Counting wildly ramified quartic extensions with fixed automorphism group

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Mass

$K/\mathbb{Q}_2$ finite field extension

S set of quartic field extensions of K

The **pre-mass** of S is

$$\tilde{m}(S) := \sum_{L \in S} \frac{1}{\#\mathrm{Aut}(L/K) \cdot q^{v_K(d_{L/K})}}$$

Discriminant valuation

Size of residue field $\#\mathbb{F}_K$

Application to arithmetic statistics

"Randomly selected" S_4-quartic extension → E

$$\begin{array}{c} E \\ | \\ F \supseteq \mathfrak{p} \end{array}$$

Prime ideal over 2 → $\mathfrak{p}$

Fixed number field → F

Theorem (Bhargava-Shankar-Wang): We have

$$\mathbb{P}(E_{\mathfrak{p}} \in S) = \frac{\tilde{m}(S)}{\tilde{m}(\text{everything})}$$

Completion at $\mathfrak{p}$ → $E_{\mathfrak{p}}$

Set of quartic extensions of $F_{\mathfrak{p}}$ → S

Serre's mass formula

$$\Sigma := \{\text{totally ramified quartic } L/K\}/\cong$$

Theorem (Serre): We have

$$\tilde{m}(\Sigma) = q^{-3}$$

Fixed Galois group

Finite group

$$\Sigma^G := \{L \in \Sigma : \mathrm{Aut}(L/K) \cong G\}$$

Goal: Find a formula for $\tilde{m}(\Sigma^G)$

Applications:

- S_4-quartics with prescribed norms
- Upcoming work by Newton-Varma
- Extending base field

My approach

$$\Sigma_m^G := \{L \in \Sigma^G : v_K(d_{L/K}) = m\}$$

Then $\quad \tilde{m}(\Sigma^G) = \frac{1}{\#G} \cdot \sum_m \frac{\#\Sigma_m^G}{q^m}$

So suffices to find $\#\Sigma_m^G$ and then add up

Four cases

Possible automorphism groups:

$$\{1\}, \quad C_2, \quad C_2 \times C_2, \quad C_4$$

- $\{1\}$: Congruence conditions
- C_2: Follows from the rest
- $C_2 \times C_2$: Class field theory (Tunnell)
- C_4: Tower of two quadratic extensions

Formula for $\#\Sigma_m^{\{1\}}$

$$\#\Sigma_m^{\{1\}} = q^{\lfloor \frac{m}{3} \rfloor - 1}(q-1)\left(1 + \mathbb{1}_{6|m} \cdot \left(\frac{1-2q}{3q}\right)\right)$$

($4 \leq m \leq 6e_K + 2$ even)

Formula for $\#\Sigma_m^{C_4}$

$6 \leq m \leq 6e_K + 2$ even $\Longrightarrow \#\Sigma_m^{C_4}$ is the sum of:

(1) $\mathbb{1}_{8\leq m\leq 5e_K-2} \cdot \mathbb{1}_{m\equiv 3 \pmod 5} \cdot 2q^{\frac{3m-14}{10}}(q-1)$.

(2) $\mathbb{1}_{4e_K+4\leq m\leq 5e_K+2} \cdot 2q^{\frac{m}{2}-e_K-2}(q-1)$.

(3) $\mathbb{1}_{5e_K<m\leq 8e_K} \cdot \mathbb{1}_{m\equiv 2e_K \pmod 3} \cdot 2q^{\frac{4e_K+m}{6}-1}(1+\mathbb{1}_{m\leq 8e_K-6t_0})(q-1-\mathbb{1}_{m=8e_K-6t_0+6})$.

(4) $\mathbb{1}_{10\leq m\leq 5e_K} \cdot 2(q-1)(q^{\lfloor\frac{3m}{10}\rfloor-1} - q^{\max\{\lceil\frac{m+2}{4}\rceil,\frac{m}{2}-e_K\}-2})$.

Only other case:

$$\#\Sigma_{8e_K+3}^{C_4} = \begin{cases} 4q^{2e_K} & \textit{if } -1 \in K^{\times 2}, \\ 2q^{2e_K} & \textit{if } K(\sqrt{-1})/K \textit{ is quadratic and totally ramified}, \\ 0 & \textit{if } K(\sqrt{-1})/K \textit{ is quadratic and unramified}. \end{cases}$$