Asymptotic behaviour of a nonlocal Fokker-Planck equation

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A nonlocal (fractional) Fokker-Planck

NLFP

$$\partial_t u = \frac{1}{\varepsilon^s}[J^s_\varepsilon * u - u] + \mathrm{div}(xu) := A^s_\varepsilon u + \mathrm{div}(xu) := L^s_\varepsilon u \qquad (1)$$

- $t \geq 0, x \in \mathbb{R}^d, \varepsilon \in (0,1), s \in (0,2]$
- $J^s : \mathbb{R}^d \to [0,\infty)$ s.t.

$$\int_{\mathbb{R}^d} J^s(x)\,\mathrm{d}x = 1, \qquad \int_{\mathbb{R}^d} J^s(x) x\,\mathrm{d}x = 0, \text{ if } s \in [1,2]$$

  and $J^s \sim G_s$ if $s \in (0,2)$ and $\int_{\mathbb{R}^d}(J^s - G_s)x_i x_j\,\mathrm{d}x = 0$ if $s = 2$, where $\hat{G}_s(\xi) = e^{-|\xi|^s}$.
- As $\varepsilon \to 0$ the operator A^2_ε approximates the Laplacian Δ and A^s_ε approximates the fractional Laplacian $-(-\Delta^{s/2})$ **[Andreu, Mazon, Rossi & Toledo, 2010]**
- Not singular, no regularization.

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Questions

- Does this equation behave like the (fractional) Fokker-Planck for large times?
- Is there a positivity estimate valid as $\varepsilon \to 0$
- Can we show exponential convergence towards the equilibrium **uniformly** in ε? Can we use entropy methods/ functional inequalities?
- Can we estimate the speed of convergence of $e^{L_\varepsilon^s t} u_0$ to $e^{L_0^s t} u_0$. What about the limit for $s \to 2^-$?
- What's the shape of this equilibrium?

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Motivation

- This type of equations are common in models arising in biology (genetic circuits **[Cañizo, Carrillo, Pajaro, 2019]**, growth fragmentation **[Caceres, Cañizo, Mischler, 2011]**) for which entropy methods work well
- The latter are not easy to make it work in the scaling. On which other tools can we rely?
- Harris's Theorem to get the correct behaviour as $\varepsilon \to 0$. Toy model for harder problems.
- Numerical methods: in the pure nonlocal diffusion case, if $J = \frac{1}{2}(\delta_{-1} + \delta_1)$, it's a numerical scheme for the heat equation. Practical importance of understanding if a numerical method preserves the long time behaviour of its limiting equation (**[Ayi, Herda, Hivert, Tristani, 2022]**, **[Dujardin, Herau, Lafitte, 2020]** etc.)
- Links with (Generalized) Central Limit Theorem.

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Previous results

- Nonlocal Diffusion **[Andreu et al., 2010]**: for every $T > 0$

$$\lim_{\varepsilon \to 0} \left\| e^{A_\varepsilon^s t} u_0 - e^{-(-\Delta^s/2)t} u_0 \right\|_{L^\infty(\mathbb{R}^d \times (0,T))} = 0$$

- Nonlocal Diffusion **[Rey & Toscani, 2012]**: Correct speed of convergence in Fourier distance
- Nonlocal Fokker Planck **[Mischler & Tristani, 2017]**: compactly supported J, different weights, splitting of the operator.
- Related equations: e.g. **[Ignat & Rossi, 2007]**, **[Molino & Rossi, 2019]**, **[Auricchio, Toscani, Zanella, 2023]**.
- Others...

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Main result

Theorem (Cañizo, T. (2024))

Under suitable hypotheses on J, there exists a unique equilibrium $F_\varepsilon^s \in L_k^1$ of equation (1) such that for $u_0 \in L_k^1$,

$$\|u(t,\cdot) - F_\varepsilon^s\|_{L_k^1} \leq Ce^{-\lambda t} \|u_0 - F_\varepsilon^s\|_{L_k^1} \qquad \text{for every } t \geq 0. \tag{2}$$

with $C \geq 1$ and $\lambda > 0$ **independent** *of ε (and $s \in [s_0, 2]$)*

- The method is constructive and the constants are explicit
- $-\lambda$ is not the first eigenvalue but provides a bound of it.

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Harris's Theorem

1. Confining Lyapunov condition: there exist $T > 0$, $0 < \lambda_L < 1$, and $K > 0$, such that

$$\|S_T \mu\|_V \leq (1 - \lambda_L) \|\mu\|_V + K \|\mu\|$$

2. A uniform positivity condition on a set $\mathcal{C}$: there exist $T > 0$, $0 < \alpha < 1$ and a probability η such that

$$S_T \mu \geq \alpha \eta \int_{\mathcal{C}} \mu$$

Harris's Theorem

If a semigroup $(S_t)_{t \geq 0}$ satisfies the previous two hypotheses with $\mathcal{C}$ "big enough", then the semigroup has a unique invariant probability measure $\mu^* \in \mathcal{P}_V$ and there exist $\lambda, C > 0$ such that

$$||S_t \mu - \mu^*||_V \leq C e^{-\lambda t} ||\mu - \mu^*||_V \qquad \text{for } t \geq 0$$

Harris's Theorem

- Roughly we require that
  - For some $\langle x\rangle^k = (1+|x|^2)^{k/2}$
    $$\frac{\mathrm{d}}{\mathrm{d}t}\int u(t,x)\,\langle x\rangle^k\,\mathrm{d}x \le C_L - \lambda_L \int u(t,x)\,\langle x\rangle^k\,\mathrm{d}x$$
  - If the initial condition is a δ_{x_0}, with x_0 "not too far"; then after a fixed time T,
    $$u(T,x) \ge \alpha > 0$$
    for all $x \in B_R$.
- References:
  - **[Harris, 1956]**
  - **[Meyn & Tweedie 1992, 1993]**
  - **[Hairer & Mattingly, 2011]**
  - **[Cañizo & Mischler, 2023]**

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We want to use Harris

We check the conditions:

- Lyapunov is fairly straightforward
- Positivity
  - No regularization effect
  - Easy for a fixed ε
  - Problem to obtain it uniformly in ε: $\lambda_\varepsilon \to 0$!

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Positivity

- Write the solution u via Wild's sums

$$u(t,x) = e^{(d-\frac{1}{\varepsilon^2})t}\left[u_0(e^t x)+\sum_{n=1}^{\infty}\left(\frac{1}{\varepsilon^2}\right)^n \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_1} J_\varepsilon^{t_1,\dots,t_n} * u_0(e^t x) dt_1 \dots dt_n.\right]$$

  where $J_\varepsilon^{t_1,\dots,t_n}(x) := J_{\varepsilon e^{t_1}} * \cdots * J_{\varepsilon e^{t_n}}(x)$.
- We want to bound the latter, independently on ε and n.
- L^∞ Berry-Esseen Central Limit Theorem

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Berry-Esseen Theorem

Assume

$$\int_{\mathbb{R}^d} xf(x)\,\mathrm{d}x = 0, \quad \int_{\mathbb{R}^d} x_i x_j f(x)\,\mathrm{d}x = \delta_{ij}, \quad \int_{\mathbb{R}^d} |x|^{2+\delta} f(x)\,\mathrm{d}x := \rho_{2+\delta} < \infty. \tag{3}$$

Theorem

Let $f \in \mathcal{P}(\mathbb{R}^d) \cap L^p(\mathbb{R})$ satisfying (3).

$$\mathfrak{f}_n(x) := (\bar{\sigma}_n^2)^{d/2} f_{\sigma_1} * f_{\sigma_2} * \cdots * f_{\sigma_n}(\bar{\sigma}_n x),$$

with $\bar{\sigma}_n^2 = \sum_{i=1}^n \sigma_i^2$. There exist $N = N(p)$ and a constant $C_{BE}(\frac{L}{l}, p, d, \rho_{2+\delta}, \|f\|_{L^p})$ such that for all $n \geq N$

$$\|\mathfrak{f}_n - G\|_{L^\infty} \leq \frac{C_{BE}}{n^{\delta/2}}.$$

Idea of the proof from **[Goudon, Junca, Toscani, 2002] [Hauray & Mischler, 2014]**

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Positivity, again

There exists explicit $\varepsilon_0(N)$ such that

- For $\varepsilon \in [\varepsilon_0, 1]$ positivity is straightforward
- For $\varepsilon < \varepsilon_0$,
$$J_\varepsilon^{t_1,\dots,t_n}(x) \geq A \qquad \text{for all } x \in B_\eta$$
for all $\varepsilon < \varepsilon_0$ and for any $t_1, \dots, t_n$ with $t \geq t_1 \geq \dots t_n \geq 0$ and n such that
$$\frac{t}{\varepsilon^2} \leq n \leq 2\frac{t}{\varepsilon^2}.$$
- Then, formally

$$u(t,x) \geq e^{(d-\frac{1}{\varepsilon^2})t} \sum_{n=\frac{t}{\varepsilon^2}}^{2\frac{t}{\varepsilon^2}} \varepsilon^{-2n} \int_0^t \cdots \int_0^{t_{n-1}} \int_{B_{R_2}} J_\varepsilon^{t_1,\dots,t_n}(e^t x - y) u_0(y)\,\mathrm{d}y\,\mathrm{d}t_n \dots \mathrm{d}t_1$$

$$\geq A e^{dt} e^{-\frac{t}{\varepsilon^2}} \sum_{n=\frac{t}{\varepsilon^2}}^{2\frac{t}{\varepsilon^2}} \left(\frac{t}{\varepsilon^2}\right)^n \frac{1}{n!} \int_{B_{R_2}} u_0(y)\,\mathrm{d}y \geq A C_L e^{dt} \int_{B_{R_2}} u_0(y)\,\mathrm{d}y$$

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The case $s < 2$

- Let J now be fat tailed,

$$\int |J^s - G_s||x|^{2+\delta}\,\mathrm{d}x < \infty$$

- The previous strategy can be followed to give a similar result, again using Wild sums formulation (replacing ε^2 with ε^s)
- We prove an updated version of Generalized Berry-Esseen Theorem for stable laws.
- Proof of positivity goes as before; for $\varepsilon < \varepsilon_0$, we use the range of n

$$\frac{t}{\varepsilon^s} \leq n \leq 2\frac{t}{\varepsilon^s}.$$

- We get rid of the dependency from s, cheating a bit, bounding G_s with G_{s_0} and G where needed.

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Convergence nonlocal to local (for $s = 2$,)

- With additional assumptions on J, for a nice fast enough decaying φ, we prove the consistency of the operator L_ε,

$$\|(L_\varepsilon - L_0)\varphi\|_{L^1_k} \le C\varepsilon$$

- Consistency + Hille Yosida $\left(\left\|L_\varepsilon^{-1}\right\| \le \frac{C}{\lambda}\right)$ give the speed of convergence of the equilibrium towards the standard Gaussian

$$\|F_\varepsilon - G\|_{L^1_k} \le C\varepsilon$$

Theorem

Convergence nonlocal to local: for every $t \ge 0$ and $\varepsilon \in (0, 1]$

$$\left\| e^{L_\varepsilon t} u_0 - e^{L_0 t} u_0 \right\|_{L^1_k} \le C\varepsilon$$

- Consistency + stability give convergence for finite time
- "Spectral gap"+ convergence of the equilibrium give convergence for large times.

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Convergence for $s < 2$

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Preliminary results

Only formal results. Under additional assumptions

- $$\|L_s^\varepsilon\varphi - L_0^s\varphi\|_{L^1_k} \leq C\varepsilon^\alpha$$

- If we assume $s \geq s_0$

  $$\|J - J^s\|_{L^1_k} \leq (2-s)$$

  then for a nice φ

  $$\|(L_\varepsilon^s - L_\varepsilon)\varphi\|_{L^1_k} \leq (2-s)^\beta$$

- Proceeding as before we should be able to prove convergence as $\varepsilon \to 0$ and $s \to 2^-$

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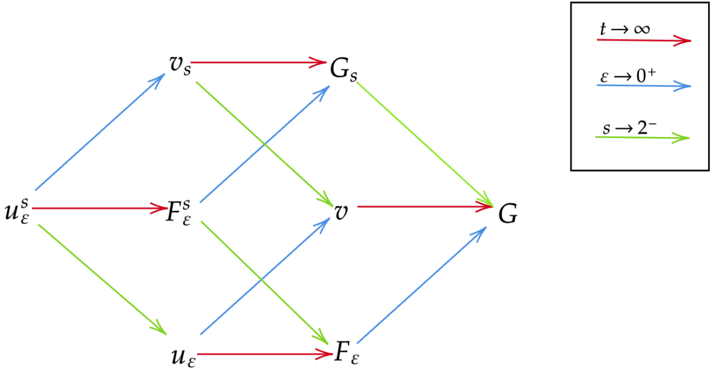
In short...



where v and v^s are the solution of the classic and fractional Fokker Planck. (Here G_s is now the standardized stable law)

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Shape of the equilibrium

- Regularity: via Fourier Analysis and boostrap arguments one can prove that the equilibrium
  - For all $\varepsilon \in (0,1]$
    1. $F_\varepsilon^s \in \mathcal{C}^{l-d}$ for all $l < \frac{1}{\varepsilon^s}$ such that $l \geq d$.
    2. $F_\varepsilon^s \notin \mathcal{C}^{l}$ for all $l > \frac{1}{\varepsilon^s}$.
  - In $d = 1$ (or for radially symmetric J), $F_\varepsilon^s \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$.
  - Non optimal
- If u_0 decays fast enough, for $s = 2$
  - F_ε has at least exponential tails.
  - For compactly supported J, F_ε has Poisson-like tails
  - We conjecture that F_ε do not have Gaussian tails (like the local case)
- For $s = 2$ we can explicitly compute the moments, via Bell's Polynomials and cumulative generating function.

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Future work

- Nonlocal Fokker-Planck with a different potential
- Kinetic Fokker-Planck
- Nonautonomous equations coming from selfsimilar scaling
  - Nonlocal diffusion: from

$$\partial_\tau w = J * w - w$$

to

$$\partial_t u = e^{st}(J_{e^{-t}} * u - u) + \mathrm{div}(xu)$$

  - Growth-Fragmentation: from

$$\partial_\tau f = \mathcal{L}^+ f - Bf$$

to

$$\partial_t g + g + \partial_x(xg) = \gamma \mathcal{L}^+_{e^-t} g - \gamma B_{e^-t} g$$

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Tapadh Leibh![1]

[1] which I have been told it means "thank you!"

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