdot,\cdot)_H$, norm $\|\cdot\|_H$
- $V_h \triangleq$ Galerkin trial/test space $H_h \subset H$, $N := \dim H_h < \infty$
- $A_h : H_h \mapsto H_h'$ induced by $a \in L(H \times H, \mathbb{C})$ with

$$\exists c_A > 0: \quad \sup_{v_h \in H_h} \frac{|a(u_h, v_h)|}{\|v_h\|_H} \geq c_A \|u_h\|_H \quad \forall u_h \in H_h, \forall h \, .$$

Abstract Framework

- V, W refl. Banach spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that
  ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable
  ❷ $\dim V_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \,.$$

- $V \mathrel{\hat{=}}$ *Hilbert space* H, inner product $(\cdot,\cdot)_H$, norm $\|\cdot\|_H$
- $V_h \mathrel{\hat{=}}$ Galerkin trial/test space $H_h \subset H$, $N := \dim H_h < \infty$
- $A_h : H_h \mapsto H_h'$ induced by $a \in L(H \times H, \mathbb{C})$ with

$$\exists c_A > 0: \quad \sup_{v_h \in H_h} \frac{|a(u_h, v_h)|}{\|v_h\|_H} \geq c_A \|u_h\|_H \quad \forall u_h \in H_h, \forall h \,.$$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that
  ❶ $A_h := A_{|H_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable
  ❷ $\dim H_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \,.$$

- $V \triangleq$ *Hilbert space* H, inner product $(\cdot, \cdot)_H$, norm $\|\cdot\|_H$
- $V_h \triangleq$ Galerkin trial/test space $H_h \subset H$, $N := \dim H_h < \infty$
- $A_h : H_h \mapsto H'_h$ induced by $a \in L(H \times H, \mathbb{C})$ with

$$\exists c_A > 0: \quad \sup_{v_h \in H_h} \frac{|a(u_h, v_h)|}{\|v_h\|_H} \geq c_A \|u_h\|_H \quad \forall u_h \in H_h, \forall h \,.$$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that

❶ $A_h := A_{|H_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable

❷ $\dim H_h = \dim W_h =: N$

- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \; .$$

- $W =$, $W_h =$,
- B
- $d(\cdot, \cdot)$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that
  ❶ $A_h := A_{|H_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable
  ❷ $\dim H_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \, .$$

- $W = H, \quad W_h = H_h,$
- B
- $d(\cdot, \cdot)$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that

  ❶ $A_h := A_{|H_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable

  ❷ $\dim H_h = \dim W_h =: N$

- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \, .$$

- ► $W = H$, $\quad W_h = H_h$,
- ► $B \in L(H, H') \quad \leftrightarrow \quad (\cdot, \cdot)_H$
- ► $d(\cdot, \cdot)$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that
  1. $A_h := A_{|H_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable
  2. $\dim H_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \, .$$

- $W = H, \quad W_h = H_h,$
- $B \in L(H, H') \quad \leftrightarrow \quad (\cdot, \cdot)_H$
- $d(\cdot, \cdot) := (\cdot, \cdot)_H \quad [B = D!]$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that

❶ $A_h := A_{|H_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable

❷ $\dim H_h = \dim W_h =: N$

- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h .$$

- $W = H, \quad W_h = H_h$,
- $B \in L(H, H') \quad \leftrightarrow \quad (\cdot,\cdot)_H$
- $d(\cdot,\cdot) := (\cdot,\cdot)_H \quad [B = D!]$

▶ **B** $=$ **D** $=$ $\mathbf{D}^H$ (Gram matrix)

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that

❶ $A_h := A_{|H_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable

❷ $\dim H_h = \dim W_h =: N$

- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \, .$$

- $W = H, \quad W_h = H_h,$
- $B \in L(H, H') \quad \leftrightarrow \quad (\cdot, \cdot)_H$
- $d(\cdot, \cdot) := (\cdot, \cdot)_H \quad [B = D!]$

▶

- $\mathbf{B} = \mathbf{D} = \mathbf{D}^H$ (Gram matrix)
- $c_D = \|d\| = \|B_h\| = \left\|B_h^{-1}\right\| = 1$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that
  ❶ $A_h := A_{|H_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable
  ❷ $\dim H_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \, .$$

- $W = H, \quad W_h = H_h,$
- $B \in L(H, H') \quad \leftrightarrow \quad (\cdot, \cdot)_H$
- $d(\cdot, \cdot) := (\cdot, \cdot)_H \quad [B = D!]$

►

- $\mathbf{B} = \mathbf{D} = \mathbf{D}^H$ (Gram matrix)
- $c_D = \|d\| = \|B_h\| = \left\|B_h^{-1}\right\| = 1$

$$\kappa(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}\mathbf{A})$$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that
  ❶ $A_h := A_{|H_h} : H_h \mapsto H_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable
  ❷ $\dim H_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h .$$

- $W = H, \quad W_h = H_h,$
- $B \in L(H, H') \quad \leftrightarrow \quad (\cdot,\cdot)_H$
- $d(\cdot,\cdot) := (\cdot,\cdot)_H \quad [B = D!]$

▶

- $\mathbf{B} = \mathbf{D} = \mathbf{D}^H$ (Gram matrix)
- $c_D = \|d\| = \|B_h\| = \left\| B_h^{-1} \right\| = 1$

$$\kappa(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}\mathbf{A}) = \kappa(\mathbf{B}^{-1}\mathbf{A})$$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that
  ❶ $A_h := A_{|H_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable
  ❷ $\dim H_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h .$$

- $W = H, \quad W_h = H_h,$
- $B \in L(H, H') \quad \leftrightarrow \quad (\cdot, \cdot)_H$
- $d(\cdot, \cdot) := (\cdot, \cdot)_H \quad [B = D!]$

▶

- $\mathbf{B} = \mathbf{D} = \mathbf{D}^H$ (Gram matrix)
- $c_D = \|d\| = \|B_h\| = \left\|B_h^{-1}\right\| = 1$

$$\kappa(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}\mathbf{A}) = \kappa(\mathbf{B}^{-1}\mathbf{A}) \leq \|A_h\| \left\|A_h^{-1}\right\| = \frac{\|a\|}{c_A}$$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.

V. Faber, T. A. Manteuffel, and S. V. Parter, *On the theory of equivalent operators and application to the numerical solution of uniformly elliptic partial differential equations*, Adv. in Appl. Math., 11 (1990), pp. 109–163.

D. Arnold, R. Falk, and R. Winther, *Preconditioning in $H(\mathrm{div})$ and applications*, Math. Comp., 66 (1997), pp. 957–984.

O. Axelsson and J. Karátson, *Equivalent operator preconditioning for elliptic problems*, Numer. Algorithms, 50 (2009), pp. 297–380.

K.-A. Mardal and R. Winther, *Preconditioning discretizations of systems of partial differential equations*, Num. Lin. Alg. Appl., 18 (2011), pp. 1–40.

$$\kappa(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}\mathbf{A}) = \kappa(\mathbf{B}^{-1}\mathbf{A}) \leq \|A_h\| \left\|A_h^{-1}\right\| = \frac{\|a\|}{c_A}$$

Abstract Framework

- H, W *Hilbert spaces*, $A \in L(H, H')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $H_h \subset H$, $W_h \subset W$ such that
  - ❶ $A_h := A_{|H_h} : H_h \mapsto H_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable
  - ❷ $\dim H_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(H \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in H_h} \frac{|d(v_h, w_h)|}{\|v_h\|_H} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h\ .$$

- $W = H, \quad W_h = H_h,$
- $B \in L(H, H') \quad \leftrightarrow \quad (\cdot, \cdot)_H$
- $d(\cdot, \cdot) := (\cdot, \cdot)_H \quad [B = D!]$

►

- $\mathbf{B} = \mathbf{D} = \mathbf{D}^H$ (Gram matrix)
- $c_D = \|d\| = \|B_h\| = \left\|B_h^{-1}\right\| = 1$

$$\kappa(\mathbf{D}^{-1}\mathbf{B}\mathbf{D}^{-H}\mathbf{A}) = \kappa(\mathbf{B}^{-1}\mathbf{A}) \leq \|A_h\| \left\|A_h^{-1}\right\| = \frac{\|a\|}{c_A}$$

Saddle Point Problems

A joke: Do you know how a mathematician . . .

Saddle Point Problems

A joke: Do you know how a mathematician . . .

pot

stove

Saddle Point Problems

A joke: Do you know how a mathematician . . .

pot

stove

Saddle Point Problems

A joke: Do you know how a mathematician . . .

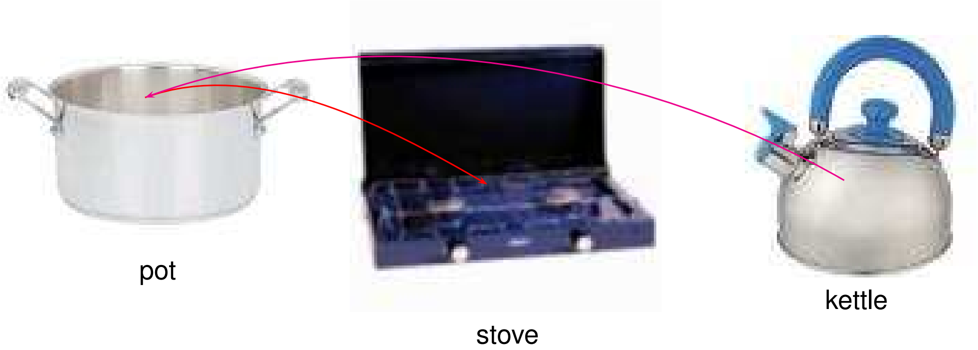
pot

stove

kettle

Saddle Point Problems

A joke: Do you know how a mathematician . . .



Saddle Point Problems

Stokes problem	Mixed magnetostatic problem
$H = (H^1_0(\Omega))^d \times L^2_0(\Omega)$	$H = \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H^1_0(\Omega)$
$\mathrm{a}\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{array}{l}(\boldsymbol{\nabla}\mathbf{u}, \boldsymbol{\nabla}\mathbf{v})_0 + \\ (\mathrm{div}\,\mathbf{u}, q)_0 + \\ (\mathrm{div}\,\mathbf{v}, p)_0\end{array}$	$\mathrm{a}\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{array}{l}(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + \\ (\mathbf{u}, \mathbf{grad}\,q)_0 + \\ (\mathbf{v}, \mathbf{grad}\,p)_0\end{array}$

Saddle Point Problems

Stokes problem	Mixed magnetostatic problem
$H = (H_0^1(\Omega))^d \times L_0^2(\Omega)$	$H = \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$
$\mathsf{a}\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix},\begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{array}{l}(\nabla\mathbf{u},\nabla\mathbf{v})_0 +\\ (\operatorname{div}\mathbf{u}, q)_0 +\\ (\operatorname{div}\mathbf{v}, p)_0\end{array}$	$\mathsf{a}\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix},\begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{array}{l}(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_0 +\\ (\mathbf{u},\mathbf{grad}\,q)_0 +\\ (\mathbf{v},\mathbf{grad}\,p)_0\end{array}$

Saddle Point Problems

Stokes problem

$$H = (H_0^1(\Omega))^d \times L_0^2(\Omega)$$

$$a\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{array}{l}(\boldsymbol{\nabla}\mathbf{u}, \boldsymbol{\nabla}\mathbf{v})_0 + \\ (\operatorname{div}\mathbf{u}, q)_0 + \\ (\operatorname{div}\mathbf{v}, p)_0\end{array}$$

Mixed magnetostatic problem

$$H = \boldsymbol{H}_0(\mathbf{curl}; \Omega) \times H_0^1(\Omega)$$

$$a\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{array}{l}(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + \\ (\mathbf{u}, \mathbf{grad}\,q)_0 + \\ (\mathbf{v}, \mathbf{grad}\,p)_0\end{array}$$

Assumption: Uniform LBB-condition for pairs of conforming FE spaces

Saddle Point Problems

Stokes problem	Mixed magnetostatic problem
$H = (H_0^1(\Omega))^d \times L_0^2(\Omega)$	$H = \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$
$\mathrm{a}\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix},\begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{matrix}(\nabla\mathbf{u},\nabla\mathbf{v})_0 + \\ (\operatorname{div}\mathbf{u},q)_0 + \\ (\operatorname{div}\mathbf{v},p)_0\end{matrix}$	$\mathrm{a}\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix},\begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{matrix}(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_0 + \\ (\mathbf{u},\mathbf{grad}\,q)_0 + \\ (\mathbf{v},\mathbf{grad}\,p)_0\end{matrix}$

Assumption: Uniform LBB-condition for pairs of conforming FE spaces

Preconditioners: Inverse Galerkin matrices for Riesz operators

$$\begin{pmatrix}-\boldsymbol{\Delta} & 0\\ 0 & Id\end{pmatrix} \qquad \begin{pmatrix}\mathbf{curl}\,\mathbf{curl}+Id & 0\\ 0 & -\Delta\end{pmatrix}$$

Saddle Point Problems

Stokes problem	Mixed magnetostatic problem
$H = (H_0^1(\Omega))^d \times L_0^2(\Omega)$	$H = \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$
$a\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{array}{l}(\nabla\mathbf{u},\nabla\mathbf{v})_0 + \\ (\operatorname{div}\mathbf{u}, q)_0 + \\ (\operatorname{div}\mathbf{v}, p)_0\end{array}$	$a\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{array}{l}(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_0 + \\ (\mathbf{u}, \mathbf{grad}\, q)_0 + \\ (\mathbf{v}, \mathbf{grad}\, p)_0\end{array}$

Assumption: Uniform LBB-condition for pairs of conforming FE spaces

Preconditioners: Inverse Galerkin matrices for Riesz operators

$$\begin{pmatrix} -\boldsymbol{\Delta} & 0 \\ 0 & Id \end{pmatrix} \qquad \begin{pmatrix} \mathbf{curl}\,\mathbf{curl} + Id & 0 \\ 0 & -\Delta \end{pmatrix}$$

Saddle Point Problems

Stokes problem

$$H = (H_0^1(\Omega))^d \times L_0^2(\Omega)$$

$$a\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{matrix}(\boldsymbol{\nabla}\mathbf{u}, \boldsymbol{\nabla}\mathbf{v})_0 + \\ (\operatorname{div}\mathbf{u}, q)_0 + \\ (\operatorname{div}\mathbf{v}, p)_0\end{matrix}$$

Mixed magnetostatic problem

$$H = \boldsymbol{H}_0(\mathbf{curl};\Omega) \times H_0^1(\Omega)$$

$$a\left(\begin{pmatrix}\mathbf{u}\\ p\end{pmatrix}, \begin{pmatrix}\mathbf{v}\\ q\end{pmatrix}\right) = \begin{matrix}(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0 + \\ (\mathbf{u}, \mathbf{grad}\,q)_0 + \\ (\mathbf{v}, \mathbf{grad}\,p)_0\end{matrix}$$

Assumption: Uniform LBB-condition for pairs of conforming FE spaces

Preconditioners: Inverse Galerkin matrices for Riesz operators

$$\begin{pmatrix}-\boldsymbol{\Delta} & 0\\ 0 & Id\end{pmatrix}$$

$$\begin{pmatrix}\mathbf{curl}\,\mathbf{curl} + Id & 0\\ 0 & -\Delta\end{pmatrix}$$

Complex Variational Problems

Complex Variational Problems

Eddy current problem in frequency domain ($\sigma > 0$):

$$H = \boldsymbol{H}_0(\mathbf{curl};\Omega) \quad , \quad c(\boldsymbol{u},\boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_0 + \imath\sigma\,(\boldsymbol{u},\boldsymbol{v})_0 \ .$$

Complex Variational Problems

Eddy current problem in frequency domain ($\sigma > 0$):

$$H = \boldsymbol{H}_0(\mathbf{curl}; \Omega) \quad , \quad c(\boldsymbol{u}, \boldsymbol{v}) = (\mathbf{curl}\, \boldsymbol{u}, \mathbf{curl}\, \boldsymbol{v})_0 + \imath\sigma\, (\boldsymbol{u}, \boldsymbol{v})_0 \ .$$

$$c(\boldsymbol{u}, \boldsymbol{v}) \leq \left(\|\mathbf{curl}\, \boldsymbol{u}\|_0^2 + \sigma\, \|\boldsymbol{u}\|_0^2\right)^{\frac{1}{2}} \left(\|\mathbf{curl}\, \boldsymbol{v}\|_0^2 + \sigma\, \|\boldsymbol{v}\|_0^2\right)^{\frac{1}{2}} \ ,$$

$$|c(\boldsymbol{u}, \boldsymbol{u})| \geq \frac{1}{\sqrt{2}}(\|\mathbf{curl}\, \boldsymbol{u}\|_0^2 + \sigma\, \|\boldsymbol{u}\|_0^2) \ .$$

Complex Variational Problems

Eddy current problem in frequency domain ($\sigma > 0$):

$$H = \boldsymbol{H}_0(\mathbf{curl};\Omega) \quad , \quad c(\boldsymbol{u},\boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_0 + \imath\sigma\,(\boldsymbol{u},\boldsymbol{v})_0 \ .$$

$$c(\boldsymbol{u},\boldsymbol{v}) \leq \left(\|\mathbf{curl}\,\boldsymbol{u}\|_0^2 + \sigma\,\|\boldsymbol{u}\|_0^2\right)^{\frac{1}{2}} \left(\|\mathbf{curl}\,\boldsymbol{v}\|_0^2 + \sigma\,\|\boldsymbol{v}\|_0^2\right)^{\frac{1}{2}} \ ,$$

$$|c(\boldsymbol{u},\boldsymbol{u})| \geq \frac{1}{\sqrt{2}}(\|\mathbf{curl}\,\boldsymbol{u}\|_0^2 + \sigma\,\|\boldsymbol{u}\|_0^2) \ .$$

Galerkin discretization: edge finite elements on triangulation of Ω ($\rightarrow$ $\mathbf{A}$)

Complex Variational Problems

Eddy current problem in frequency domain ($\sigma > 0$):

$$H = \boldsymbol{H}_0(\mathbf{curl};\Omega) \quad , \quad c(\boldsymbol{u},\boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_0 + \imath\sigma\,(\boldsymbol{u},\boldsymbol{v})_0 \ .$$

$$c(\boldsymbol{u},\boldsymbol{v}) \leq \left(\|\mathbf{curl}\,\boldsymbol{u}\|_0^2 + \sigma\,\|\boldsymbol{u}\|_0^2\right)^{\frac{1}{2}} \left(\|\mathbf{curl}\,\boldsymbol{v}\|_0^2 + \sigma\,\|\boldsymbol{v}\|_0^2\right)^{\frac{1}{2}} \ ,$$

$$|c(\boldsymbol{u},\boldsymbol{u})| \geq \frac{1}{\sqrt{2}}(\|\mathbf{curl}\,\boldsymbol{u}\|_0^2 + \sigma\,\|\boldsymbol{u}\|_0^2) \ .$$

Galerkin discretization: edge finite elements on triangulation of Ω ($\rightarrow$ **A**)

Preconditioner: Inverse of Galerkin matrix ($\rightarrow$ **B**)for

$$(\boldsymbol{u},\boldsymbol{v})_H := (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_0 + \sigma\,(\boldsymbol{u},\boldsymbol{v})_0 \ , \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \ .$$

Complex Variational Problems

Eddy current problem in frequency domain ($\sigma > 0$):

$$H = \boldsymbol{H}_0(\mathbf{curl};\Omega) \quad , \quad c(\boldsymbol{u},\boldsymbol{v}) = (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_0 + \imath\sigma\,(\boldsymbol{u},\boldsymbol{v})_0 \ .$$

$$c(\boldsymbol{u},\boldsymbol{v}) \leq \left(\|\mathbf{curl}\,\boldsymbol{u}\|_0^2 + \sigma\,\|\boldsymbol{u}\|_0^2\right)^{\frac{1}{2}} \left(\|\mathbf{curl}\,\boldsymbol{v}\|_0^2 + \sigma\,\|\boldsymbol{v}\|_0^2\right)^{\frac{1}{2}} \ ,$$

$$|c(\boldsymbol{u},\boldsymbol{u})| \geq \frac{1}{\sqrt{2}}(\|\mathbf{curl}\,\boldsymbol{u}\|_0^2 + \sigma\,\|\boldsymbol{u}\|_0^2) \ .$$

Galerkin discretization: edge finite elements on triangulation of Ω ($\rightarrow$ $\mathbf{A}$)

Preconditioner: Inverse of Galerkin matrix ($\rightarrow$ $\mathbf{B}$)for

$$(\boldsymbol{u},\boldsymbol{v})_H := (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_0 + \sigma\,(\boldsymbol{u},\boldsymbol{v})_0 \ , \quad \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \ .$$

Abstract theory ► $\kappa(\mathbf{BA}) \leq \sqrt{2}$

What Next ?

1. (Simple) Abstract Framework
2. Finite Element Applications: Equivalent Operator Preconditioning
3. Boundary Element Applications: Calderón Preconditioning
4. Calderón Preconditioning for Screen Problems

Electric Field Integral Equation (EFIE)

Electric Field Integral Equation (EFIE)

- ▷ $\Gamma \hat{=}$ boundary of a domain $\Omega \subset \mathbb{R}^3$, wave number $k > 0$
- ▷ Hilbert space $V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\boldsymbol{v} \in H_{\mathbf{t}}^{-\frac{1}{2}}(\Gamma), \mathrm{div}_\Gamma \boldsymbol{v} \in H^{-\frac{1}{2}}(\Gamma)\}$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x} - \boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \, .$$

Electric Field Integral Equation (EFIE)

▷ $\Gamma \hat{=}$ boundary of a domain $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

▷ Hilbert space $V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\mathbf{v} \in H_{\mathbf{t}}^{-\frac{1}{2}}(\Gamma), \mathrm{div}_\Gamma \mathbf{v} \in H^{-\frac{1}{2}}(\Gamma)\}$

$$a(\mathbf{u}, \mathbf{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|} \left(\mathbf{u}(\mathbf{y}) \cdot \mathbf{v}(\mathbf{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \mathbf{u}(\mathbf{y})\, \mathrm{div}_\Gamma \mathbf{v}(\mathbf{x}) \right) \mathrm{d}S(\mathbf{y}, \mathbf{x}) \,.$$

trace space of $\boldsymbol{H}(\mathbf{curl}; \Omega)$

Electric Field Integral Equation (EFIE)

▷ $\Gamma \hat{=}$ boundary of a domain $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

▷ Hilbert space $V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\boldsymbol{v} \in H_{\mathbf{t}}^{-\frac{1}{2}}(\Gamma), \mathrm{div}_\Gamma \boldsymbol{v} \in H^{-\frac{1}{2}}(\Gamma)\}$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \, .$$

Galerkin discretization ➢ surface edge/RWG elements (space V_h)

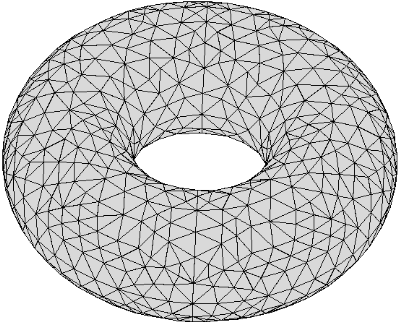
Electric Field Integral Equation (EFIE)

▷ $\Gamma \triangleq$ boundary of a domain $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

▷ Hilbert space $V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\boldsymbol{v} \in \boldsymbol{H}_{\mathbf{t}}^{-\frac{1}{2}}(\Gamma), \mathrm{div}_\Gamma \boldsymbol{v} \in H^{-\frac{1}{2}}(\Gamma)\}$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \,.$$

Galerkin discretization ➢ surface edge/RWG elements (space V_h)



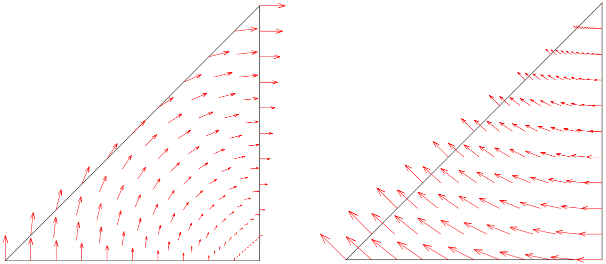
Electric Field Integral Equation (EFIE)

▷ $\Gamma \hat{=}$ boundary of a domain $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

▷ Hilbert space $V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\boldsymbol{v} \in H_{\mathbf{t}}^{-\frac{1}{2}}(\Gamma), \mathrm{div}_\Gamma \boldsymbol{v} \in H^{-\frac{1}{2}}(\Gamma)\}$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \, .$$

Galerkin discretization ➢ surface edge/RWG elements (space V_h)



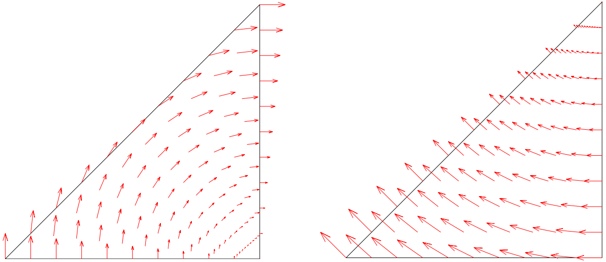
Electric Field Integral Equation (EFIE)

- ▷ $\Gamma \hat{=}$ boundary of a domain $\Omega \subset \mathbb{R}^3$, wave number $k > 0$
- ▷ Hilbert space $V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) := \{\boldsymbol{v} \in H_{\mathbf{t}}^{-\frac{1}{2}}(\Gamma), \mathrm{div}_\Gamma \boldsymbol{v} \in H^{-\frac{1}{2}}(\Gamma)\}$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \, .$$

Galerkin discretization ➢ surface edge/RWG elements (space V_h)

If $k \neq$ interior resonant frequency of Ω, then $a_{|V_h}$ satisfies h-uniform inf-sup condition on sufficiently fine and shape regular meshes.



“Self Duality” of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

“Self Duality” of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

- V, W Hilbert spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that
  - ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable
  - ❷ $\dim V_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \,.$$

“Self Duality” of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

- V, W Hilbert spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that
  ❶ $A_h := A_{|V_h} : V_h \mapsto V'_h$, $B_h := B_{|W_h} : W_h \mapsto W'_h$ (h-uniformly) stable
  ❷ $\dim V_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \; .$$

“Self Duality” of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

- V, W Hilbert spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that

❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable

❷ $\dim V_h = \dim W_h =: N$

- Stable discrete duality pairing: sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \,.$$

“Self Duality” of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

- V, W Hilbert spaces, $A \in L(V, V')$, $B \in L(W, W')$ isomorphisms.
- Finite dimensional trial/test spaces $V_h \subset V$, $W_h \subset W$ such that
  ❶ $A_h := A_{|V_h} : V_h \mapsto V_h'$, $B_h := B_{|W_h} : W_h \mapsto W_h'$ (h-uniformly) stable
  ❷ $\dim V_h = \dim W_h =: N$
- Stable discrete duality pairing: sesqui-linear form $d \in L(V \times W, \mathbb{C})$

$$\exists c_D > 0: \quad \sup_{v_h \in V_h} \frac{|d(v_h, w_h)|}{\|v_h\|_V} \geq c_D \|w_h\|_W \quad \forall w_h \in W_h \ .$$

Thm.: $\exists c > 0$: $\displaystyle\sup_{\mathbf{v} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)} \frac{|\int_\Gamma \mathbf{u} \cdot (\bar{\mathbf{v}} \times \mathbf{n})\, \mathrm{d}S|}{\|\mathbf{v}\|_{-\frac{1}{2}, \mathrm{div}_\Gamma}} \geq c \|\mathbf{u}\|_{-\frac{1}{2}, \mathrm{div}_\Gamma} \quad \forall \mathbf{u} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

"Self Duality" of $H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

Thm.: $\exists c>0$: $\displaystyle\sup_{\mathbf{v}\in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \frac{\left|\int_\Gamma \mathbf{u}\cdot(\bar{\mathbf{v}}\times\mathbf{n})\,\mathrm{d}S\right|}{\|\mathbf{v}\|_{-\frac{1}{2},\mathrm{div}_\Gamma}} \geq c\,\|\mathbf{u}\|_{-\frac{1}{2},\mathrm{div}_\Gamma} \quad \forall \mathbf{u}\in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

- $W = V = H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

“Self Duality” of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

Thm.: $\exists c > 0$: $\sup\limits_{\mathbf{v}\in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \dfrac{\left|\int_\Gamma \mathbf{u}\cdot(\bar{\mathbf{v}}\times\mathbf{n})\,\mathrm{d}S\right|}{\|\mathbf{v}\|_{-\frac{1}{2},\mathrm{div}_\Gamma}} \geq c\,\|\mathbf{u}\|_{-\frac{1}{2},\mathrm{div}_\Gamma}\quad \forall \mathbf{u}\in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

- $W = V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$
- operator $B : W \mapsto W'$ induced by same bilinear form $a(\cdot,\cdot)$

"Self Duality" of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

Thm.: $\exists c > 0$: $\displaystyle\sup_{\mathbf{v}\in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \frac{\left|\int_\Gamma \mathbf{u}\cdot(\bar{\mathbf{v}}\times\mathbf{n})\,\mathrm{d}S\right|}{\|\mathbf{v}\|_{-\frac{1}{2},\mathrm{div}_\Gamma}} \geq c\,\|\mathbf{u}\|_{-\frac{1}{2},\mathrm{div}_\Gamma} \quad \forall \mathbf{u} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

- $W = V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$
- operator $B : W \mapsto W'$ induced by same bilinear form $a(\cdot,\cdot)$
- Duality pairing $d(\mathbf{u},\mathbf{v}) := \int_\Gamma \mathbf{u}\cdot(\bar{\mathbf{v}}\times\mathbf{n})\,\mathrm{d}S, \quad \mathbf{u},\mathbf{v} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

"Self Duality" of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

Thm.: $\exists c > 0$: $\sup_{\mathbf{v} \in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \dfrac{\left|\int_\Gamma \mathbf{u} \cdot (\bar{\mathbf{v}} \times \mathbf{n})\,\mathrm{d}S\right|}{\|\mathbf{v}\|_{-\frac{1}{2},\mathrm{div}_\Gamma}} \geq c\,\|\mathbf{u}\|_{-\frac{1}{2},\mathrm{div}_\Gamma}\quad \forall \mathbf{u} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

- $W = V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$
- operator $B : W \mapsto W'$ induced by same bilinear form $a(\cdot,\cdot)$
- Duality pairing $d(\mathbf{u}, \mathbf{v}) := \int_\Gamma \mathbf{u} \cdot (\bar{\mathbf{v}} \times \mathbf{n})\,\mathrm{d}S$, $\quad \mathbf{u}, \mathbf{v} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

"Self Duality" of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

Thm.: $\exists c > 0$: $\displaystyle\sup_{\boldsymbol{v} \in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \frac{\left|\int_\Gamma \boldsymbol{u} \cdot (\bar{\boldsymbol{v}} \times \boldsymbol{n})\,\mathrm{d}S\right|}{\|\boldsymbol{v}\|_{-\frac{1}{2},\mathrm{div}_\Gamma}} \geq c\,\|\boldsymbol{u}\|_{-\frac{1}{2},\mathrm{div}_\Gamma} \quad \forall \boldsymbol{u} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

- $W = V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$
- operator $B : W \mapsto W'$ induced by same bilinear form $a(\cdot, \cdot)$
- Duality pairing $\displaystyle d(\boldsymbol{u}, \boldsymbol{v}) := \int_\Gamma \boldsymbol{u} \cdot (\bar{\boldsymbol{v}} \times \boldsymbol{n})\,\mathrm{d}S, \quad \boldsymbol{u}, \boldsymbol{v} \in H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$
- "Temptation": $W_h = V_h$

"Self Duality" of $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$

Thm.: $\exists c > 0$: $\displaystyle\sup_{\mathbf{v}\in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \frac{|\int_\Gamma \mathbf{u}\cdot(\bar{\mathbf{v}}\times\mathbf{n})\,\mathrm{d}S|}{\|\mathbf{v}\|_{-\frac{1}{2},\mathrm{div}_\Gamma}} \geq c\,\|\mathbf{u}\|_{-\frac{1}{2},\mathrm{div}_\Gamma} \quad \forall \mathbf{u}\in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$

- $W = V = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$
- operator $B : W \mapsto W'$ induced by same bilinear form $a(\cdot,\cdot)$
- Duality pairing $\displaystyle d(\mathbf{u},\mathbf{v}) := \int_\Gamma \mathbf{u}\cdot(\bar{\mathbf{v}}\times\mathbf{n})\,\mathrm{d}S, \quad \mathbf{u},\mathbf{v}\in H^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$
- "Temptation": $W_h = V_h$

This is flawed !

S. Christiansen and J.-C. Nédélec, *A preconditioner for the electric field integral equation based on Calderón formulas*, SIAM J. Numer. Anal., 40 (2002), pp. 1100–1135.

$\exists$ subspace $N_h \subset V_h$, $C, c > 0$: $\dim N_h \geq c \dim V_h$ such that

$$\forall \mathbf{u}_h \in N_h: \quad \sup_{\mathbf{v}_h\in V_h} \frac{d(\mathbf{u}_h,\mathbf{v}_h)}{\|\mathbf{v}_h\|_{-\frac{1}{2},\mathrm{div}_\Gamma}} \leq Ch^{1/2}\,\|\mathbf{u}_h\|_{-\frac{1}{2},\mathrm{div}_\Gamma} \quad \Rightarrow \quad c_D \to 0 \quad \text{for} \quad h\to 0\,.$$

Scalar Single Layer BIE

- $V = H^{-\frac{1}{2}}(\Gamma)$ (dual of trace space for $H^1(\Omega)$)

Scalar Single Layer BIE

- $V = H^{-\frac{1}{2}}(\Gamma)$ (dual of trace space for $H^1(\Omega)$)
- Single layer boundary integral operator $A : V \mapsto V'$ for $-\Delta$

$$\longleftrightarrow \qquad a(\varphi, \psi) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \varphi(\mathbf{x})\, \psi(\mathbf{y})\, \mathrm{d}S(\mathbf{x}, \mathbf{y}) \; .$$

Scalar Single Layer BIE

- $V = H^{-\frac{1}{2}}(\Gamma)$ (dual of trace space for $H^1(\Omega)$)
- Single layer boundary integral operator $A : V \mapsto V'$ for $-\Delta$

$$\longleftrightarrow \qquad a(\varphi, \psi) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \varphi(\mathbf{x})\,\psi(\mathbf{y})\,\mathrm{d}S(\mathbf{x},\mathbf{y}) \,.$$

- Duality pairing between $V = H^{-\frac{1}{2}}(\Gamma)$ and $W = H^{\frac{1}{2}}(\Gamma)$:

$$d(u, v) := \int_\Gamma u\, v\, \mathrm{d}S$$

► trivially stable

Scalar Single Layer BIE

- $V = H^{-\frac{1}{2}}(\Gamma)$ (dual of trace space for $H^1(\Omega)$)
- Single layer boundary integral operator $A : V \mapsto V'$ for $-\Delta$

$$\longleftrightarrow \qquad a(\varphi,\psi) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\varphi(\mathbf{x})\,\psi(\mathbf{y})\,\mathrm{d}S(\mathbf{x},\mathbf{y}) \ .$$

- Duality pairing between $V = H^{-\frac{1}{2}}(\Gamma)$ and $W = H^{\frac{1}{2}}(\Gamma)$:

$$d(u,v) := \int_\Gamma u\,v\,\mathrm{d}S$$

  ► trivially stable

- Hypersingular boundary integral operator $B : W \to W'$

$$\leftrightarrow b(u,v) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\mathbf{curl}_\Gamma u(\mathbf{x}) \cdot \mathbf{curl}_\Gamma v(\mathbf{y})\,\mathrm{d}S(\mathbf{x},\mathbf{y}) + \int_\Gamma u\,\mathrm{d}S \int_\Gamma v\,\mathrm{d}S \ .$$

Scalar Single Layer BIE

- $V = H^{-\frac{1}{2}}(\Gamma)$ (dual of trace space for $H^1(\Omega)$)
- Single layer boundary integral operator $A : V \mapsto V'$ for $-\Delta$

$$\longleftrightarrow \qquad a(\varphi, \psi) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \varphi(\mathbf{x})\, \psi(\mathbf{y})\, \mathrm{d}S(\mathbf{x}, \mathbf{y}) \;.$$

- Duality pairing between $V = H^{-\frac{1}{2}}(\Gamma)$ and $W = H^{\frac{1}{2}}(\Gamma)$:

$$d(u, v) := \int_\Gamma u\, v\, \mathrm{d}S \qquad \blacktriangleright \qquad \text{trivially stable}$$

- Hypersingular boundary integral operator $B : W \to W'$

$$\leftrightarrow b(u, v) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \mathbf{curl}_\Gamma\, u(\mathbf{x}) \cdot \mathbf{curl}_\Gamma\, v(\mathbf{y})\, \mathrm{d}S(\mathbf{x}, \mathbf{y}) + \int_\Gamma u\, \mathrm{d}S \int_\Gamma v\, \mathrm{d}S \;.$$

- Simplest Galerkin space: $V_h \;\hat{=}$ piecewise constants

Scalar Single Layer BIE

- $V = H^{-\frac{1}{2}}(\Gamma)$ (dual of trace space for $H^1(\Omega)$)
- Single layer boundary integral operator $A : V \mapsto V'$ for $-\Delta$

$$\longleftrightarrow \qquad a(\varphi,\psi) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\varphi(\mathbf{x})\,\psi(\mathbf{y})\,\mathrm{d}S(\mathbf{x},\mathbf{y}) \; .$$

- Duality pairing between $V = H^{-\frac{1}{2}}(\Gamma)$ and $W = H^{\frac{1}{2}}(\Gamma)$:

$$d(u,v) := \int_\Gamma u\,v\,\mathrm{d}S$$ ► trivially stable

- Hypersingular boundary integral operator $B : W \to W'$

$$\leftrightarrow b(u,v) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\mathbf{curl}_\Gamma u(\mathbf{x}) \cdot \mathbf{curl}_\Gamma v(\mathbf{y})\,\mathrm{d}S(\mathbf{x},\mathbf{y}) + \int_\Gamma u\,\mathrm{d}S \int_\Gamma v\,\mathrm{d}S \; .$$

- Simplest Galerkin space: $V_h \hat{=}$ piecewise constants

$V_h \not\subset W$ ➢ $W_h = V_h$ not an option: What is W_h **?**

Scalar Single Layer BIE

- $V = H^{-\frac{1}{2}}(\Gamma)$ (dual of trace space for $H^1(\Omega)$)
- Single layer boundary integral operator $A : V \mapsto V'$ for $-\Delta$

$$\longleftrightarrow \qquad a(\varphi, \psi) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \varphi(\mathbf{x})\,\psi(\mathbf{y})\,\mathrm{d}S(\mathbf{x}, \mathbf{y}) \;.$$

- Duality pairing between $V = H^{-\frac{1}{2}}(\Gamma)$ and $W = H^{\frac{1}{2}}(\Gamma)$:

$$d(u, v) := \int_\Gamma u\,v\,\mathrm{d}S \qquad \blacktriangleright \qquad \text{trivially stable}$$

- Hypersingular boundary integral operator $B : W \to W'$

$$\leftrightarrow b(u, v) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \mathbf{curl}_\Gamma u(\mathbf{x}) \cdot \mathbf{curl}_\Gamma v(\mathbf{y})\,\mathrm{d}S(\mathbf{x}, \mathbf{y}) + \int_\Gamma u\,\mathrm{d}S \int_\Gamma v\,\mathrm{d}S \;.$$

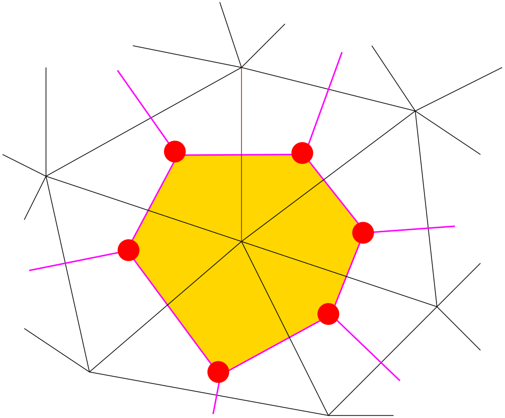
- Simplest Ga

Required $\dim V_h = \dim W_h$, $W_h \subset C^0(\Gamma)$

$V_h \not\subset W$ ➢ $W_h = V_h$ not an option: What is W_h **?**

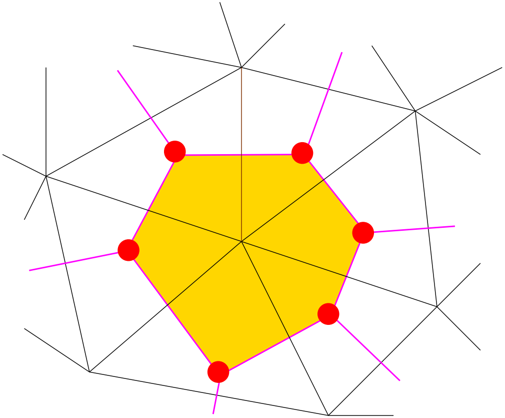
Dual Meshes

Dual Meshes

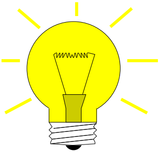


mesh $\mathcal{M}$	$\leftrightarrow$	dual mesh $\widetilde{\mathcal{M}}$
nodes	$\leftrightarrow$	cells
edges	$\leftrightarrow$	edges
cells	$\leftrightarrow$	nodes

Dual Meshes



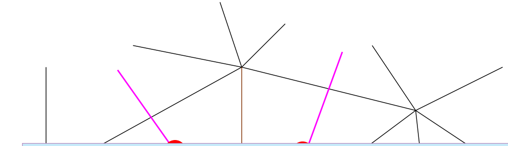
mesh $\mathcal{M}$	$\leftrightarrow$	dual mesh $\widetilde{\mathcal{M}}$
nodes	$\leftrightarrow$	cells
edges	$\leftrightarrow$	edges
cells	$\leftrightarrow$	nodes



Idea:

$V_h \leftrightarrow$ cells of $\widetilde{\mathcal{M}}$ $\quad\Leftrightarrow\quad$ $W_h \leftrightarrow$ nodes of $\mathcal{M}$

Dual Meshes

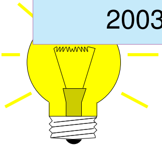


mesh $\mathcal{M} \leftrightarrow$ dual mesh $\widetilde{\mathcal{M}}$

O. STEINBACH AND W. WENDLAND, *The construction of some efficient preconditioners in the boundary element method*, Adv. Comput. Math., 9 (1998), pp. 191–216.

O. STEINBACH, *On a generalized L_2 projection and some related stability estimates in Sobolev spaces*, Numer. Math., 90 (2002), pp. 775–786.

O. STEINBACH, *Stability estimates for hybrid coupled domain decomposition methods*, vol. 1809 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2003.



$V_h \leftrightarrow$ cells of $\widetilde{\mathcal{M}}$ $\Leftrightarrow$ $W_h \leftrightarrow$ nodes of $\mathcal{M}$

Duality via Dual Meshes: Scalar Case

Duality via Dual Meshes: Scalar Case

For $V = H^{-\frac{1}{2}}(\Gamma)$:

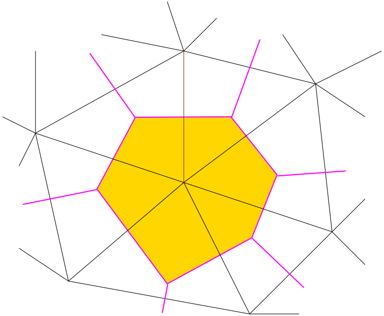
Duality via Dual Meshes: Scalar Case

For $V = H^{-\frac{1}{2}}(\Gamma)$:

V_h

$\updownarrow$

p.w. constant on cells of $\mathcal{M}$

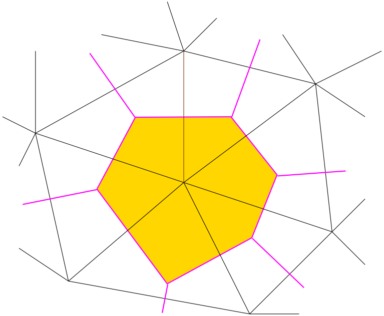


$\leftrightarrow$

Duality via Dual Meshes: Scalar Case

For $V = H^{-\frac{1}{2}}(\Gamma)$:

$$\begin{array}{ccc} V_h & \longleftrightarrow & W_h \subset H^{\frac{1}{2}}(\Omega) \\ \updownarrow & & \updownarrow \\ \text{p.w. constant on cells of } \mathcal{M} & \longleftrightarrow & \text{“p.w. linear” \& } C^0 \text{ on } \widetilde{\mathcal{M}} \end{array}$$

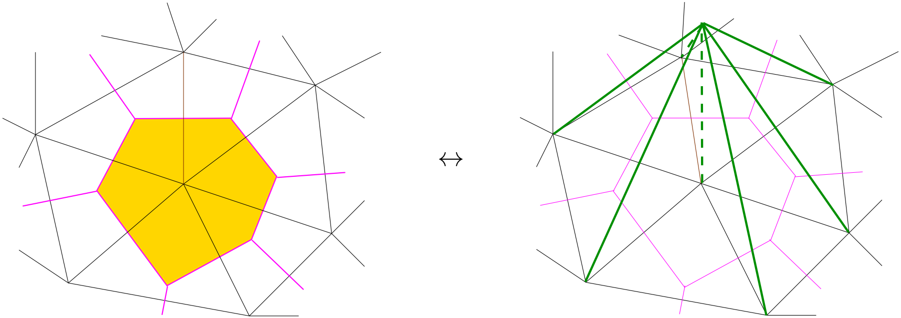


$\leftrightarrow$

Duality via Dual Meshes: Scalar Case

For $V = H^{-\frac{1}{2}}(\Gamma)$:

$$
\begin{array}{ccc}
V_h & \longleftrightarrow & W_h \subset H^{\frac{1}{2}}(\Omega) \\
\updownarrow & & \updownarrow \\
\text{p.w. constant on cells of } \mathcal{M} & \longleftrightarrow & \text{"p.w. linear" \& } C^0 \text{ on } \widetilde{\mathcal{M}}
\end{array}
$$



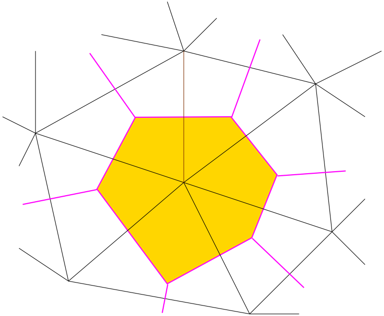
Duality via Dual Meshes: Scalar Case

For $V = H^{-\frac{1}{2}}(\Gamma)$:

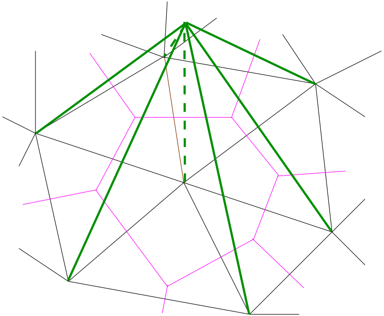
$$V_h \longleftrightarrow W_h \subset H^{\frac{1}{2}}(\Omega)$$

$$\updownarrow \qquad\qquad \updownarrow$$

p.w. constant on cells of $\mathcal{M}$ $\longleftrightarrow$ "p.w. linear" **&** C^0 on $\widetilde{\mathcal{M}}$



$\leftrightarrow$



h-uniform stability of discrete duality paiting $(u, v) \mapsto \int uv \, \mathrm{d}S$ on $V_h \times W_h$

Duality via Dual Meshes: Vectorial Case

Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

$$V_h$$
$$\updownarrow$$
edge elements on $\mathcal{M}$

Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

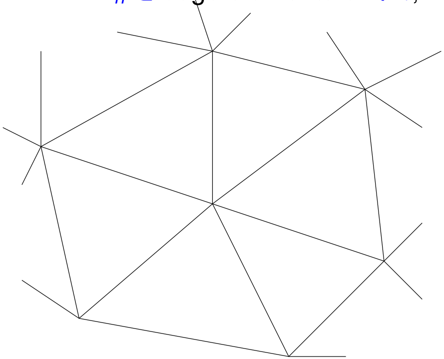
$$\begin{array}{ccc} V_h & \longleftrightarrow & W_h \\ \updownarrow & & \updownarrow \\ \text{edge elements on } \mathcal{M} & \longleftrightarrow & \text{edge elements on } \widetilde{\mathcal{M}} \text{ (?)} \end{array}$$

Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

$$\begin{array}{ccc} V_h & \longleftrightarrow & W_h \\ \updownarrow & & \updownarrow \\ \text{edge elements on } \mathcal{M} & \longleftrightarrow & \text{edge elements on } \widetilde{\mathcal{M}} \text{ (?)} \end{array}$$

Idea: $W_h \subset$ edge elements on $\widehat{\mathcal{M}}$, $\widehat{\mathcal{M}}$ = barycentric refinement of $\mathcal{M}$

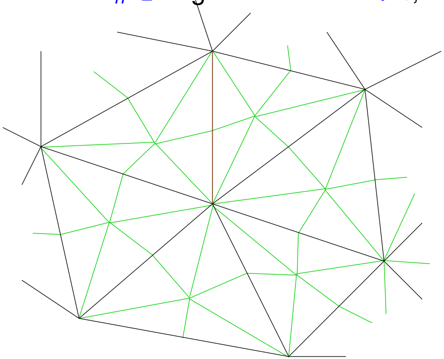


Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

$$\begin{array}{ccc} V_h & \longleftrightarrow & W_h \\ \updownarrow & & \updownarrow \\ \text{edge elements on } \mathcal{M} & \longleftrightarrow & \text{edge elements on } \widetilde{\mathcal{M}} \text{ (?)} \end{array}$$

Idea: $W_h \subset$ edge elements on $\widehat{\mathcal{M}}$, $\widehat{\mathcal{M}}$ = barycentric refinement of $\mathcal{M}$

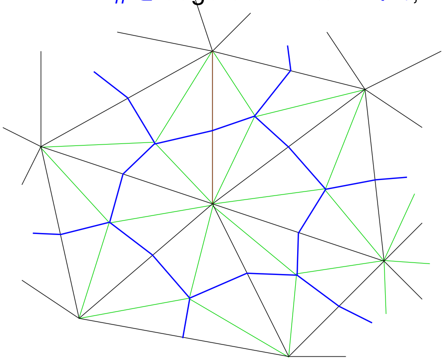


Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

$$\begin{array}{ccc} V_h & \longleftrightarrow & W_h \\ \updownarrow & & \updownarrow \\ \text{edge elements on } \mathcal{M} & \longleftrightarrow & \text{edge elements on } \widetilde{\mathcal{M}} \text{ (?)} \end{array}$$

Idea: $W_h \subset$ edge elements on $\widehat{\mathcal{M}}$, $\widehat{\mathcal{M}}$ = barycentric refinement of $\mathcal{M}$

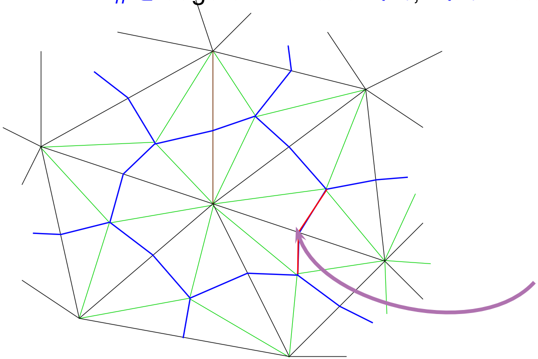


Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

$$\begin{array}{ccc} V_h & \longleftrightarrow & W_h \\ \updownarrow & & \updownarrow \\ \text{edge elements on } \mathcal{M} & \longleftrightarrow & \text{edge elements on } \widetilde{\mathcal{M}} \text{ (?)} \end{array}$$

Idea: $W_h \subset$ edge elements on $\widehat{\mathcal{M}}$, $\widehat{\mathcal{M}}$ = barycentric refinement of $\mathcal{M}$



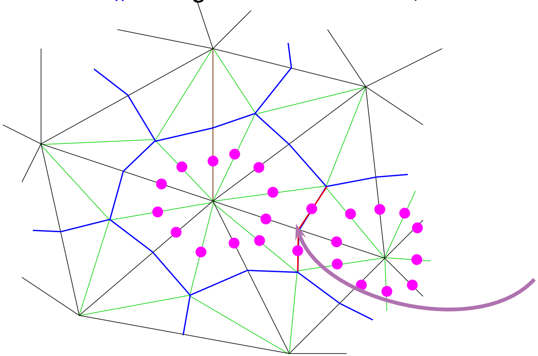
supporting edge of $\widetilde{\mathcal{M}}$

Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

$$\begin{array}{ccc} V_h & \longleftrightarrow & W_h \\ \updownarrow & & \updownarrow \\ \text{edge elements on } \mathcal{M} & \longleftrightarrow & \text{edge elements on } \widetilde{\mathcal{M}} \text{ (?)} \end{array}$$

Idea: $W_h \subset$ edge elements on $\widehat{\mathcal{M}}$, $\widehat{\mathcal{M}}$ = barycentric refinement of $\mathcal{M}$



◁ Basis function of W_h

•: edges of barycentric refinement with nonzero contribution to basis function at —

supporting edge of $\widetilde{\mathcal{M}}$

Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

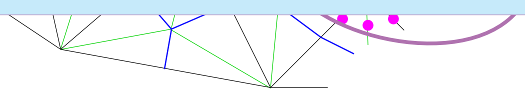
$$V_h \qquad \longleftrightarrow \qquad W_h$$

A. BUFFA AND S. CHRISTIANSEN, *A dual finite element complex on the barycentric refinement*, Math. Comp., 76 (2007), pp. 1743–1769.

Ide

ne-

to



Duality via Dual Meshes: Vectorial Case

For $V = W = H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$: ($\leftrightarrow$ electric field integral equation)

$V_h \quad \longleftrightarrow \quad W_h$

A. BUFFA AND S. CHRISTIANSEN, *A dual finite element complex on the barycentric refinement*, Math. Comp., 76 (2007), pp. 1743–1769.

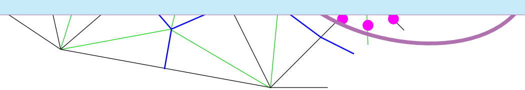
F. ANDRIULLI, K. COOLS, H. BAGCI, F. OLYSLAGER, A. BUFFA, S. CHRISTIANSEN, AND E. MICHIELSSEN, *A multiplicative Calderon preconditioner for the electric field integral equation*, IEEE Trans. Antennas and Propagation, 56 (2008), pp. 2398–2412.

F. P. ANDRIULLI, K. COOLS, I. BOGAERT, AND E. MICHIELSSEN, *On a well-conditioned electric field integral operator for multiply connected geometries*, IEEE Transactions on Antennas and Propagation, 61 (2013), pp. 2077–2087.

Ide

ne-

n to

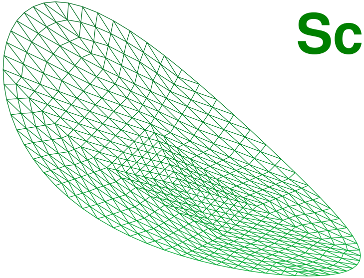


What Next ?

1. (Simple) Abstract Framework
2. Finite Element Applications: Equivalent Operator Preconditioning
3. Boundary Element Applications: Calderón Preconditioning
4. Calderón Preconditioning for Screen Problems

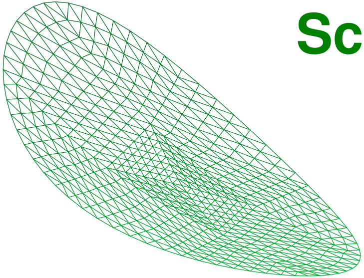
Scalar Screen BIE

Scalar Screen BIE



Screen $\hat{=}$ open orientable surface

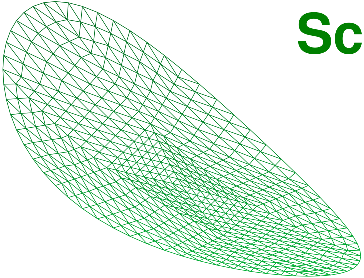
Scalar Screen BIE



Screen $\triangleq$ open orientable surface

"Reference Screen" = disk

$$\mathbb{D} := \{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| = 1,\ x_3 = 0\}$$



Scalar Screen BIE

Screen $\hat{=}$ open orientable surface

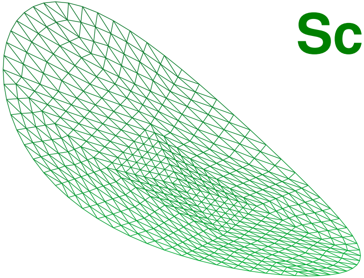
"Reference Screen" = disk

$$\mathbb{D} := \{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| = 1,\ x_3 = 0\}$$

- Scalar single layer boundary integral operator A : bilinear form

$$a(\varphi, \psi) := \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \varphi(\mathbf{x})\, \psi(\mathbf{y})\, \mathrm{d}S(\mathbf{x}, \mathbf{y}) \ ,\ \varphi, \psi \in \widetilde{H}^{-\frac{1}{2}}(\mathbb{D}) \ .$$

Scalar Screen BIE



Screen $\triangleq$ open orientable surface

"Reference Screen" = disk

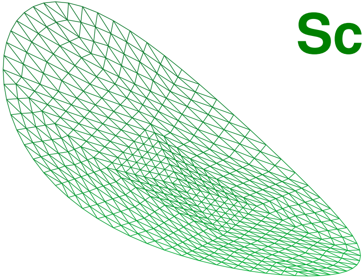
$$\mathbb{D} := \{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| = 1,\ x_3 = 0\}$$

- Scalar single layer boundary integral operator A : bilinear form

$$a(\varphi, \psi) := \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \varphi(\mathbf{x})\, \psi(\mathbf{y})\, \mathrm{d}S(\mathbf{x}, \mathbf{y}),\ \varphi, \psi \in \widetilde{H}^{-\frac{1}{2}}(\mathbb{D})\,.$$

- Trace space $V = \widetilde{H}^{-\frac{1}{2}}(\mathbb{D})[= H_{00}^{-\frac{1}{2}}(\mathbb{D})] \subset H^{-\frac{1}{2}}(\mathbb{D})$, dense with strictly stronger norm

Scalar Screen BIE



Screen $\hat{=}$ open orientable surface

"Reference Screen" = disk

$$\mathbb{D} := \{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| = 1,\ x_3 = 0\}$$

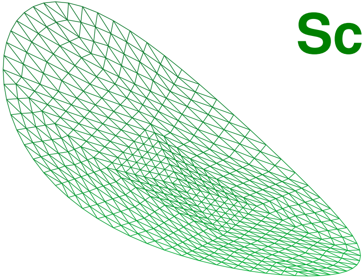
- Scalar single layer boundary integral operator A : bilinear form

$$a(\varphi,\psi) := \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\varphi(\mathbf{x})\,\psi(\mathbf{y})\,\mathrm{d}S(\mathbf{x},\mathbf{y}) \ ,\ \varphi,\psi \in \widetilde{H}^{-\frac{1}{2}}(\mathbb{D}) \ .$$

- Trace space $V = \widetilde{H}^{-\frac{1}{2}}(\mathbb{D})[= H_{00}^{-\frac{1}{2}}(\mathbb{D})] \subset H^{-\frac{1}{2}}(\mathbb{D})$, dense with strictly stronger norm
- Hypersingular boundary integral operator $\widetilde{B}$: bilinear form

$$(u,v) \mapsto \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\,\mathbf{curl}_\Gamma u(\mathbf{x})\,\mathbf{curl}_\Gamma v(\mathbf{y})\,\mathrm{d}S(\mathbf{x},\mathbf{y}) \ ,\ u,v \in \widetilde{H}^{\frac{1}{2}}(\mathbb{D}) \ .$$

Scalar Screen BIE



Screen $\hat{=}$ open orientable surface

"Reference Screen" = disk

$$\mathbb{D} := \{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| = 1,\ x_3 = 0\}$$

- Scalar single layer boundary integral operator A : bilinear form

$$a(\varphi, \psi) := \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \varphi(\mathbf{x})\, \psi(\mathbf{y})\, \mathrm{d}S(\mathbf{x}, \mathbf{y}) \ , \ \varphi, \psi \in \widetilde{H}^{-\frac{1}{2}}(\mathbb{D}) \ .$$

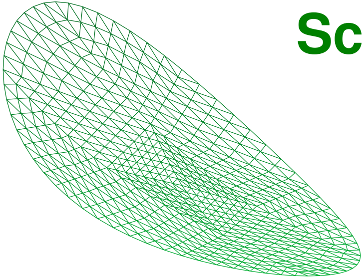
- Trace space $V = \widetilde{H}^{-\frac{1}{2}}(\mathbb{D})[= H_{00}^{-\frac{1}{2}}(\mathbb{D})] \subset H^{-\frac{1}{2}}(\mathbb{D})$, dense with strictly stronger norm
- Hypersingular boundary integral operator $\widetilde{B}$: bilinear form

$$(u, v) \mapsto \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \mathbf{curl}_\Gamma u(\mathbf{x})\, \mathbf{curl}_\Gamma v(\mathbf{y})\, \mathrm{d}S(\mathbf{x}, \mathbf{y}) \ , \ u, v \in \widetilde{H}^{\frac{1}{2}}(\mathbb{D}) \ .$$

- L^2-duality: $(\widetilde{H}^{-\frac{1}{2}}(\mathbb{D}))' = H^{\frac{1}{2}}(\mathbb{D}) \neq \widetilde{H}^{\frac{1}{2}}(\mathbb{D})$,

NOTE $\widetilde{H}^{\frac{1}{2}}(\mathbb{D}) \subset H^{\frac{1}{2}}(\mathbb{D})$ dense with strictly stronger norm

Scalar Screen BIE



Screen $\triangleq$ open orientable surface

"Reference Screen" = disk

$$\mathbb{D} := \{\boldsymbol{x} \in \mathbb{R}^3 \colon \|\boldsymbol{x}\| = 1,\ x_3 = 0\}$$

- Scalar single layer boundary integral operator A : bilinear form

$$a(\varphi, \psi) := \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \varphi(\mathbf{x})\, \psi(\mathbf{y})\, \mathrm{d}S(\mathbf{x}, \mathbf{y}) \ ,\ \varphi, \psi \in \widetilde{H}^{-\frac{1}{2}}(\mathbb{D}) \ .$$

- Trace space $V = \widetilde{H}^{-\frac{1}{2}}(\mathbb{D})[= H_{00}^{-\frac{1}{2}}(\mathbb{D})] \subset H^{-\frac{1}{2}}(\mathbb{D})$, dense with strictly stronger norm
- Hypersingular boundary integral operator $\widetilde{B}$: bilinear form

$$(u, v) \mapsto \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \mathbf{curl}_{\Gamma} u(\mathbf{x})\, \mathbf{curl}_{\Gamma} v(\mathbf{y})\, \mathrm{d}S(\mathbf{x}, \mathbf{y}) \ ,\ u, v \in \widetilde{H}^{\frac{1}{2}}(\mathbb{D}) \ .$$

- L^2-duality: $\quad (\widetilde{H}^{-\frac{1}{2}}(\mathbb{D}))' = H^{\frac{1}{2}}(\mathbb{D}) \neq \widetilde{H}^{\frac{1}{2}}(\mathbb{D})$,

NOTE $\quad \widetilde{H}^{\frac{1}{2}}(\mathbb{D}) \subset H^{\frac{1}{2}}(\mathbb{D})$ dense with strictly stronger norm

Screens: Modified Hypersingular BIO

Screens: Modified Hypersingular BIO

? Bijective boundary integral operator $B : H^{\frac{1}{2}}(\mathbb{D}) \to \widetilde{H}^{-\frac{1}{2}}(\mathbb{D}) = (H^{\frac{1}{2}}(\mathbb{D}))'$

$\leftrightarrow$ inf-sup-stable bounded bilinear form $b : H^{\frac{1}{2}}(\mathbb{D}) \times H^{\frac{1}{2}}(\mathbb{D}) \to \mathbb{R}$

Screens: Modified Hypersingular BIO

? Bijective boundary integral operator $B : H^{\frac{1}{2}}(\mathbb{D}) \to \widetilde{H}^{-\frac{1}{2}}(\mathbb{D}) = (H^{\frac{1}{2}}(\mathbb{D}))'$

$\Leftrightarrow$ inf-sup stable bounded bilinear form $b : H^{\frac{1}{2}}(\mathbb{D}) \times H^{\frac{1}{2}}(\mathbb{D}) \to \mathbb{R}$

- T. BOGGIO, *Sulle funzioni di Green d'ordine m*, Rendiconti del Circolo Matematico di Palermo (1884-1940), 20 (1905), pp. 97–135.
- X.-F. Li and E.-Q. Rong, *Solution of a class of two-dimensional integral equations* J. Comput. Appl. Math. **145** (2002), 335–343.
- R. HIPTMAIR, C. JEREZ-HANCKES, AND C. URZÚA-TORRES, *Closed-form inverses of the weakly singular and hypersingular operators on disks*, Integral Equations and Operator Theory, 90 (2018).
- H. GIMPERLEIN, J. STOCEK, AND C. URZÚA-TORRES, *Optimal operator preconditioning for pseudodifferential boundary problems*, Numer. Math., 148 (2021), pp. 1–41.

Screens: Modified Hypersingular BIO

? Bijective boundary integral operator $B : H^{\frac{1}{2}}(\mathbb{D}) \to \widetilde{H}^{-\frac{1}{2}}(\mathbb{D}) = (H^{\frac{1}{2}}(\mathbb{D}))'$

$\leftrightarrow$ inf-sup-stable bounded bilinear form $b : H^{\frac{1}{2}}(\mathbb{D}) \times H^{\frac{1}{2}}(\mathbb{D}) \to \mathbb{R}$

Screens: Modified Hypersingular BIO

? Bijective boundary integral operator $B : H^{\frac{1}{2}}(\mathbb{D}) \to \widetilde{H}^{-\frac{1}{2}}(\mathbb{D}) = (H^{\frac{1}{2}}(\mathbb{D}))'$

$\leftrightarrow$ inf-sup-stable bounded bilinear form $b : H^{\frac{1}{2}}(\mathbb{D}) \times H^{\frac{1}{2}}(\mathbb{D}) \to \mathbb{R}$

Thm. *Equivalent inner product* on $H^{\frac{1}{2}}(\mathbb{D})$:

$$b(u,v) := \frac{2}{\pi^2}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{S(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\, \mathbf{curl}_\Gamma u(\mathbf{x}) \cdot \mathbf{curl}_\Gamma v(\mathbf{y})\, \mathrm{d}S(\mathbf{x})\mathrm{d}S(\mathbf{y}) + \frac{2}{\pi^2}\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{u(\mathbf{x})v(\mathbf{y})}{\omega(\mathbf{x})\omega(\mathbf{y})}\, \mathrm{d}S(\mathbf{x})\mathrm{d}S(\mathbf{y}), \quad u,v \in H^{\frac{1}{2}}(\mathbb{D}),$$

$$S(\mathbf{x},\mathbf{y}) := \tan^{-1}\left(\frac{\omega(\mathbf{x})\omega(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\right), \mathbf{x} \neq \mathbf{y}, \quad \omega(\mathbf{z}) := \sqrt{1-\|\mathbf{z}\|^2}.$$

Screens: Modified Hypersingular BIO

? Bijective boundary integral operator $B : H^{\frac{1}{2}}(\mathbb{D}) \to \widetilde{H}^{-\frac{1}{2}}(\mathbb{D}) = (H^{\frac{1}{2}}(\mathbb{D}))'$

$\leftrightarrow$ inf-sup-stable bounded bilinear form $b : H^{\frac{1}{2}}(\mathbb{D}) \times H^{\frac{1}{2}}(\mathbb{D}) \to \mathbb{R}$

Thm. *Equivalent inner product* on $H^{\frac{1}{2}}(\mathbb{D})$:

$$b(u,v) := \frac{2}{\pi^2} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{S(\mathbf{x},\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|} \, \mathbf{curl}_\Gamma u(\mathbf{x}) \cdot \mathbf{curl}_\Gamma v(\mathbf{y}) \, \mathrm{d}S(\mathbf{x}) \mathrm{d}S(\mathbf{y}) + \frac{2}{\pi^2} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{u(\mathbf{x}) v(\mathbf{y})}{\omega(\mathbf{x})\omega(\mathbf{y})} \, \mathrm{d}S(\mathbf{x}) \mathrm{d}S(\mathbf{y}) \,, \quad u,v \in H^{\frac{1}{2}}(\mathbb{D}) \,,$$

$$S(\mathbf{x},\mathbf{y}) := \tan^{-1}\left(\frac{\omega(\mathbf{x})\omega(\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|}\right) , \ \mathbf{x} \neq \mathbf{y} \,, \quad \omega(\mathbf{z}) := \sqrt{1 - \|\mathbf{z}\|^2} \,.$$

Then: (Adapted) *dual-mesh-based* Calderón preconditioning

Compact-Equivalent OPC

Compact-Equivalent OPC

- *Hilbert space framework*: H, $H_h \subset H$, $\dim H_h < \infty$

Compact-Equivalent OPC

- *Hilbert space framework*: $H,\ H_h \subset H,\ \dim H_h < \infty$
- $B \in L(H, H')$ isomorphism, $C \in L(H, H')$ **compact**,

Compact-Equivalent OPC

- *Hilbert space framework*: $H,\ H_h \subset H,\ \dim H_h < \infty$
- $B \in L(H, H')$ isomorphism, $C \in L(H, H')$ **compact**,
- $A := B + C \in L(H, H')$ bijective,

Compact-Equivalent OPC

- *Hilbert space framework*: H, $H_h \subset H$, $\dim H_h < \infty$
- $B \in L(H, H')$ isomorphism, $C \in L(H, H')$ **compact**,
- $A := B + C \in L(H, H')$ bijective,
- $A_h := A_{|V_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : H_h \mapsto H'_h$ (asymptotically) h-uniformly stable

Compact-Equivalent OPC

- *Hilbert space framework*: H, $H_h \subset H$, $\dim H_h < \infty$
- $B \in L(H, H')$ isomorphism, $C \in L(H, H')$ **compact**,
- $A := B + C \in L(H, H')$ bijective,
- $A_h := A_{|V_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : H_h \mapsto H'_h$ (asymptotically) h-uniformly stable

Thm.: **GMRES** in H for $\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \mathbf{B}^{-1}\mathbf{b}$ produces residuals $\mathbf{r}_0, \mathbf{r}_1, \ldots$ with

$$\left(\frac{\|\mathbf{r}_k\|_H}{\|\mathbf{r}_0\|_H}\right)^{1/k} \leq \left\|\mathbf{A}^{-1}\mathbf{B}\right\|_H \left\|\mathbf{B}^{-1}\right\|_H \cdot \frac{1}{k}\sum_{j=1}^{k} \sigma_j(C) \ ,$$

Compact-Equivalent OPC

- *Hilbert space framework*: H, $H_h \subset H$, $\dim H_h < \infty$
- $B \in L(H, H')$ isomorphism, $C \in L(H, H')$ **compact**,
- $A := B + C \in L(H, H')$ bijective,
- $A_h := A_{|V_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : H_h \mapsto H'_h$ (asymptotically) h-uniformly stable

Thm.: **GMRES** in H for $\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \mathbf{B}^{-1}\mathbf{b}$ produces residuals $\mathbf{r}_0, \mathbf{r}_1, \ldots$ with

$$\left(\frac{\|\mathbf{r}_k\|_H}{\|\mathbf{r}_0\|_H}\right)^{1/k} \leq \|\mathbf{A}^{-1}\mathbf{B}\|_H \, \|\mathbf{B}^{-1}\|_H \cdot \frac{1}{k}\sum_{j=1}^{k} \sigma_j(C) \,,$$

$\mathbf{A}, \mathbf{B} \triangleq$ any Galerkin matrices

singular values

Compact-Equivalent OPC

- *Hilbert space framework*: H, $H_h \subset H$, $\dim H_h < \infty$
- $B \in L(H, H')$ isomorphism, $C \in L(H, H')$ **compact**,
- $A := B + C \in L(H, H')$ bijective,
- $A_h := A_{|V_h} : H_h \mapsto H'_h$, $B_h := B_{|W_h} : H_h \mapsto H'_h$ (asymptotically) h-uniformly stable

Thm.: **GMRES** in H for $\mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \mathbf{B}^{-1}\mathbf{b}$ produces residuals $\mathbf{r}_0, \mathbf{r}_1, \ldots$ with

$$\left(\frac{\|\mathbf{r}_k\|_H}{\|\mathbf{r}_0\|_H}\right)^{1/k} \leq \|\mathbf{A}^{-1}\mathbf{B}\|_H \, \|\mathbf{B}^{-1}\|_H \cdot \frac{1}{k}\sum_{j=1}^{k} \sigma_j(C) \, ,$$

$\mathbf{A}, \mathbf{B} \triangleq$ any Galerkin matrices   singular values

C compact ► $\sigma_j(C) \to 0$ for $j \to \infty$ ► asymptotically *super-linearly* convergent!

Compact-Equivalent OPC

I. Moret, *A note on the superlinear convergence of GMRES*, SIAM J. Numer. Anal., 34 (1997), pp. 513–516.

J. Blechta, *Stability of linear GMRES convergence with respect to compact perturbations*, SIAM J. Matrix Anal. Appl., 42 (2021), pp. 436–447.

O. Axelsson and J. Karátson, *Mesh independent superlinear PCG rates via compact-equivalent operators*, SIAM J. Numer. Anal., 45 (2007), pp. 1495–1516.

O. Axelsson, J. Karátson, and F. Magoulès, *Superlinear convergence using block preconditioners for the real system formulation of complex Helmholtz equations*, J. Comput. Appl. Math., 340 (2018), pp. 424–431.

T

A, **B** $\hat{=}$ any Galerkin matrices   singular values

C compact ► $\sigma_j(C) \to 0$ for $j \to \infty$ ► asymptotically *super-linearly* convergent!

EFIE on Screens: Splitting

EFIE on Screens: Splitting

- ▷ $\Gamma \triangleq$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$
- ▷ Hilbert space $H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

EFIE on Screens: Splitting

▷ $\Gamma \triangleq$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

▷ Hilbert space $H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \, .$$

EFIE on Screens: Splitting

- ▷ $\Gamma \triangleq$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$
- ▷ Hilbert space $H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) .$$

Galerkin discretization ➢ from surface edge/RWG elements $\rightarrow$ H_h

EFIE on Screens: Splitting

▷ $\Gamma \triangleq$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

▷ Hilbert space $H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \, .$$

Galerkin discretization ➢ from surface edge/RWG elements → H_h

a does not induce isomorphism on $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$!

EFIE on Screens: Splitting

▷ $\Gamma \mathrel{\hat{=}}$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

▷ Hilbert space $\quad H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \; .$$

Galerkin discretization ➢ from surface edge/RWG elements $\rightarrow$ H_h

a does not induce isomorphism on $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$!

Tool: **Hodge decomposition**

$$\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = X_0(\Gamma) \oplus X_\perp(\Gamma) := \mathbf{curl}_\Gamma \widetilde{H}^{\frac{1}{2}}(\Omega) + \mathbf{grad}_\Gamma (\Delta_\Gamma)^{-1} \mathrm{div}_\Gamma \widetilde{H}_*^{-\frac{1}{2}}(\Omega) \; .$$

EFIE on Screens: Splitting

- ▷ $\Gamma \triangleq$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$
- ▷ Hilbert space $H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \, .$$

Galerkin discretization ➢ from surface edge/RWG elements $\to$ H_h

a does not induce isomorphism on $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$!

Tool: **Hodge decomposition**

$$\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = X_0(\Gamma) \oplus X_\perp(\Gamma) := \mathbf{curl}_\Gamma \widetilde{H}^{\frac{1}{2}}(\Omega) + \mathbf{grad}_\Gamma (\Delta_\Gamma)^{-1} \mathrm{div}_\Gamma \widetilde{H}_*^{-\frac{1}{2}}(\Omega) \, .$$

EFIE on Screens: Splitting

▷ $\Gamma \triangleq$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

▷ Hilbert space $H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y})\, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \; .$$

Galerkin discretization ➢ from surface edge/RWG elements $\rightarrow$ H_h

a does not induce isomorphism on $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$!

Tool: **Hodge decomposition**

$$\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = X_0(\Gamma) \oplus X_\perp(\Gamma) := \mathbf{curl}_\Gamma \widetilde{H}^{\frac{1}{2}}(\Omega) + \mathbf{grad}_\Gamma (\Delta_\Gamma)^{-1} \mathrm{div}_\Gamma \widetilde{H}_*^{-\frac{1}{2}}(\Omega) \; .$$

EFIE on Screens: Splitting

- ▷ $\Gamma \triangleq$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$
- ▷ Hilbert space $\quad H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \, .$$

Galerkin discretization ➢ from surface edge/RWG elements $\rightarrow$ H_h

a does not induce isomorphism on $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$!

Tool: **Hodge decomposition**

$$\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = X_0(\Gamma) \oplus X_\perp(\Gamma) := \mathbf{curl}_\Gamma \widetilde{H}^{\frac{1}{2}}(\Omega) + \mathbf{grad}_\Gamma (\Delta_\Gamma)^{-1} \mathrm{div}_\Gamma \widetilde{H}_*^{-\frac{1}{2}}(\Omega) \, .$$

EFIE on Screens: Splitting

- $\Gamma \triangleq$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$
- Hilbert space $\quad H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \, .$$

Galerkin discretization ➢ from surface edge/RWG elements $\rightarrow$ H_h

a does not induce isomorphism on $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$!

Tool: **Hodge decomposition**

$$\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = X_0(\Gamma) \oplus X_\perp(\Gamma) := \mathbf{curl}_\Gamma \widetilde{H}^{\frac{1}{2}}(\Omega) + \mathbf{grad}_\Gamma (\Delta_\Gamma)^{-1} \mathrm{div}_\Gamma \widetilde{H}_*^{-\frac{1}{2}}(\Omega) \, .$$

EFIE on Screens: Splitting

▷ $\Gamma \triangleq$ *screen* $\Omega \subset \mathbb{R}^3$, wave number $k > 0$

▷ Hilbert space $H = \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$ [dense in $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma)$, strictly stronger norm]

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Gamma \int_\Gamma \frac{e^{ik|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\boldsymbol{u}(\boldsymbol{y}) \cdot \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y}) \, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y}, \boldsymbol{x}) \,.$$

Galerkin discretization ➢ from surface edge/RWG elements $\rightarrow$ H_h

a does not induce isomorphism on $H^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$!

Tool: **Hodge decomposition**

$$\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) = X_0(\Gamma) \oplus X_\perp(\Gamma) := \mathbf{curl}_\Gamma \widetilde{H}^{\frac{1}{2}}(\Omega) + \mathbf{grad}_\Gamma (\Delta_\Gamma)^{-1} \mathrm{div}_\Gamma \widetilde{H}_*^{-\frac{1}{2}}(\Omega) \,.$$

► Projections: $\mathsf{P}_0 : \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \rightarrow X_0$, $\mathsf{P}_\perp : \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \rightarrow X_\perp$

EFIE on Screens: C.-E. OPC

EFIE on Screens: C.-E. OPC

Principal part: $a(\boldsymbol{u},\boldsymbol{v}) = d(u,v) + \{\text{compact}\}$,

$$d(\boldsymbol{u},\boldsymbol{v}) := \int_\Gamma\int_\Gamma \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left(\mathsf{P}_o\boldsymbol{u}(\boldsymbol{y})\cdot\mathsf{P}_0\boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2}\operatorname{div}_\Gamma\boldsymbol{u}(\boldsymbol{y})\operatorname{div}_\Gamma\boldsymbol{v}(\boldsymbol{x})\right)\mathrm{d}S(\boldsymbol{y},\boldsymbol{x})\ .$$

Inverse of $d \leftrightarrow \mathsf{D} : \widetilde{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma) \rightarrow (\widetilde{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma))'$
through Hodge decomposition

EFIE on Screens: C.-E. OPC

Principal part: $a(\boldsymbol{u},\boldsymbol{v}) = d(u,v) + \{\text{compact}\}$,

$$d(\boldsymbol{u},\boldsymbol{v}) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left(\mathsf{P}_o\boldsymbol{u}(\boldsymbol{y})\cdot\mathsf{P}_0\boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2}\mathrm{div}_\Gamma\boldsymbol{u}(\boldsymbol{y})\,\mathrm{div}_\Gamma\boldsymbol{v}(\boldsymbol{x})\right)\mathrm{d}S(\boldsymbol{y},\boldsymbol{x}) \,.$$

Inverse of $d \leftrightarrow \mathsf{D} : \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \rightarrow (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma))'$
through Hodge decomposition

On $\mathbb{D}$: $\mathsf{D}^{-1} = \mathsf{B} := \mathbf{curl}_\Gamma \circ \overline{\mathsf{V}} \circ \mathbf{curl}_\Gamma' - k^2\,\mathbf{grad}\circ(\Delta_\Gamma)^{-1} \circ \overline{\mathsf{W}} \circ \left(\mathbf{grad}\circ(\Delta_\Gamma)^{-1}\right)' \,.$

EFIE on Screens: C.-E. OPC

Principal part: $a(\boldsymbol{u},\boldsymbol{v}) = d(u,v) + \{\text{compact}\}$,

$$d(\boldsymbol{u},\boldsymbol{v}) := \int_\Gamma\int_\Gamma \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left(\mathsf{P}_o\boldsymbol{u}(\boldsymbol{y})\cdot\mathsf{P}_0\boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2}\mathrm{div}_\Gamma\boldsymbol{u}(\boldsymbol{y})\,\mathrm{div}_\Gamma\boldsymbol{v}(\boldsymbol{x})\right)\,\mathrm{d}S(\boldsymbol{y},\boldsymbol{x})\,.$$

Inverse of $d \leftrightarrow \mathsf{D} : \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \rightarrow (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma))'$
through Hodge decomposition

On $\mathbb{D}$: $\quad \mathsf{D}^{-1} = \mathsf{B} := \mathbf{curl}_\Gamma \circ \overline{\mathsf{V}} \circ \mathbf{curl}'_\Gamma - k^2\,\mathbf{grad}\circ(\Delta_\Gamma)^{-1}\circ\overline{\mathsf{W}}\circ\left(\mathbf{grad}\circ(\Delta_\Gamma)^{-1}\right)'\,.$

modified single layer/hypersingular BI-Ops.

EFIE on Screens: C.-E. OPC

Principal part: $a(\boldsymbol{u},\boldsymbol{v}) = d(u,v) + \{\text{compact}\}$,

$$d(\boldsymbol{u},\boldsymbol{v}) := \int_\Gamma\int_\Gamma \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left(\mathsf{P}_o\boldsymbol{u}(\boldsymbol{y})\cdot\mathsf{P}_0\boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2}\mathrm{div}_\Gamma\boldsymbol{u}(\boldsymbol{y})\,\mathrm{div}_\Gamma\boldsymbol{v}(\boldsymbol{x})\right)\mathrm{d}S(\boldsymbol{y},\boldsymbol{x})\,.$$

Inverse of $d \leftrightarrow \mathsf{D} : \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) \to (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma))'$
through Hodge decomposition

On $\mathbb{D}$: $\quad \mathsf{D}^{-1} = \mathsf{B} := \mathbf{curl}_\Gamma \circ \overline{\mathsf{V}} \circ \mathbf{curl}'_\Gamma - k^2\,\mathbf{grad} \circ(\Delta_\Gamma)^{-1} \circ \overline{\mathsf{W}} \circ \left(\mathbf{grad} \circ(\Delta_\Gamma)^{-1}\right)'\,.$

preconditioning operator

EFIE on Screens: C.-E. OPC

Principal part: $a(\boldsymbol{u},\boldsymbol{v}) = d(u,v) + \{\text{compact}\}$,

$$d(\boldsymbol{u},\boldsymbol{v}) := \int_\Gamma \int_\Gamma \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \left(\mathsf{P}_o \boldsymbol{u}(\boldsymbol{y}) \cdot \mathsf{P}_0 \boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2} \mathrm{div}_\Gamma \boldsymbol{u}(\boldsymbol{y})\, \mathrm{div}_\Gamma \boldsymbol{v}(\boldsymbol{x}) \right) \mathrm{d}S(\boldsymbol{y},\boldsymbol{x}) \,.$$

Inverse of $d \leftrightarrow \mathsf{D} : \widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma) \rightarrow (\widetilde{H}^{-1/2}(\mathrm{div}_\Gamma, \Gamma))'$
through Hodge decomposition

On $\mathbb{D}$: $\mathsf{D}^{-1} = \mathsf{B} := \mathbf{curl}_\Gamma \circ \overline{\mathsf{V}} \circ \mathbf{curl}'_\Gamma - k^2\, \mathbf{grad} \circ (\Delta_\Gamma)^{-1} \circ \overline{\mathsf{W}} \circ \left(\mathbf{grad} \circ (\Delta_\Gamma)^{-1}\right)' \,.$

preconditioning operator

- Mixed BE Galerkin discretization primal/dual meshes

EFIE on Screens: C.-E. OPC

Principal part: $a(\boldsymbol{u},\boldsymbol{v}) = d(u,v) + \{\text{compact}\}$,

$$d(\boldsymbol{u},\boldsymbol{v}) := \int_\Gamma\int_\Gamma \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left(\mathsf{P}_o\boldsymbol{u}(\boldsymbol{y})\cdot\mathsf{P}_0\boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2}\operatorname{div}_\Gamma\boldsymbol{u}(\boldsymbol{y})\operatorname{div}_\Gamma\boldsymbol{v}(\boldsymbol{x})\right)\mathrm{d}S(\boldsymbol{y},\boldsymbol{x})\ .$$

Inverse of $d \leftrightarrow \mathsf{D} : \widetilde{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma) \to (\widetilde{H}^{-1/2}(\operatorname{div}_\Gamma,\Gamma))'$
through Hodge decomposition

On $\mathbb{D}$: $\mathsf{D}^{-1} = \mathsf{B} := \mathbf{curl}_\Gamma \circ \overline{\mathsf{V}} \circ \mathbf{curl}'_\Gamma - k^2\,\mathbf{grad}\circ(\Delta_\Gamma)^{-1}\circ\overline{\mathsf{W}}\circ\left(\mathbf{grad}\circ(\Delta_\Gamma)^{-1}\right)'\ .$

preconditioning operator

- Mixed BE Galerkin discretization primal/dual meshes
- $h \to 0$ & $k \to 0$-*robust* OPC

EFIE on Screens: C.-E. OPC

Principal part: $a(\boldsymbol{u},\boldsymbol{v}) = d(u,v) + \{\text{compact}\}$,

$$d(\boldsymbol{u},\boldsymbol{v}) := \int_\Gamma\int_\Gamma \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\left(\mathsf{P}_o\boldsymbol{u}(\boldsymbol{y})\cdot\mathsf{P}_0\boldsymbol{v}(\boldsymbol{x}) - \frac{1}{k^2}\mathrm{div}_\Gamma\boldsymbol{u}(\boldsymbol{y})\,\mathrm{div}_\Gamma\boldsymbol{v}(\boldsymbol{x})\right)\mathrm{d}S(\boldsymbol{y},\boldsymbol{x})\,.$$

R. HIPTMAIR AND C. URZÚA-TORRES, *Compact Equivalent Inverse of the Electric Field Integral Operator on Screens*, Integral Equations Operator Theory, 92 (2020), p. Paper No. 9.

R. HIPTMAIR AND C. URZÚA-TORRES, *Preconditioning the EFIE on screens*, Math. Models Methods Appl. Sci., 30 (2020), pp. 1705–1726.

preconditioning operator

- Mixed BE Galerkin discretization primal/dual meshes
- $h \to 0$ & $k \to 0$-*robust* OPC

Summary and Conclusion

Summary and Conclusion

In a nutshell . . .:

Summary and Conclusion

In a nutshell . . .:

- To solve: discretized operator equation $Au = f$ for $A : V \to V'$

Summary and Conclusion

In a nutshell ...:

- To solve: discretized operator equation $Au = f$ for $A : V \to V'$
- Known: "preconditioning operator" $B : W \to W'$

Summary and Conclusion

In a nutshell . . .:

- To solve: discretized operator equation $Au = f$ for $A : V \to V'$
- Known: "preconditioning operator" $B : W \to W'$
- Available: discretely stable (local) pairing $d : V \times W \to \mathbb{C}$

Summary and Conclusion

In a nutshell ...:

- To solve: discretized operator equation $Au = f$ for $A : V \rightarrow V'$
- Known: "preconditioning operator" $B : W \rightarrow W'$
- Available: discretely stable (local) pairing $d : V \times W \rightarrow \mathbb{C}$

➤ *Operator preconditioning possible*

Summary and Conclusion

In a nutshell . . .:

- To solve: discretized operator equation $Au = f$ for $A : V \to V'$
- Known: “preconditioning operator” $B : W \to W'$
- Available: discretely stable (local) pairing $d : V \times W \to \mathbb{C}$

▶ *Operator preconditioning possible*

- General preconditioning paradigm

Summary and Conclusion

In a nutshell . . .:

- To solve: discretized operator equation $Au = f$ for $A : V \to V'$
- Known: "preconditioning operator" $B : W \to W'$
- Available: discretely stable (local) pairing $d : V \times W \to \mathbb{C}$

▶ *Operator preconditioning possible*

- General preconditioning paradigm
- Mesh-independent performance

Summary and Conclusion

In a nutshell ...:

- To solve: discretized operator equation $Au = f$ for $A : V \to V'$
- Known: “preconditioning operator” $B : W \to W'$
- Available: discretely stable (local) pairing $d : V \times W \to \mathbb{C}$

► *Operator preconditioning possible*

- General preconditioning paradigm
- Mesh-independent performance

- (Often) inferior to tailored approaches

Summary and Conclusion

In a nutshell . . .:

- To solve: discretized operator equation $Au = f$ for $A : V \to V'$
- Known: “preconditioning operator” $B : W \to W'$
- Available: discretely stable (local) pairing $d : V \times W \to \mathbb{C}$

► *Operator preconditioning possible*

- General preconditioning paradigm
- Mesh-independent performance

- (Often) inferior to tailored approaches
- Lack of robustness

Summary and Conclusion

In a nutshell . . .:

- To solve: discretized operator equation $Au = f$ for $A : V \to V'$
- Known: "preconditioning operator" $B : W \to W'$
- Available: discretely stable (local) pairing $d : V \times W \to \mathbb{C}$

➤ *Operator preconditioning possible*

THANK YOU

- General preconditioning paradigm
- Mesh-independent performance

- (Often) inferior to tailored approaches
- Lack of robustness