

Counting wildly ramified quartic extensions with fixed automorphism group

Sebastian Monnet

Mass

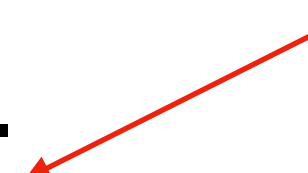
K/\mathbb{Q}_2 finite field extension

S set of quartic field extensions of K

The **pre-mass** of S is

$$\tilde{m}(S) := \sum_{L \in S} \frac{1}{\#\text{Aut}(L/K) \cdot q^{v_K(d_{L/K})}}$$

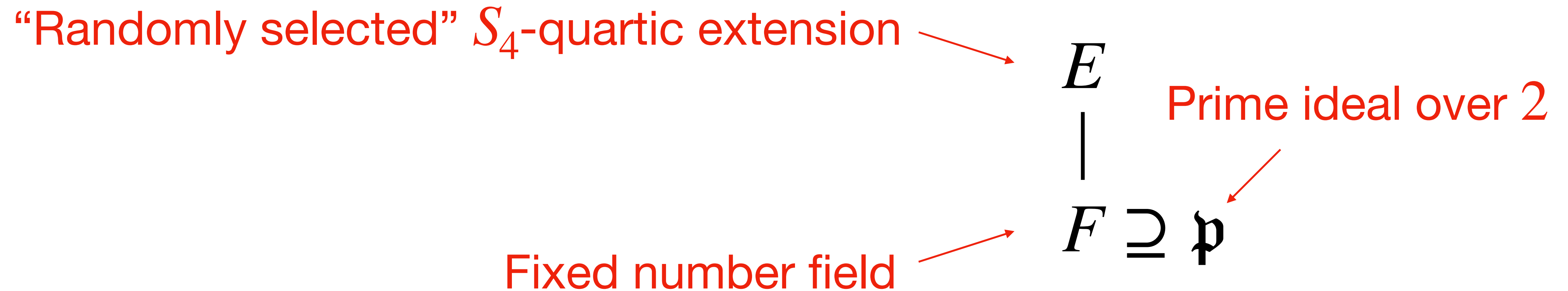
Discriminant valuation



Size of residue field $\#\mathbb{F}_K$



Application to arithmetic statistics

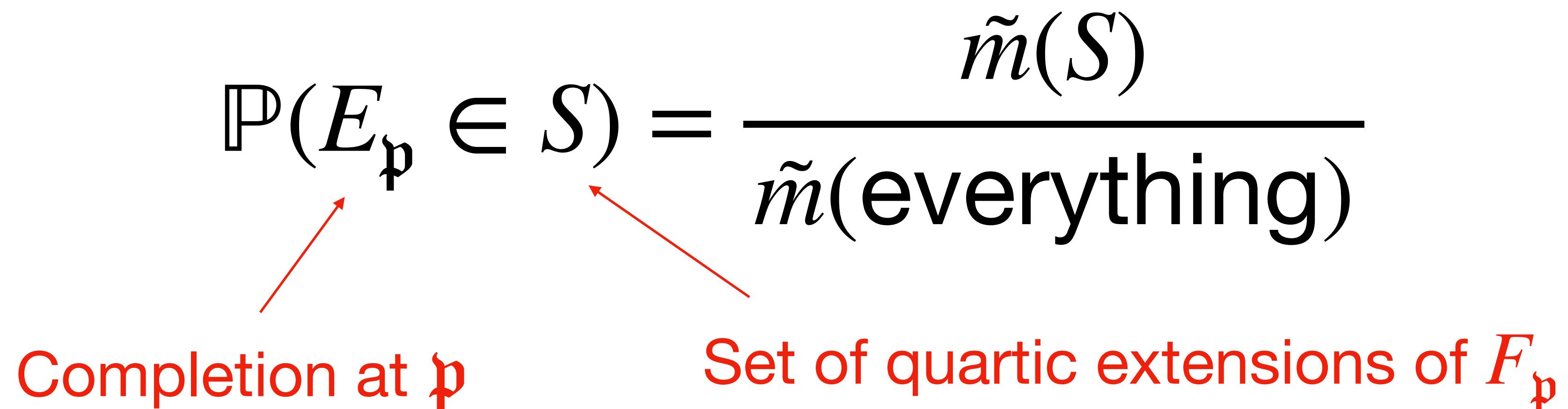


Theorem (Bhargava-Shankar-Wang): We have

$$\mathbb{P}(E_{\mathfrak{p}} \in S) = \frac{\tilde{m}(S)}{\tilde{m}(\text{everything})}$$

Completion at \mathfrak{p} \rightarrow

Set of quartic extensions of $F_{\mathfrak{p}}$ \rightarrow



Serre's mass formula

$$\Sigma := \{\text{totally ramified quartic } L/K\} / \cong$$

Theorem (Serre): We have

$$\tilde{m}(\Sigma) = q^{-3}$$

Fixed Galois group

Finite group

$$\Sigma^G := \{L \in \Sigma : \text{Aut}(L/K) \cong G\}$$

Goal: Find a formula for $\tilde{m}(\Sigma^G)$

Applications:

- S_4 -quartics with prescribed norms
- Upcoming work by Newton-Varma
- Extending base field

My approach

$$\Sigma_m^G := \{L \in \Sigma^G : v_K(d_{L/K}) = m\}$$

$$\text{Then } \tilde{m}(\Sigma^G) = \frac{1}{\#G} \cdot \sum_m \frac{\#\Sigma_m^G}{q^m}$$

So suffices to find $\#\Sigma_m^G$ and then add up

Four cases

Possible automorphism groups:

$\{1\}$, C_2 , $C_2 \times C_2$, C_4

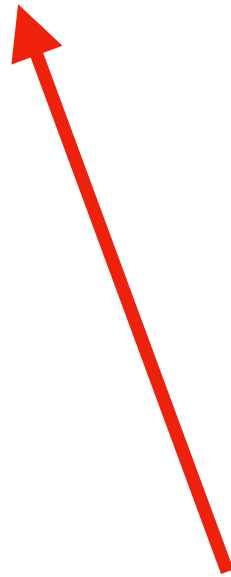
Congruence
conditions



Follows from
the rest



Class field
theory (Tunnell)



Tower of two
quadratic extensions



Formula for $\#\Sigma_m^{\{1\}}$

$$\#\Sigma_m^{\{1\}} = q^{\lfloor \frac{m}{3} \rfloor - 1} (q - 1) \left(1 + \mathbb{1}_{6|m} \cdot \left(\frac{1 - 2q}{3q} \right) \right)$$

$$(4 \leq m \leq 6e_K + 2 \text{ even})$$

Formula for $\#\Sigma_m^{C_4}$

$6 \leq m \leq 6e_K + 2$ even $\implies \#\Sigma_m^{C_4}$ is the sum of:

- (1) $\mathbb{1}_{8 \leq m \leq 5e_K - 2} \cdot \mathbb{1}_{m \equiv 3 \pmod{5}} \cdot 2q^{\frac{3m-14}{10}} (q-1)$.
- (2) $\mathbb{1}_{4e_K + 4 \leq m \leq 5e_K + 2} \cdot 2q^{\frac{m}{2} - e_K - 2} (q-1)$.
- (3) $\mathbb{1}_{5e_K < m \leq 8e_K} \cdot \mathbb{1}_{m \equiv 2e_K \pmod{3}} \cdot 2q^{\frac{4e_K + m}{6} - 1} (1 + \mathbb{1}_{m \leq 8e_K - 6t_0}) (q-1 - \mathbb{1}_{m=8e_K - 6t_0 + 6})$.
- (4) $\mathbb{1}_{10 \leq m \leq 5e_K} \cdot 2(q-1) (q^{\lfloor \frac{3m}{10} \rfloor - 1} - q^{\max\{\lceil \frac{m+2}{4} \rceil, \frac{m}{2} - e_K\} - 2})$.

Only other case:

$$\#\Sigma_{8e_K+3}^{C_4} = \begin{cases} 4q^{2e_K} & \text{if } -1 \in K^{\times 2}, \\ 2q^{2e_K} & \text{if } K(\sqrt{-1})/K \text{ is quadratic and totally ramified,} \\ 0 & \text{if } K(\sqrt{-1})/K \text{ is quadratic and unramified.} \end{cases}$$