Introduction to BPS Spectra and q-series Invariants.

$T[M_3]$ theories

- $T[M_3]$ theories are compactification of the 6d (0,2) theory, the world-volume theory of M5 branes, on M_3 (with a topological twist along M_3).

$$\mathbb{R} \times \Sigma \times M_3$$

- The 6d theory contains 5 scalars, 4 chiral fermions and a 2-form gauge field with a self dual field strength.

- This theory has $SO(5)_R$ ($Spin(5)$) R symmetry.

$$SO(5)_R \supset SO(3)_R \times SO(2)_R$$

We identify $SO(3)_R$ with local rotations $SO(3)$ of M_3. This is called a (partial) topological twist.

- After such a twist the 6d theory becomes independent of the metric on M_3.

- 11d perspective

Space time	$\mathbb{R}^5$	$\times\ CY_3$
N M5 branes	$\mathbb{R}^3$	$\times\ M_3$

M_3 is a Lagrangian in CY_3 such that, locally near M_3, CY_3 looks like $T^* M_3$.

$T[M_3]$ is the low energy effective theory living on $\mathbb{R}^3$

- $T[M_3]$ is generically a strongly coupled theory.

- Examples of $T[M_3]$
- $T[S^3]$
  - We can give a UV description of $T[S^3]$
  - $T[S^3, U(N)]$ is the IR of 3d $\mathcal{N}=2$ Chern Simons theory at level 1 with one chiral multiplet in adjoint with R-charge $R(\Phi)=2$.
  - This theory has $U(1)$ flavour symmetry. $\Phi \longrightarrow e^{i\alpha}\Phi$.

- $T[L(p,1)]$ $L(p,1)$ are Lens spaces
  - $T[L(p,1), U(N)]$ is the IR of 3d $\mathcal{N}=2$ Chern Simons theory at level p with one chiral multiplet in adjoint with R-charge $R(\Phi)=2$.
  - This theory has $U(1)$ flavour symmetry. $\Phi \longrightarrow e^{i\alpha}\Phi$.

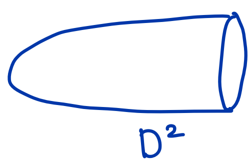
- We don't know any Lagrangian description of $T[M_3]$ for other 3-Manifolds.

Can we say anything about $T[M_3]$ for other 3-Manifolds?

$\rightarrow$ We can write down half indices of $T[M_3]$

- Half index of $T[M_3]$.

Half index of a 3d $\mathcal{N}=2$ theory is a supersymmetric partition function on $S^1 \times D^2$, with Cigar geometry on D^2.



$\times$ S^1

$S^1 \times D^2$ has a boundary $\partial S^1 \times D^2 = T^2$. Therefore, to write down the partition function on $S^1 \times D^2$, we have to specify a boundary condition that preserves part of supersymmetry. A boundary condition can be a 2d $\mathcal{N}=(0,2)$ theory.

The half index labeled by boundary condition is defined as follows,

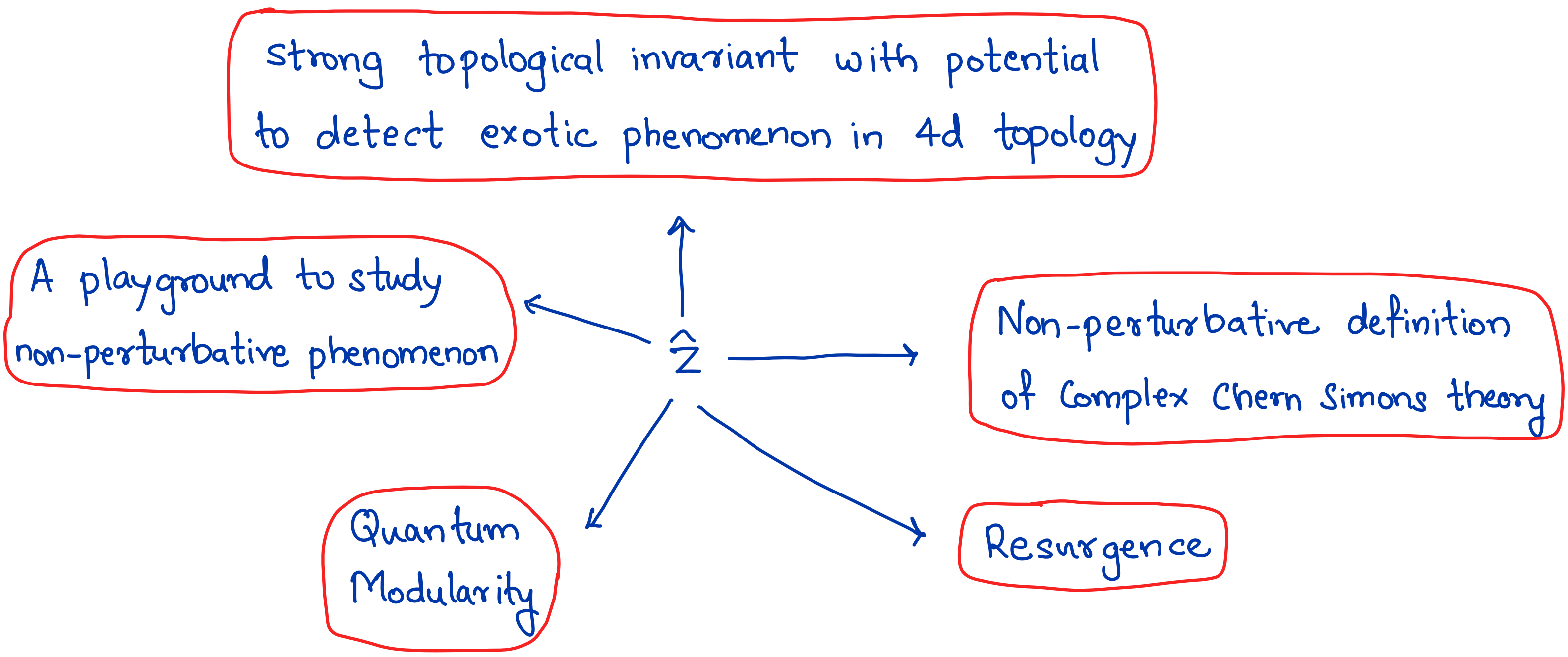
$$Z(S^1 \times_q D^2, B) := \mathrm{Tr}_{\mathcal{H}_{D^2,B}}\left[(-1)^F e^{-\beta H} q^{\frac{\Delta+J_3}{2}}\right]$$

Only states with $\Delta - R - J_3 = \{Q^\dagger, Q\} = H = 0$ contribute,

$$Z(S^1 \times_q D^2, B) = \mathrm{Tr}_{\mathcal{H}_{D^2,B}}\left[(-1)^F q^{\Delta - R/2}\right] = \sum_n a_{n,B}\, q^n ,$$

where $\mathcal{H}_{D^2,B}$ is the Hilbert space associated to D^2 with boundary condition B in the 3d $N=2$ theory under consideration.

- $a_{n,B}$ is a signed count of number of BPS states with boundary condition B and $J_3 + R/2 = \Delta - R/2 = n$.
- $\hat{Z}_b(M_3, G; q)$ is the normalized half index of $T[M_3, G]$ with some special choice of boundary conditions b.
- Connection between $\hat{Z}$ and different worlds of Mathematical Physics.



- Half index of $T[L(p,1), U(N)]$

$$Z_{T[L(p,1)]}(S^1 \times_q D^2, a; q)$$

$$= \frac{1}{W_a} \int_{|z_j|=1} \prod_{i=1}^{N} \frac{dz_i}{2\pi i z_i} \prod_{i\neq j} (1 - z_i/z_j) \prod_{i,j=1}^{N} \frac{(q z_i/z_j, q)_\infty}{(q^t z_i/z_j, q)_\infty} \frac{\Theta^N_{a,p}(z,q)}{(q,q)^N_\infty}$$

Where,

$$(x,q)_k = \prod_{i=1}^{k}(1-xq^{i-1}) \quad \text{is called the } q\text{-Pochhammer symbol}$$

$$\Theta^N_{a,p}(z,q) = \sum_{n\in p\mathbb{Z}^N+a} q^{\frac{n^2}{2p}} \prod_{i=1}^{N} z_i^{n_i} \qquad \text{Rank } N \text{ lattice theta}$$

$$t \longrightarrow \text{Fugacity for Flavour symmetry}$$

$$\prod_{i\neq j}(1-z_i/z_j) \longrightarrow \text{Vandermonde determinant}$$

$$\prod_{i,j=1}^{N}(qz_i/z_j,q)_\infty \longrightarrow \text{Fermion from gauge multiplet}$$

$$\prod_{i,j=1}^{N}\frac{1}{(qt\, z_i/z_j,q)_\infty} \longrightarrow \text{Scalar from chiral multiplet}$$

$$\frac{\Theta^N_{a,p}(z,q)}{(q,q)^N_\infty} \longrightarrow \text{Boundary } \mathcal{N}=(0,2) \text{ theory.}$$

Normalized $\hat{Z}$ for $SU(2)$ (with $t\to 1$)

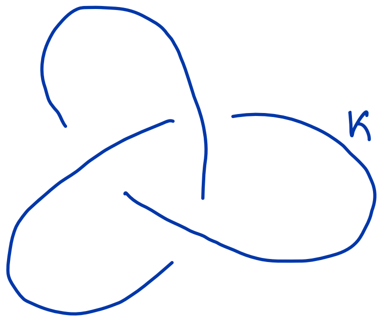
$$\hat{Z}_a(L(p,1)) = \oint_{|z|=1}\frac{dz}{2\pi i z}(z-z^{-1})^2 \sum_{n\in b+2p\mathbb{Z}} q^{\frac{n^2}{4p}} z^n$$

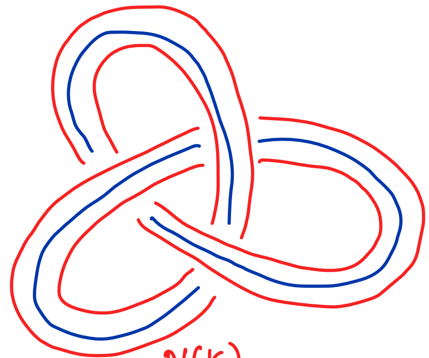
What about other 3-Manifolds? We can write down $\hat{Z}$ for negative definite plumbed 3-Manifolds.

What are plumbed 3-Manifolds?

Before we look at plumbed 3-Manifolds let's quickly review Dehn Surgery.

Suppose we have a knot K in some 3-manifold M. Excavate the tubular neighbourhood of K, $N(K)$, from M. $(M \setminus N(K))$





N(K)

$N(K)$ is topologically a solid torus $N(K) \simeq S^1 \times D^2$, the boundary of $S^1 \times D^2$ and $M \setminus N(K)$ is T^2. We can glue back $S^1 \times D^2$ to $M \setminus N(K)$ using an element of $MCG(T^2)$ $(SL(2,\mathbb{Z}))$ to get a new 3-manifold. Non-trivial elements of $SL(2,\mathbb{Z})$ are diffeomorphisms of T^2 not connected to identity.

$$SL(2,\mathbb{Z}) \ni \varphi = \begin{pmatrix} p & s \\ r & t \end{pmatrix}$$

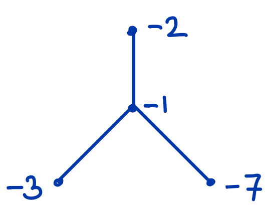
p/r is called the surgery slope or surgery coefficient

This process of getting new 3-manifolds from knots or links is called Dehn Surgery.

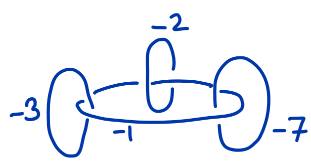
Plumbed manifold

Given a graph tree Γ with vertices labeled by integers we can get a 3-manifold as follows:

Put an unknot on each vertex of graph. Two unknots are linked if the associated vertices share an edge between them.



$\longrightarrow$



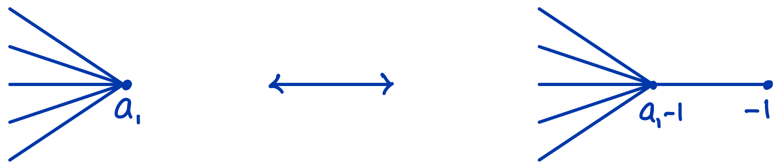
The integer label of a vertex gives us the surgery coefficient on that unknot.

3-manifold obtained by performing Dehn surgery on such linked unknots are called plumbed manifold.

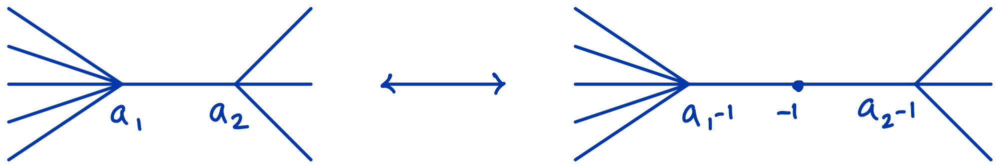
Neumann moves

Two plumbed manifolds obtained from plumbing graphs Γ_1 and Γ_2 are same if Γ_1 and Γ_2 are related by the following moves.

1)



2)



- Since $\hat{Z}$ is a topological invariant of a 3-manifold, it should be invariant under Neumann moves.

- Using invariance of $\hat{Z}$ under Neumann moves, we can bootstrap a formula of $\hat{Z}$ for negative definite plumbed manifolds.

$$\hat{Z}_b(M_3(\Gamma), q) = q^{f} \oint \left(\prod_{v \in V} \frac{dz_v}{2\pi i z_v} \, (z_v - z_v^{-1})^{2-\deg(v)} \right) \sum_{n \in b + 2Q\mathbb{Z}^V} q^{-\frac{n^T Q^{-1} n}{4}} \prod_{i=1}^{V} z_v^{n_v},$$

where Q is the adjacency matrix of Γ, $b_v \in \deg(v) + 2\mathbb{Z}^V/2Q\mathbb{Z}^V$, that is $b \in \mathrm{Spin}^c(M_3(\Gamma))$.

$\hat{Z}$ for $L(p,1)$

$$\hat{Z}_a(L(p,1); q) = \begin{cases} q^{\frac{p^2-3p+4}{4p}} & 2a = \pm 2 \mod 2p \\ -q^{\frac{p-3}{4}} & 2a = 0 \mod 2p \\ 0 & \text{otherwise.} \end{cases}$$

BPS cohomology of $T[M_3]$

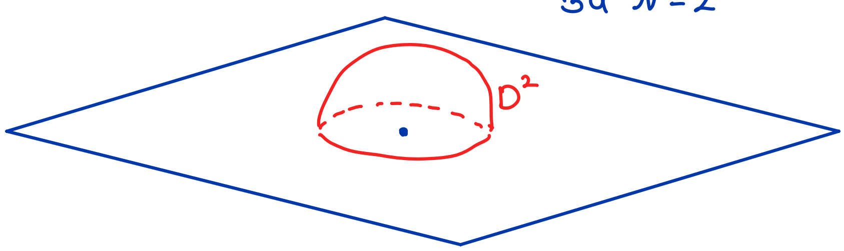
We will now look at the BPS cohomology of $T[M_3]$.

What is a BPS cohomology or a Q-cohomology?

Just like the exterior derivative d squares to zero ($d^2=0$), supersymmetric charges square to zero $Q^2=0$. Therefore, we can use Q to construct a cohomology.

$$\text{BPS cohomology} = \frac{Q\text{-closed local operators}}{Q\text{-exact local operators}}$$

If you have 3d-2d set up we can consider another kind of BPS cohomology



$$\mathcal{H}_B^{BPS} = \frac{Q\text{-closed local operators at boundary}}{Q\text{-exact local operators at boundary}}$$

How is this related to $\hat{Z}(q)$ or half-index?

$\longrightarrow$ Half-index $Z(S^1 \times_q D^2, B)$ is the graded Euler characteristic of $\mathcal{H}_B^{BPS}$

$$Z(S^1 \times_q D^2, B) = \sum_{n,j} (-1)^j q^n \dim \mathcal{H}^{BPS}_{B,n,j}$$

The BPS cohomologies of $T[M_3]$ gives us a cohomology valued topological invariant.

$$M_3 \longrightarrow T[M_3] \longrightarrow \mathcal{H}_B^{BPS}(M_3).$$

Let's look at a concrete example of $\mathcal{H}_{B}^{BPS}(S^3)$.

- Recall $T[S^3, SU(2)]$ is the IR of 3d $\mathcal{N}=2$ Chern Simons theory at level 1 with one chiral multiplet in adjoint with R-charge $R(\Phi)=2$.

- Jafferis-Yin duality.

$$\begin{array}{c}\text{3d } \mathcal{N}=2 \text{ Chern Simons theory at level 1} \\ \text{with one chiral multiplet in adjoint} \\ \updownarrow \\ \text{Free chiral} \quad \left(\phi = \mathrm{Tr}(\varphi^2)\right)\end{array}$$

The free chiral has R-charge 4, and flavour charge 2

$$\varphi \to e^{i\alpha}\varphi \Rightarrow \phi \to e^{2i\alpha}\phi.$$

The supersymmetric transformation of free chiral

Integrating out F

$$\delta\phi = \sqrt{2}\,\epsilon\Psi \qquad\qquad \delta\Psi = \sqrt{2}\,i\sigma\tilde{\epsilon}\phi - \sqrt{2}\,i\gamma^{\mu}\tilde{\epsilon}D_{\mu}\phi$$

The 3d N=2 SUSY algebra has 4 super charges $\{Q_\pm, \tilde{Q}_\pm\}$. We will look at the $\tilde{Q}_-$ cohomology.

The action of $\tilde{Q}_-$ on ϕ, Ψ is

$$[\tilde{Q}_-, \phi] = 0 \qquad \{\tilde{Q}_-, \tilde{\Psi}_\pm\} = 0$$

$$[\tilde{Q}_-, \tilde{\phi}] = \sqrt{2}\tilde{\Psi}_- \qquad \{\tilde{Q}_-, \Psi_+\} = -\sqrt{2}\,i\,\partial_0\phi \qquad \{\tilde{Q}_-, \Psi_-\} = \sqrt{2}\,i\,\partial_+\phi$$

where $\partial_+ = \partial_1 + i\partial_2 \qquad \partial_- = \partial_1 - i\partial_2$.

The equation of motion gives us

$$(-\partial_0^2 + \partial_+\partial_-)\phi = 0 \qquad \partial_+\tilde{\Psi}_+ = -\partial_0\tilde{\Psi}_- \qquad \partial_0\tilde{\Psi}_+ = \partial_-\tilde{\Psi}_-$$

Boundary condition $\tilde{\Psi}_+|_\partial = 0$

Local operators on boundary is generated by

$\phi, \tilde{\Phi}, \Psi_+, \Psi_-, \tilde{\Psi}_+, \tilde{\Psi}_-$ and their derivatives.

$\tilde{Q}_-$ closed local operators on boundary is generated by ϕ, $\tilde{\Psi}_-$ and derivatives.

$\tilde{Q}_-$ exact local operators on boundary is generated by $\partial_0\phi$, $\partial_+\phi$, $\tilde{\Psi}_-$ and derivatives

$\mathcal{H}_B^{BPS}(S^3)$ generated by $\partial_-^n\phi$

$$\mathcal{H}_B^{BPS}(S^3) = \bigotimes_{j=1}^{\infty} \mathbb{C}[\partial_-^j\phi]$$

$$\begin{array}{lll} \text{R-charge of } \partial_-^n\phi & = 4 & \\ J_3 + R/2 \text{ of } \partial_-^n\phi & = 2+n & \longrightarrow \dfrac{1}{(1-q^{2+n}z^4)} \end{array}$$

$$\text{Graded Euler characteristic} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{2+n}z^4)} = \frac{1}{(q^2z^4, q)_\infty}$$

Putting $z \to 1$ we get the Half index of $T[S^3]$.

Finding H^{BPS} for other manifold is an important problem and solving it might lead to fascinating results in Low-dimensional topology.

Can we say anything else about $\mathcal{H}^{BPS}$ for other 3-manifolds?

→ We can make a qualitative statement about the size of $\mathcal{H}^{BPS}$.

$$\dim \mathcal{H}^{BPS}_{B,n} \sim e^{\sqrt{cn}}$$

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c_{eff} for 3d $N=2$ theories

An important tool in understanding, space of quantum field theories or RG flows beyond perturbative regime is a measure of degrees of freedom.

Such measures are often called c-functions.

We have c-function in 2d which has been studied extensively and the a-function in 4d.

We propose a measure of BPS degrees of freedom in 3d $N=2$ theories.

The growth of supersymmetric (BPS) states in 3d $N=2$ theories is given by the following asymptotic formula

$$a_n \sim \mathrm{Re}\left[\exp\left(\sqrt{\frac{2\pi^2}{3} c_{eff}\, n} + 2\pi i r n\right)\right]$$

where $r \in \mathbb{Q}/\mathbb{Z}$ and $c_{eff} \in \mathbb{C}$.

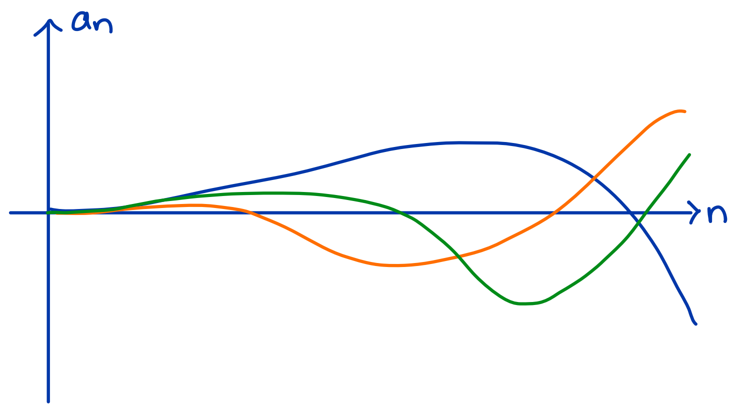
$$Z_B(S^1 \times_q D^2, q) = \sum_n a_{n,B}\, q^n \quad \text{or} \quad Z(S^1 \times_q S^2, q) = \sum_n a_n q^n$$

When $r=0$ and $c_{eff} \in \mathbb{R}_+$, it is exactly the same as Cardy formula.

When $r \neq 0$ or when $c_{eff} \notin \mathbb{R}_+$ the above formula captures two curious features of density of BPS states that we call branching and oscillations.

When $r = \frac{l}{k} \neq 0$ then BPS states organize themselves into k branches

When $\mathrm{Im}(\sqrt{c_{eff}}) \neq 0$ then the density of BPS states oscillates.



Supersymmetric partition functions have the following asymptotic behaviour near roots of unity $q = e^{-2\pi i r}$ $r \in \mathbb{Q}/\mathbb{Z}$

$$Z(q) \sim \exp\left[-\frac{1}{\hbar_r}\left(\widetilde{W}_r^{(0)} + \hbar_r \widetilde{W}_r^{(1)} + \hbar_r^2 W_r^{(2)} + \cdots\right)\right]$$

$\hbar_r = 2\pi i(\tau + r)$

Using this asymptotic behaviour of SUSY partition functions we can derive the asymptotic formula for a_n.

By Cauchy's theorem

$$a_n = \oint \frac{dq}{2\pi i q} q^{-n} Z(q) .$$

Near unit circle $Z(q)$ vanishes or blows up (has a singularity).

Suppose $Z(q)$ has a dominant singularity near $q = e^{-2\pi i r}$

Then

$$a_n \sim \int_{-r-\frac{1}{2}+i\epsilon}^{-r+\frac{1}{2}+i\epsilon} d\tau \, \exp\left[-\frac{1}{\hbar_r}\left(\widetilde{W}_r^{(0)} + \hbar_r\left(\widetilde{W}_r^{(1)} - 2\pi i r n\right) + \hbar_r^2\left(\widetilde{W}_r^{(2)} - n\right)\right)\right]$$

Saddle point approximation gives

$$\exp\left(\pm 2i\sqrt{\widetilde{W}_r^{(0)} n - \widetilde{W}_r^{(0)}\widetilde{W}_r^{(2)}} + 2\pi i r n - \widetilde{W}_r^{(1)}\right)$$

Since a_n are real, we expect an equally dominant singularity near $q = e^{2\pi i r}$, which has same real contribution but opposite imaginary contribution.

$$a_n \sim \mathrm{Re}\left[\exp\left(\sqrt{\frac{2\pi^2}{3}\left(-\frac{6}{\pi^2}\widetilde{W}_r^{(0)}\right)n} + 2\pi i r n\right)\right]$$

Example:

Chiral multiplet with superpotential $W = \Phi^3$

$$I(q) = \frac{(q^2, q^3)_\infty}{(q, q^3)_\infty}$$

It has dominant singularity at $q = e^{\pm\frac{2\pi i}{3}}$

$$I(q) \underset{q \to e^{\pm\frac{2\pi i}{3}}}{\sim} \exp\left(-\frac{\mathrm{Li}_2(e^{\pm\frac{2\pi i}{3}}) - \mathrm{Li}_2(e^{\pm\frac{4\pi i}{3}})}{3\hbar_{\pm\frac{1}{3}}}\right)$$

$$a_n \sim \mathrm{Re}\left[\exp\left(\sqrt{\frac{4}{3}\left(\mathrm{Li}_2(e^{\frac{2\pi i}{3}}) - \mathrm{Li}_2(e^{-\frac{2\pi i}{3}})\right)n} + \frac{2\pi i n}{3}\right)\right]$$

$$c_{\text{eff}} = \frac{2}{\pi^2}\left(\mathrm{Li}_2(e^{\frac{2\pi i}{3}}) - \mathrm{Li}_2(e^{-\frac{2\pi i}{3}})\right) \approx 0.274227\, i$$

Purely imaginary!

- c_{eff} of Half index of $T[M_3]$.

For M_3 with $H_1(M_3, \mathbb{Z}) = 0$,

$$\hat{Z}(q) = \sqrt{\frac{\pi i}{\hbar}} \sum_{\alpha \in \mathcal{M}_{flat}} \sum_{m \in \mathbb{Z}} n_{(\alpha,m),0} \, e^{-\frac{4\pi^2}{\hbar}(CS(\alpha)+m)} \, Z_\alpha^{Pert}(\hbar)$$

$\mathcal{M}_{flat} \longrightarrow$ Moduli space of $SL(2,\mathbb{C})$ flat connections

$n_{(\alpha,m),0} \longrightarrow$ trans-series coefficient.

$CS(\alpha) \longrightarrow$ Chern-Simons value of α.

$Z_\alpha^{Pert}(\hbar) \longrightarrow$ Perturbative series around α.

For half index of $T[M_3]$

$$Z_{T[M_3]}(S^1 \times_q D^2, B_0) \underset{q \to 1}{\sim} \exp\left[\frac{\pi^2}{6\hbar}\left(1 + 24(CS(\alpha)+m)\right)\right],$$

for some flat connection α.

$$c_{eff} = 1 + 24\,(CS(\alpha) + m)$$

$CS(\alpha)$ is defined modulo integers. Therefore, $CS(\alpha)+m$ is a lift of $CS(\alpha)$ from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$.

Example: $\overline{\Sigma(2,3,5)}$

For $\overline{\Sigma(2,3,5)}$

$$Z(S^1 \times_q D^2, B_0) = \frac{q^{3/2}\,\chi_0(q)}{\eta(q)}$$, where χ_0 is the Ramanujan's order 5 mock theta function.

$$\chi_0(q) = \# \, q^{-\frac{1}{120}} \chi_0(\tilde{q}) + \# \, q^{-\frac{49}{120}+1} \chi_1(\tilde{q}) + \text{Borel Mordell integral}$$

$$\tilde{q} = e^{-\frac{4\pi^2}{\hbar}}$$

$$\chi_0(q) \sim e^{\frac{\pi^2}{30\hbar}} \longrightarrow c_{\text{eff}} = 1 + 24\left(\frac{1}{120}\right) = \frac{6}{5}.$$