THE STATISTICAL MECHANICS OF FORESTS AND FERMIONIC σ-MODELS

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1. Abstract

I plan to overview previous and ongoing work regarding the statistical mechanics of unrooted spanning forests. This model turns out to be connected through a graphical representation to the ferromagnetic $\mathbb{H}(0|2)$ nonlinear $\sigma-$models with inverse temperature β playing the role of the chemical potential for number of edges in the random forests. The single spin space in the $\sigma-$model consists of a single even generator z and 2 Grassman generators.

Using this connection in the case $\mathrm{N} = 1$, with R. Bauerschmidt and T. Helmuth we showed that when formulated over the graph $\mathbb{Z}^d, d \geqslant 3$, random unrooted spanning forests exhibits a percolative phase transition and has massless decay of truncated correlations in its supercritical phase (for β large enough). The proof is obtained by performing a renormalization group analysis of Grassman variable lattice field theory with a mass term and quartic interactions and then using Ward identities to show that the $\mathbb{H}(0|2)$ model of interest sits on the stable manifold for the flow in the limit as the mass tends to 0.

I will also discuss work in progress which considers the behavior of this model on $\mathbb{Z}^2$. It is expected expect that, in two dimensions, there is no percolative phase for any value of β and exponential decay of connection probabilities. We show that this model exhibits ultraviolet asymptotic freedom. In other words, we set up and construct the continuum limit for this model on the torus $\mathbb{T}^2$. Starting from a discrete approximation of $\mathbb{T}^2$ with lattice spacing ε, we choose the initial mass $m_\varepsilon = \bar{m}\varepsilon \log(1/\varepsilon)^{1/6}$ and the initial quartic coupling $\beta_\varepsilon = \bar{\beta} + \frac{3}{2\pi}\log(1/\varepsilon) + \frac{1}{2\pi}\log\log(1/\varepsilon)$. With these choices, the connection probability for a pair of points $x \neq y \in \mathbb{T}^2$, $\mathbb{P}_\varepsilon(x \leftrightarrow y)$ converges, after centering and rescaling, to a function $g_{\bar{\beta},\bar{h}}(x-y)$ which blows up as $c_{\bar{\beta},\bar{h}} \log(1/|x-y|)^{1/3}$ as $x \to y$. The exponent $1/3$ should be contrasted with the more conventional power 1 appearing in the free field correlation function.