AWG, an algebraic solver for symmetric positive definite problems

ICMS@Strathclyde: Solvers for frequency-domain wave problems and applications

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Objectives of this talk

Provide a black box solver that will efficiently solve:

$$\mathbf{A}\mathbf{x} = \mathbf{b}; \quad \mathbf{A} \in \mathbb{R}^{n\times n}; \quad \mathbf{b} \in \mathbb{R}^{n}.$$

Matrix **A** is:

- of very high order n
- symmetric positive definite
- sparse (*i.e.*, finite element discretization of PDE)
- ill-conditioned

Domain Decomposition w/ spectral coarse space (GenEO).

i.e., PCG with a preconditioner **H** that approximates $\mathbf{A}^{-1}$ with solutions to subproblems in subdomains and a coarse problem.

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- already assembled.

Algebraic Domain Decomposition w/ spectral coarse space (GenEO).

i.e., PCG with a preconditioner **H** that approximates $\mathbf{A}^{-1}$ with solutions to subproblems in subdomains and a coarse problem.

Abstract DD

Algebraic Subdomains

Partition of $\Omega = [\![1, n]\!]$ (set of all indices in $\mathbb{R}^n$)

$$\Omega = \bigcup_{s=1}^{N} \Omega^s; \text{ with } \Omega^s \subset \Omega \text{ for any } s = 1, \ldots, N.$$

Minimal overlap assumption

For $(i, j) \in [\![1, n]\!]^2$:

$$A_{ij} \neq 0 \Rightarrow (\exists\, s \in [\![1, N]\!] \text{ such that } \{i, j\} \subset \Omega^s)\,.$$

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Restriction Operators $\mathbf{R}^s \in \{0; 1\}^{n^s \times n}$; $\quad n^s = \#\Omega^s$

$$\Omega^s \underset{\mathbf{R}^s}{\overset{\mathbf{R}^{s\top}}{\rightleftarrows}} \Omega.$$

One-level abstract Schwarz preconditioner

Local solver for each $s = 1, \dots, N$

$\tilde{\mathbf{A}}^s \in \mathbb{R}^{n^s \times n^s}$ is an spsd matrix, and $\tilde{\mathbf{A}}^{s\,\dagger}$ is the pseudo-inverse of $\tilde{\mathbf{A}}^s$.

One-level preconditioner

$$\mathbf{H} := \sum_{s=1}^{N} \mathbf{R}^{s\top} \tilde{\mathbf{A}}^{s\,\dagger} \mathbf{R}^s.$$

Examples

- Additive Schwarz if $\tilde{\mathbf{A}}^s = \mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top}$,
- Inexact Schwarz if $\tilde{\mathbf{A}}^s =$ incomplete factorization$(\mathbf{R}^s \mathbf{A} \mathbf{R}^{s\top})$,
- Neumann Neumann if $\tilde{\mathbf{A}}^s$ is the *Neumann* matrix.

Two-level abstract Schwarz preconditioner for $\tilde{\mathbf{A}}$

Ingredients

- $\mathbf{H}$: one-level preconditioner,
- $\mathbf{R}^0$: coarse restriction operator (lines of $\mathbf{R}^0$: coarse basis V^0),
- $\mathbf{R}^0\tilde{\mathbf{A}}\mathbf{R}^{0\top}$: coarse problem.

Additive two-level preconditioner

$$\mathbf{H}_{\mathrm{ad}} := \mathbf{H} + \mathbf{R}^{0\top}(\mathbf{R}^0\tilde{\mathbf{A}}\mathbf{R}^{0\top})^{-1}\mathbf{R}^0,$$

Hybrid two-level preconditioner

$$\mathbf{H}_{\mathrm{hyb}} := \mathbf{\Pi}\mathbf{H}\mathbf{\Pi}^\top + \mathbf{R}^{0\top}(\mathbf{R}^0\tilde{\mathbf{A}}\mathbf{R}^{0\top})^{-1}\mathbf{R}^0, \quad \mathbf{\Pi} := \mathbf{I} - \mathbf{R}^{0\top}(\mathbf{R}^0\tilde{\mathbf{A}}\mathbf{R}^{0\top})^{-1}\mathbf{R}^0\tilde{\mathbf{A}}.$$

Remark: $\mathbf{\Pi}$ is an $\tilde{\mathbf{A}}$-orthogonal projection operator.

Abstract GenEO

Coarse Spaces of the GenEO family

- In each subdomain, 1 or 2 generalized eigenvalue problems.
- Eigenvectors corresponding to eigenvalues smaller than a chosen threshold τ form the basis for the coarse space V^0.
- Guaranteed convergence of PCG:

$$\lambda_{\min} < \lambda(\mathbf{H}_{\text{ad}}\mathbf{A}) < \lambda_{\max} \text{ with } \lambda_{\min} \text{ and } \lambda_{\max} \text{chosen by user.}$$

- If τ grows, larger coarse space, smaller condition number of prec. operator and faster convergence (in iterations).

It all relies on this Assumption

There exist $\mathbf{N}^s \in \mathbb{R}^{n^s \times n^s}$ and $\mathcal{N}'$ such that:

$$\sum_{s=1}^{N} \langle \mathbf{x}, \mathbf{R}^{s\top} \mathbf{N}^s \mathbf{R}^s \mathbf{x} \rangle \leqslant \mathcal{N}' \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle \quad \forall\, \mathbf{x} \in \mathbb{R}^n \text{ with } \mathbf{N}^s \text{ spsd } \forall\, s.$$

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$\rightarrow$ Do this algebraically ?

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New matrix $\mathbf{A}_+$ and its GenEO preconditioners

Let's relax the splitting assumption

Symmetric splitting of A

Assume that there exists a family of symmetric (possibly indefinite) matrices $\mathbf{B}^s \in \mathbb{R}^{n^s \times n^s}$ for $s = 1, \dots, N$ such that

$$\mathbf{A} = \sum_{s=1}^{N} \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s.$$

Possible choice of matrice $\mathbf{B}^s$

Let $\mathbf{B} \in \mathbb{R}^{n \times n}$ be defined by its entries

$$\mathbf{B}_{ij} := \begin{cases} \frac{A_{ij}}{\#\{s; \{i,j\} \subset \Omega^s\}} & \text{if } A_{ij} \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $s = 1, \dots, N$, let $\mathbf{B}^s := \mathbf{R}^s \mathbf{B} \mathbf{R}^{s\top}$.

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Then, for $s = 1, \dots, N$, let $\mathbf{B}^s := \mathbf{R}^s \mathbf{B} \mathbf{R}^{s\top}$. $\leftarrow$ Algebraic construction !

A splitting of the splitting

Recall that $\mathbf{A} = \sum_{s=1}^{N} \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s$.

1. Diagonalize $\mathbf{B}^s$ (for each $s = 1, \dots, N$)

$$\mathbf{B}^s = \mathbf{V}^s \mathbf{\Lambda}^s \mathbf{V}^{s\top}; \text{ with } \mathbf{V}^s \text{ orthogonal and } \mathbf{\Lambda}^s \text{diagonal.}$$

2. Separate positive and non-positive eigenvalues of $\mathbf{B}^s$

$$\mathbf{\Lambda}^s = \begin{pmatrix} \mathbf{\Lambda}_-^s & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_+^s \end{pmatrix}, \quad \mathbf{V}^s = \left[\mathbf{V}_-^s | \mathbf{V}_+^s\right], \quad \mathbf{\Lambda}_+^s \text{ is spd,} \quad -\mathbf{\Lambda}_-^s \text{ is spsd.}$$

3. For each $s = 1, \dots, N$, define (in $\mathbb{R}^{n^s \times n^s}$):

$$\mathbf{A}_+^s := \mathbf{V}_+^s \mathbf{\Lambda}_+^s \mathbf{V}_+^{s\top} \text{ and } \mathbf{A}_-^s := -\mathbf{V}_-^s \mathbf{\Lambda}_-^s \mathbf{V}_-^{s\top}.$$

By definition:

- $\mathbf{B}^s = \mathbf{A}_+^s - \mathbf{A}_-^s$,
- $\mathbf{A}_-^s$ is spsd,
- $\mathbf{A}_+^s$ is spsd.

Definition of $\mathbf{A}_+$

New global matrices $\mathbf{A}_+$ and $\mathbf{A}_-$

$$\mathbf{A}_+ := \sum_{s=1}^{N} \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{R}^s, \text{ and } \mathbf{A}_- := \sum_{s=1}^{N} \mathbf{R}^{s\top} \mathbf{A}_-^s \mathbf{R}^s.$$

Properties:

1. $\mathbf{A} = \mathbf{A}_+ - \mathbf{A}_-$
2. $\mathbf{A}_-$ is spsd
3. $\mathbf{A}_+$ is spd

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2. $\mathbf{A}_-$ is spsd $\quad$ (indeed: $\mathbf{A}_- = -\sum_{s=1}^{N} \mathbf{R}^{s\top} \mathbf{V}_-^s \mathbf{\Lambda}_-^s \mathbf{V}_-^{s\top} \mathbf{R}^s$).
3. $\mathbf{A}_+$ is spd $\quad$ (indeed: $\langle \mathbf{x}, \mathbf{A}_+ \mathbf{x} \rangle \geqslant \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle \; \forall \mathbf{x}$).

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3. $\mathbf{A}_+$ is spd (indeed: $\langle \mathbf{x}, \mathbf{A}_+ \mathbf{x} \rangle \geqslant \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle \, \forall \mathbf{x}$).

Remark: splitting of $\mathbf{A}_+$ with spsd local matrices

$$\sum_{s=1}^{N} \langle \mathbf{x}, \mathbf{R}^{s\top} \mathbf{N}^s \mathbf{R}^s \mathbf{x} \rangle \leqslant \mathcal{N}' \langle \mathbf{x}, \mathbf{A}_+ \mathbf{x} \rangle \text{ with } \mathcal{N}' = 1 \text{ and } \mathbf{N}^s = \mathbf{A}_+^s \text{ spsd.}$$

$\rightarrow$ The GenEO theory applies to $\mathbf{A}_+$!

Recap: Definition of $\mathbf{A}_+$

1. Algebraic splitting of $\mathbf{A}$ with symmetric matrices:

$$\mathbf{A} = \sum_{s=1}^{N} \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s.$$

2. For each $s = 1, \dots, N$, splitting of $\mathbf{B}^s$ into

$$\mathbf{B}^s = \mathbf{A}_+^s - \mathbf{A}_-^s \text{ with } \mathbf{A}_+^s \text{ and } \mathbf{A}_-^s \text{ spsd.}$$

3. Assemble $\mathbf{A}_+^s$ into a global matrix

$$\mathbf{A}_+ := \sum_{s=1}^{N} \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{R}^s.$$

$\mathbf{A}_+$ is spd and the GenEO theory applies

$\rightarrow$ Two-level preconditioners for $\mathbf{A}_+$ with adaptive spectral bounds.

One-level preconditioners for $\mathbf{A}_+ = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{A}_+^s \mathbf{R}^s$.

Definition

- Exact local Solvers:

$$\mathbf{H}_+^{\mathrm{AS}} := \sum_{s=1}^{N} \mathbf{R}^{s\top} (\mathbf{R}^s \mathbf{A}_+ \mathbf{R}^{s\top})^{-1} \mathbf{R}^s,$$

- *Neumann-Neumann*:

$$\mathbf{H}^{\mathrm{NN}} := \sum_{s=1}^{N} \mathbf{R}^{s\top} \mathbf{D}^s (\mathbf{A}_+^s)^{\dagger} \mathbf{D}^s \mathbf{R}^s,$$

  with $\mathbf{D}^s$ a partition of unity (*i.e.*, $\mathbf{I} = \sum_{s=1}^N \mathbf{R}^{s\top} \mathbf{D}^s \mathbf{R}^s$).
- Other choices possible…

Two-level preconditioners for $\mathbf{A}_+$ with GenEO

Let $\tau > 1$, the coarse space for $\mathbf{A}_+$ $\left(= \sum_{s=1}^N \mathbf{R}^{s\top}\mathbf{A}_+^s\mathbf{R}^s\right)$ is

$$V^0(\tau) := \sum_{s=1}^{N} \operatorname{span}\left\{ \mathbf{R}^{s\top}\mathbf{y}^s;\ \underbrace{(\lambda^s, \mathbf{y}^s)}_{\in\mathbb{R}^+\times\mathbb{R}^{n^s}} \text{ solution of (1) and } \lambda^s < \tau^{-1} \right\},$$

where the generalized eigenvalue problem is

$$(\mathbf{D}^s)^{-1}\mathbf{A}_+^s(\mathbf{D}^s)^{-1}\mathbf{y}^s = \lambda^s\mathbf{R}^s\mathbf{A}_+\mathbf{R}^{s\top}\mathbf{y}^s;\ \text{w/ } \mathbf{D}^s\text{: partition of unity.} \quad (1)$$

Spectral bounds with $\mathcal{N}(\mathbf{A}_+)$: coloring constant for $\mathbf{A}_+$

$$\begin{aligned} 1/\tau \leqslant \quad & \lambda(\mathbf{H}^{\mathrm{AS}}_{+,\mathrm{hyb}}\mathbf{A}_+) \quad \leqslant \mathcal{N}(\mathbf{A}_+), \\ 1/((1 + 2\mathcal{N}(\mathbf{A}_+))\tau) \leqslant \quad & \lambda(\mathbf{H}^{\mathrm{AS}}_{+,\mathrm{ad}}\mathbf{A}_+) \quad \leqslant \mathcal{N}(\mathbf{A}_+) + 1, \\ 1 \leqslant \quad & \lambda(\mathbf{H}^{\mathrm{NN}}_{\mathrm{hyb}}\mathbf{A}_+) \quad \leqslant \mathcal{N}(\mathbf{A}_+)\tau. \end{aligned}$$

AWG preconditioners

AWG (Algebraic Woodbury GenEO) as a solver

Theorem for $\mathbf{A}_- = \mathbf{A}_+ - \mathbf{A}$

$$n_- = \operatorname{rank}(\mathbf{A}_-) \leqslant \sum_{s=1}^{N} n^s - n$$

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Theorem for $\mathbf{A}_- = \mathbf{A}_+ - \mathbf{A}$

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Woodbury matrix identity for $\mathbf{A} = \mathbf{A}_+ - \mathbf{A}_-$

If $\mathbf{A}_- = \mathbf{V}_- \mathbf{\Lambda}_- \mathbf{V}_-^\top$ with $\mathbf{\Lambda}_-$ spd and $\mathbf{V}_-$ full-rank

$$\mathbf{A}^{-1} = \mathbf{A}_+^{-1} + \mathbf{A}_+^{-1}\mathbf{V}_- \underbrace{\left(\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1} \mathbf{V}_-\right)^{-1}} \mathbf{V}_-^\top \mathbf{A}_+^{-1},$$

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$$\mathbf{A}^{-1} = \mathbf{A}_+^{-1} + \mathbf{A}_+^{-1}\mathbf{V}_- \underbrace{\left(\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-\right)^{-1}}_{\text{at most } n_- \text{ iterations of PCG}} \mathbf{V}_-^\top \mathbf{A}_+^{-1},$$

$\rightarrow$ Possible to approximate $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ by solving at most $(n_- + 2)$ linear systems with $\mathbf{A}_+$ for which a good preconditioner is known.

First AWG preconditioner for $\mathbf{A}$

AWG preconditioner for A with inexact coarse space

Given a preconditioner $\mathbf{H}_2$ for $\mathbf{A}_+$ such that the eigenvalues of $\mathbf{H}_2\mathbf{A}_+$ are in the interval $[\lambda_{\min}(\mathbf{H}_2\mathbf{A}_+), \lambda_{\max}(\mathbf{H}_2\mathbf{A}_+)]$. Let

$$\mathbf{H}_{3,\text{inex}} := \mathbf{H}_2 + \mathbf{A}_+^{-1}\mathbf{V}_- \left(\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-\right)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}.$$

The eigenvalues of the new preconditioned operator satisfy

$$\min(1, \lambda_{\min}(\mathbf{H}_2\mathbf{A}_+)) \leqslant \lambda(\mathbf{H}_{3,\text{inex}}\mathbf{A}) \leqslant \max(1, \lambda_{\max}(\mathbf{H}_2\mathbf{A}_+)).$$

For the proof, recall that

$$\mathbf{A}^{-1} = \mathbf{A}_+^{-1} + \mathbf{A}_+^{-1}\mathbf{V}_- \left(\mathbf{\Lambda}_-^{-1} - \mathbf{V}_-^\top \mathbf{A}_+^{-1}\mathbf{V}_-\right)^{-1} \mathbf{V}_-^\top \mathbf{A}_+^{-1}.$$

Toward two more AWG preconditioners for A

Lemma:

$$\text{range}\left(\mathbf{A}_+^{-1}\mathbf{V}_-\right) = \text{range}\left(\mathbf{A}^{-1}\mathbf{V}_-\right),$$

since, by definition, $\mathbf{A} = \mathbf{A}_+ - \mathbf{V}_-\mathbf{\Lambda}_-\mathbf{V}_-^\top$.

Definition of an A-orthogonal projection operator:

$$\mathbf{\Pi}_3 := \mathbf{I} - \mathbf{W}(\mathbf{W}^\top\mathbf{A}\mathbf{W})^{-1}\mathbf{W}^\top\mathbf{A}, \text{ with } \text{range}(\mathbf{W}) = \text{range}(\mathbf{A}_+^{-1}\mathbf{V}_-).$$

Property:

$$\text{range}(\mathbf{\Pi}_3) = \text{Ker}\left(\mathbf{A}_-\right) \text{ so } \mathbf{A}\mathbf{\Pi}_3 = \mathbf{A}_+\mathbf{\Pi}_3.$$

Toward two more AWG preconditioners for $\mathbf{A}$

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"Corollary":

Good preconditioners for $\mathbf{A}$ once restricted to $\mathrm{range}(\mathbf{\Pi}_3)$ are known.

Additive and Hybrid AWG preconditioners

Definition

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$$\mathbf{H}_{3,\mathrm{ad}} := \mathbf{H}_2 + \mathbf{W}(\mathbf{W}^\top\mathbf{A}\mathbf{W})^{-1}\mathbf{W}^\top \text{ (Additive AWG preconditioner)},$$

$$\mathbf{H}_{3,\mathrm{hyb}} := \mathbf{\Pi}_3\mathbf{H}_2\mathbf{\Pi}_3^\top + \mathbf{W}(\mathbf{W}^\top\mathbf{A}\mathbf{W})^{-1}\mathbf{W}^\top \text{ (Hybrid AWG preconditioner)}.$$

Spectral bounds

$$\begin{aligned}
\min(1, \lambda_{\min}(\mathbf{H}_2\mathbf{A}_+)) \leqslant \quad & \lambda(\mathbf{H}_{3,\mathrm{hyb}}\mathbf{A}) \quad \leqslant \max(1, \lambda_{\max}(\mathbf{H}_2\mathbf{A}_+)),\\
\min(1, \lambda_{\min}(\mathbf{H}_2\mathbf{A}_+)) \leqslant \quad & \lambda(\mathbf{H}_{3,\mathrm{ad}}\mathbf{A}) \quad \leqslant (\lambda_{\max}(\mathbf{H}_2\mathbf{A}_+) + 1).
\end{aligned}$$

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\min(1, \lambda_{\min}(\mathbf{H}_2\mathbf{A}_+)) \leqslant \quad & \lambda(\mathbf{H}_{3,\mathrm{ad}}\mathbf{A}) \quad \leqslant (\lambda_{\max}(\mathbf{H}_2\mathbf{A}_+) + 1).
\end{aligned}$$

These are abstract DD preconditioners with two coarse spaces

- One in $\mathbf{H}_2$ computed by GenEO for $\mathbf{A}_+$. It includes range$(\mathbf{V}_-)$.
- The second is range$(\mathbf{W})$ = range$(\mathbf{A}_+^{-1}\mathbf{V}_-)$.

Some remarks on implementation

1. In the splitting of $\mathbf{B}^s$, only the non-positive eigenvalues $\mathbf{\Lambda}_-^s$ and eigenvectors $\mathbf{V}_-^s$ are computed.
2. $\mathbf{A}_- = \mathbf{V}_-\mathbf{\Lambda}_-\mathbf{V}_-^\top$ if:
   $\mathbf{V}_- = [\mathbf{R}^{1\top}\mathbf{V}_-^1 \mid \dots \mid \mathbf{R}^{N\top}\mathbf{V}_-^N]$; $\mathbf{\Lambda}_- = \operatorname{diag}(\mathbf{\Lambda}_-^1, \dots \mathbf{\Lambda}_-^N)$.
3. The computation of $\mathbf{A}_+^{-1}\mathbf{V}_-$ is one of the bottlenecks of the algorithm. Currently solved with PCG for each column of $\mathbf{V}_-$.
4. $$\mathbf{A}_+^s = \mathbf{B}^s - \mathbf{V}_-^s\mathbf{\Lambda}_-^s\mathbf{V}_-^{s\,\top} = (\mathbf{I} - \mathbf{V}_-^s\mathbf{V}_-^{s\,\top})\mathbf{B}^s(\mathbf{I} - \mathbf{V}_-^s\mathbf{V}_-^{s\,\top})$$
   which also implies that
   $$\mathbf{A}_+^{s\,\dagger} = (\mathbf{I} - \mathbf{V}_-^s\mathbf{V}_-^{s\,\top})\mathbf{B}^{s\dagger}(\mathbf{I} - \mathbf{V}_-^s\mathbf{V}_-^{s\,\top}).$$
   Since $\mathbf{B}^s$ is symmetric, it can be factorized using MUMPS.
5. $$\mathbf{R}^s\mathbf{A}_+\mathbf{R}^{s\top} = \mathbf{R}^s\mathbf{A}\mathbf{R}^{s\top} + \mathbf{R}^s\mathbf{A}_-\mathbf{R}^{s\top},$$
   And $(\mathbf{R}^s\mathbf{A}_+\mathbf{R}^{s\top})^{-1}$ is computed by the Woodbury matrix identity.

Numerical Results

Our code (collaboration with Loïc Gouarin)

- Available on Github:

  https://github.com/gouarin/GenEO

- Implemented in petsc4py, a Python port to the PETSc libraries

L. D. Dalcin, R. R. Paz, P. A. Kler, and A. Cosimo.
Parallel distributed computing using python.
Advances in Water Resources, 34(9):1124–1139, 2011.

S. Balay, S. Abhyankar, M. F. Adams, J. Brown, P. Brune, K. Buschelman, L. Dalcin, A. Dener, V. Eijkhout, W. D. Gropp, D. Karpeyev, D. Kaushik, M. G. Knepley, D. A. May, L. C. McInnes, R. T. Mills, T. Munson, K. Rupp, P. Sanan, B. F. Smith, S. Zampini, H. Zhang, and H. Zhang.
PETSc users manual.
Technical Report ANL-95/11 - Revision 3.15, Argonne National Laboratory, 2021.

- Eigensolves with SLEPc

V. Hernandez, J. E. Roman, and V. Vidal.
SLEPc: A scalable and flexible toolkit for the solution of eigenvalue problems.
ACM Trans. Math. Software, 31(3):351–362, 2005.

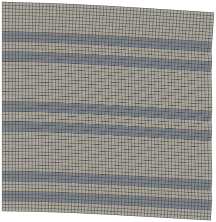
- Matrix factorizations (local and coarse problems) performed by MUMPS

P. Amestoy, A. Buttari, J.-Y. L'Excellent, and T. Mary.
Performance and Scalability of the Block Low-Rank Multifrontal Factorization on Multicore Architectures.
ACM Transactions on Mathematical Software, 45:2:1–2:26, 2019.

P. Amestoy, I. S. Duff, J. Koster, and J.-Y. L'Excellent.
A fully asynchronous multifrontal solver using distributed dynamic scheduling.
SIAM Journal on Matrix Analysis and Applications, 23(1):15–41, 2001.

Numerical Results (1/9): Testcase

- Two-dimensional linear elasticity on $\omega = [0,3] \times [0,3]$,
- Poisson's ratio $\nu = 0.3$, Young's modulus: see Figure,
- discretized by Q_1 finite elements,
- regular mesh of size $h = 1/21$ so that $n = 8064$,
- Partition into $N = 9$ unit squares, overlap only on boundaries,
- PCG solves (for $\mathbf{A}$ and $\mathbf{A}_+$) up to rel. residual tolerance of 10^{-10}.
- In GenEO $\tau = 10$.



Dark: $E = E_1 = 10^{11}$ (harder material)
Light : $E = E_2 = 10^7$ (softer material).

Numerical Results (2/9):

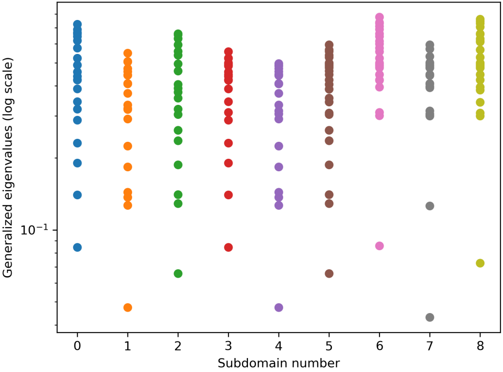
κ: condition number of preconditioned operator; *It*: iteration count; $\#V_0$: dim of GenEO coarse space; n_-: rank of $\mathbf{A}_-$.

	κ	*It*	$\#V_0$	n_-
New AWG preconditioners:				
$\mathbf{H}_{3,\mathrm{ad}}$ with $\mathbf{H}_2 = \mathbf{H}^{\mathrm{NN}}_{\mathrm{hyb}}$	9.09	26	57	48
$\mathbf{H}_{3,\mathrm{ad}}$ with $\mathbf{H}_2 = \mathbf{H}^{\mathrm{AS}}_{+,\mathrm{hyb}}$	12.3	25	57	48
$\mathbf{H}_{3,\mathrm{ad}}$ with $\mathbf{H}_2 = \mathbf{H}^{\mathrm{AS}}_{+,\mathrm{ad}}$	16.8	31	57	48
$\mathbf{H}_{3,\mathrm{hyb}}$ with $\mathbf{H}_2 = \mathbf{H}^{\mathrm{NN}}_{\mathrm{hyb}}$	9.09	27	57	48
$\mathbf{H}_{3,\mathrm{hyb}}$ with $\mathbf{H}_2 = \mathbf{H}^{\mathrm{AS}}_{+,\mathrm{hyb}}$	12.2	25	57	48
$\mathbf{H}_{3,\mathrm{hyb}}$ with $\mathbf{H}_2 = \mathbf{H}^{\mathrm{AS}}_{+,\mathrm{ad}}$	16.7	29	57	48
Non-algebraic methods:				
Hybrid AS + GenEO ($\tau = 10$)	26.5	43	55	0
Additive AS + GenEO ($\tau = 10$)	50.0	58	55	0
BNN with GenEO ($\tau = 0.1$)	11.1	29	55	0
One-level AS	34772	> 150	0	0

$$\mathbf{H}^{\mathrm{AS}}_{+} := \sum_{s=1}^{N} \mathbf{R}^{s\top}(\mathbf{R}^s\mathbf{A}_+\mathbf{R}^{s\top})^{-1}\mathbf{R}^s;\ \mathbf{H}^{\mathrm{NN}} := \sum_{s=1}^{N} \mathbf{R}^{s\top}\mathbf{D}^s(\mathbf{A}^s_+)^{\dagger}\mathbf{D}^s\mathbf{R}^s,$$

$$\mathbf{H}_{3,\mathrm{ad}} := \mathbf{H}_2 + \mathbf{W}(\mathbf{W}^\top\mathbf{A}\mathbf{W})^{-1}\mathbf{W}^\top;\ \mathbf{H}_{3,\mathrm{hyb}} := \mathbf{\Pi}_3\mathbf{H}_2\mathbf{\Pi}_3^\top + \mathbf{W}(\mathbf{W}^\top\mathbf{A}\mathbf{W})^{-1}\mathbf{W}^\top.$$

Numerical Results (3/9) - Spectrum of GenEO gevp

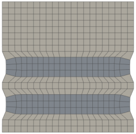


Twenty smallest non-zero eigenvalues for GenEO gevp

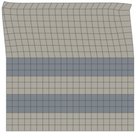
$$(\mathbf{D}^s)^{-1}\mathbf{A}_+^s(\mathbf{D}^s)^{-1}\mathbf{y}^s = \lambda^s \mathbf{R}^s\mathbf{A}_+\mathbf{R}^{s\top}\mathbf{y}^s; \text{ for } \mathbf{D}^s \text{ partition of unity.}$$

$\rightarrow$ There is a gap.

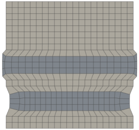
Numerical Results (4/9) – GenEO eigenvectors



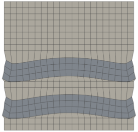
$(\lambda_-^s)_0 = -1073375689.03$



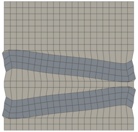
$(\lambda_-^s)_1 = -1073375684.54$



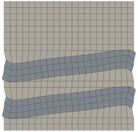
$(\lambda_-^s)_2 = -837860641.49$



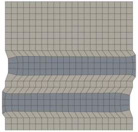
$(\lambda_-^s)_3 = -837859551.56$



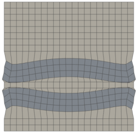
$(\lambda_-^s)_4 = -801312797.80$



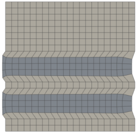
$(\lambda_-^s)_5 = -801311586.34$



$(\lambda_-^s)_6 = -121545449.39$



$(\lambda_-^s)_7 = -121544161.49$



$(\lambda_\sharp^s)_8 = 0.0473$

Numerical Results (5/9) – Influence of ν

From now on the preconditioner is $\mathbf{H}_{3,\text{ad}}$ with $\mathbf{H}_2 = \mathbf{H}_{\text{hyb}}^{\text{NN}}$ and $\tau = 10$.

AWG					Classical GenEO (not algebraic)				
ν	κ	*It*	$\#V_0$	n_-	ν	κ	*It*	$\#V_0$	n_-
0.20	19.7	33	21	12	0.20	17.2	33	21	0
0.30	20.3	32	29	19	0.30	17.6	36	21	0
0.35	18.6	32	47	25	0.35	19.1	37	21	0
0.40	25.8	39	98	70	0.40	20.1	39	24	0
0.45	27.1	29	115	110	0.45	33.7	46	28	0
0.49	16.8	25	362	357	0.49	34.9	51	94	0

- ν: Poisson's ratio,
- κ: condition number of preconditioned operator,
- *It*: number of iterations,
- $\#V_0$: dimension of GenEO coarse space,
- $n_- = \text{rank}(\mathbf{A}_-)$: dimension of second coarse space.

Numerical Results (6/9) – Influence of E

	AWG				Classical GenEO			
(E_1, E_2)	κ	It	$\#V_0$	n_-	κ	It	$\#V_0$	n_-
$(10^5, 10^{11})$	10.8	22	95	75	8.6	23	90	0
$(10^7, 10^{11})$	10.8	23	95	75	8.6	26	87	0
$(10^9, 10^{11})$	10.4	24	94	73	8.5	25	85	0
$(10^{11}, 10^{11})$	12.2	29	35	19	13.7	32	28	0
$(10^{11}, 10^9)$	8.0	26	59	48	11.2	30	52	0
$(10^{11}, 10^7)$	9.0	26	57	48	11.1	29	55	0
$(10^{11}, 10^5)$	8.4	29	57	48	12.7	30	55	0

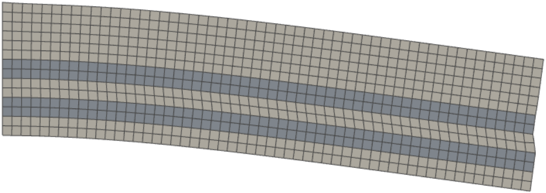
- E_1: Young's modulus in the dark layers, E_2 elsewhere,
- κ: condition number of preconditioned operator,
- It: number of iterations,
- $\#V_0$: dimension of GenEO coarse space,
- $n_- = \text{rank}(\mathbf{A}_-)$: dimension of second coarse space.

Numerical Results (7/9) - influence of *rtol*

	$\nu = 0.3$				$\nu = 0.4$			
rtol	κ	*It*	$\#V_0$	n_-	κ	*It*	$\#V_0$	n_-
10^{-10}	9.0	26	57	48	9.4	29	100	74
10^{-2}	9.0	27	57	48	9.4	30	100	74
0.05	11.1	31	57	48	12.0	33	100	74
0.1	12.2	32	57	48	17.4	36	100	74
0.5	400.8	40	57	48	1563.3	88	100	74
0.9	706.8	64	57	48	2142.1	100	100	74

- *rtol*: relative residual accuracy up to which the linear systems with $\mathbf{A}_+$ preconditioned by $\mathbf{H}_2$ are solved during the setup of the second coarse basis $\mathbf{W}$.
- κ: condition number of preconditioned operator,
- *It*: number of iterations,
- $\#V_0$: dimension of GenEO coarse space,
- $n_- = \operatorname{rank}(\mathbf{A}_-)$: dimension of second coarse space.

Numerical Results (8/9) – Weak scalability Testcase



- $N \in \{2; 4; 8; 15; 29\}$ (number of unit-square subdomains),
- $\omega = [N, 1]$,
- $h = 1/14$ (mesh size),
- $\nu = 0.3$ (Poisson's ratio),
- $E = \begin{cases} E_1 = 10^{11} \text{ if } y \in [1/7; 2/7] \cup [3/7; 4/7], \\ E_2 = 10^7 \text{ otherwise,} \end{cases}$
- $\mathbf{H}_{3,\mathrm{ad}}$ with $\mathbf{H}_2 = \mathbf{H}_{\mathrm{hyb}}^{\mathrm{NN}}$ and $\tau = 10$,
- $rtol = 10^{-10}$ (relative residual tolerance for the linear solves with $\mathbf{A}_+$ and the linear solve with $\mathbf{A}$).

Numerical Results (9/9) – Weak scalability results

AWG					Classical GenEO				
N	κ	It	$\#V_0$	n_-	N	κ	It	$\#V_0$	n_-
2	12.6	15	8	8	2	9.5	15	7	0
4	9.8	16	26	20	4	11.9	19	19	0
8	9.0	15	62	44	8	12.6	23	43	0
15	8.8	15	125	86	15	12.8	27	85	0
29	8.7	17	251	170	29	12.8	28	169	0

- N: number of subdomains (also proportional to problem size),
- κ: condition number of preconditioned operator,
- It: number of iterations,
- $\#V_0$: dimension of GenEO coarse space,
- $n_- = \mathrm{rank}(\mathbf{A}_-)$: dimension of second coarse space.

Conclusion – we introduced:

- a new algebraic splitting $\mathbf{A} = \sum_{s=1}^{N} \mathbf{R}^{s\top} \mathbf{B}^s \mathbf{R}^s$ ($\mathbf{B}^s$ symmetric).
- $\mathbf{A}_+$ and its preconditioners with adaptive spectral bounds,
- a new formula for $\mathbf{A}^{-1}$ based on the Woodbury matrix identity applied to $\mathbf{A}$ viewed as a modification of $\mathbf{A}_+$,
- new fully algebraic preconditioners for $\mathbf{A}$ with adaptive spectral bounds and two coarse spaces (one rather expensive).

N. Spillane.
Toward a new fully algebraic preconditioner for symmetric positive definite problems.
https://hal.archives-ouvertes.fr/hal-03187092, 2021.

L. Gouarin and N. Spillane.
Fully algebraic domain decomposition preconditioners with adaptive spectral bounds.
https://hal.archives-ouvertes.fr/hal-03258644, 2021.