Normal L-theory and topological cyclic homology

Yonatan Harpaz

CNRS, Université Sorbonne Paris Nord

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For a ring R, the cyclotomic trace map (Bökstedt–Hsiang–Madsen)

$$\mathrm{K}(R) \longrightarrow \mathrm{TC}(R)$$

algebraic K-theory   topological cyclic homology

is a very useful tool in the study of algebraic K-theory.

- $\mathrm{TC}(R)$ is generally easier to compute.
- The cyclotomic trace map induces an equivalence on *relative* theories for quotients $R \twoheadrightarrow R/I$ by nilpotent two-sided ideals (Dundas-Goodwillie-McCarthy).

What about the K-theory of quadratic forms?

Recall

For a commutative ring R, its *real* symmetric K-theory $\mathrm{K}\mathbb{R}^{s}(R)$ is a (genuine) C_2-spectrum with

- underlying spectrum algebraic K-theory;
- genuine fixed points the symmetric Grothendieck-Witt spectrum; and
- geometric fixed points the symmetric L-spectrum.

Goal: understand the real analogue $\mathrm{K}\mathbb{R}^{s}(R) \to \mathrm{TC}\mathbb{R}^{s}(R)$ of the cyclotomic trace map.

Main results

Theorem (Nikolaus–Sha–H.)

On the level of geometric fixed points the map $\mathrm{K}\mathbb{R}^s(R) \to \mathrm{TC}\mathbb{R}^s(R)$ *is*

$$\mathrm{L}^{\mathrm{s}}(R) \to \mathrm{L}^{\mathrm{nor}}(R) := \mathrm{cof}[\mathrm{L}^{\mathrm{q}}(R) \to \mathrm{L}^{\mathrm{s}}(R)].$$

- The statement holds in the wider context of ring spectra, and more generally Poincaré ∞-categories.
- Generalizes computations of normal L-theory of spherical group rings by Weiss–Williams and corresponding computations of $\mathrm{TC}\mathbb{R}$ (Høgenhaven, Dotto–Moi–Patchkoria).

$\Rightarrow$

A hermitian Dundas-Goodwillie-McCarthy theorem for nilpotent extensions of commutative rings.

Overview:

- K-theory of rings and trace maps.
- K-theory of stable ∞-categories and trace maps.
- Real K-theory of Poincaré ∞-categories and real trace maps.
- The trace of L-theory.

Back to algebraic K-theory

Classical idea

Study a ring R via its category $\mathrm{Proj}(R)$ of finitely generated projective (left) modules.

$\Rightarrow$ the groupoid core $\mathrm{coreProj}(R)$ inherits an E_∞-monoid structure via direct sums.

Identifying ∞-groupoids and spaces (or homotopy types), we may consider $\mathrm{coreProj}(R)$ as an E_∞-space (with non-trivial homotopy groups only in dimension 0 and 1).

Definition (Quillen)

The algebraic K-theory space of R is given by the group completion

$$\mathsf{K}(R) := \mathrm{coreProj}(R)^{\mathrm{grp}}$$

of the E_∞-space $\mathrm{coreProj}(R)$. It is an E_∞-group.

- The K-group $\mathsf{K}_0(R) = \pi_0 \mathsf{K}(R)$ is the Grothendieck group of $\mathrm{Proj}(R)$.
- The higher K-group $\mathsf{K}_n(R) := \pi_n \mathsf{K}(R)$ contain rich information on R, and are generally hard to compute.

Idea

Projective modules can be studied via their ranks

- For $P \in \mathrm{Proj}(R)$ one has $\mathrm{D}P := \mathsf{Hom}_R(P, R) \in \mathrm{Proj}(R^{\mathrm{op}})$.
- For $P, Q \in \mathrm{Proj}(R)$ an isomorphism of abelian groups $\mathrm{D}P \otimes_R Q \xrightarrow{\cong} \mathsf{Hom}_R(P, Q)$.

Definition

For $P \in \mathrm{Proj}(R)$ define the trace map by

$$\mathsf{End}_R(P) = \mathrm{D}P \otimes_R P \xrightarrow{f \otimes v \mapsto f(v)} R/[R, R]$$

The *rank* of P is $\mathrm{Rank}(P) := \mathsf{Tr}(\mathrm{id}_P)$.

Examples

- The rank of R^n is $[n] \in R/[R, R]$.
- If $a \in R$ is an idempotent element then $\mathrm{Rank}(Ra) = [a]$.

Observation

Ranks are additive: $\mathrm{Rank}(P \oplus Q) = \mathrm{Rank}(P) + \mathrm{Rank}(Q)$ in $R/[R,R]$

$\Rightarrow$ The formation of ranks descends to a homomorphism $\mathrm{Tr}_0 \colon \mathrm{K}_0(R) \to R/[R,R]$, known as the *trace map*. What about higher K-groups?

Dennis 76'

For every $n \geq 0$ there is a natural trace map

$$\mathrm{Tr}_n \colon \mathrm{K}_n(R) \to \mathrm{HH}_n(R)$$

where $\mathrm{HH}_n(R)$ is the n'th Hochschild homology group.

The groups $\mathrm{HH}_n(R)$ are the homology groups of the derived tensor product

$$R \otimes^{\mathsf{L}}_{R \otimes R^{\mathrm{op}}} R \simeq [\cdots \to R \otimes R \otimes R \to R \otimes R \xrightarrow{a \otimes b \mapsto ab - ba} R],$$

computed via the cyclic bar construction. Here $\mathrm{HH}_0(R) = R \otimes_{R \otimes R^{\mathrm{op}}} R = R/[R,R]$.

$\mathrm{HH}_n(R)$ is are much easier to compute then $\mathrm{K}_n(R)$, but also can be quite far from it in practice (e.g., $\mathrm{HH}_n(\mathbb{Z}) = 0$ for $n \geq 1$).

$\mathrm{K}(R)$ is an $\mathbb{E}_\infty$-group, so we can also faithfully encode it as a (connective) spectrum.
$\Rightarrow$ Obtain a refined trace map by replacing Hochschild homology by its spectral avatar.

Definition

For a ring spectrum A the topological Hochschild homology

$$\mathrm{THH}(A) := A \otimes_{A \otimes A^{\mathrm{op}}} A \in \mathcal{S}p$$

where $\otimes$ now stands for smash product of spectra. For an ordinary ring R one sets $\mathrm{THH}(R) = \mathrm{THH}(\mathrm{H}R)$, where $\mathrm{H}R$ is the associated Eilenberg-MacLane ring spectrum.

For $n \geq 0$ there are natural maps $\pi_n \mathrm{THH}(R) \to \mathrm{HH}_n(R)$. They are generally not isomorphisms, except for $n = 0$.

Theorem (Bökstedt)

There exists a canonical map of spectra

$$\mathrm{Tr} \colon \mathrm{K}(R) \to \mathrm{THH}(R)$$

such that the composites $\mathrm{K}_n(R) \to \pi_n \mathrm{THH}(R) \to \mathrm{HH}_n(R)$ *are the trace maps constructed by Dennis.*

The spectrum $\mathrm{THH}(R)$ admits a canonical *circle action*.

Even more, it is a *cyclotomic spectrum* (Hesselholt–Madsen, Nikolaus–Scholze)

Definition (Nikolaus-Scholze)

A cyclotomic spectrum is a spectrum X equipped with a $\mathbb{T}$-action ($\mathbb{T} = S^1$) and with a collection of $\mathbb{T}$-equivariant *Frobenius maps*

$$\varphi_p \colon X \to X^{\mathrm{t}\mu_p} = \mathrm{cof}[X_{\mathrm{h}\mu_p} \to X^{\mathrm{h}\mu_p}]$$

for each prime p, where $\mu_p \subseteq \mathbb{T}$ is the subgroup of p-torsion points.

For a cyclotomic spectrum X there are two distinguished maps $X^{\mathrm{h}\mu_p} \to X^{\mathrm{t}\mu_p}$, the canonical quotient map and the composite

$$X^{\mathrm{h}\mu_p} \to X \xrightarrow{\varphi_p} X^{\mathrm{t}\mu_p}.$$

Both are equivariant with respect to the residual action of $\mathbb{T}/\mu_p \simeq \mathbb{T}$.

Definition (Topological cyclic homology of a cyclotomic spectrum)

$$\mathrm{TC}(X) := \mathrm{Eq}\Big[X^{\mathrm{h}\mathbb{T}} \rightrightarrows \textstyle\prod_p (X^{\mathrm{t}\mu_p})^{\mathrm{h}\mathbb{T}/\mu_p}\Big]$$

Agrees with the notion of Hesselholt–Madsen when X is bounded below.

Trace methods

The trace map canonically factors through the *cyclotomic trace*

$$\mathsf{K}(R) \to \mathrm{TC}(R) := \mathrm{TC}(\mathrm{THH}(R)) \to \mathrm{THH}(R) \qquad \text{[Bökstedt–Hsiang–Madsen]}.$$

- $\mathrm{TC}(R)$ is generally easier to compute.
- The cyclotomic trace map is often close to being an equivalence.

Theorem (Dundas–McCarthy-Goodwillie)

Let R be a ring and $I \subseteq R$ a two-sided nilpotent ideal. Then the square of spectra

$$\begin{array}{ccc} \mathsf{K}(R) & \longrightarrow & \mathrm{TC}(R) \\ \downarrow & & \downarrow \\ \mathsf{K}(R/I) & \longrightarrow & \mathrm{TC}(R/I) \end{array}$$

is (homotopy) cartesian.

The algebraic K-theory of R depends only on $\mathrm{Proj}(R)$ as an additive category.

In fact, $\mathsf{K}(R)$ only depends on the perfect derived ∞-category $\mathcal{D}^{\mathrm{p}}(R) \simeq \mathrm{Mod}^{\mathrm{perf}}_{\mathrm{H}R}$

Definition (Barwick–Rognes)

For a stable ∞-category $\mathcal{C}$ the algebraic K-theory spectrum is

$$\mathsf{K}(\mathcal{C}) := \Omega|\mathrm{Span}(\mathcal{C})|$$

Gillet–Waldhausen

For a ring R one has

$$\mathsf{K}(\mathcal{D}^{\mathrm{p}}(R)) \simeq \mathsf{K}(R).$$

Algebraic K-theory is *localizing*: for every fibre-cofibre (or Verdier) sequence

$$\mathcal{C} \hookrightarrow \mathcal{D} \to \mathcal{E}$$

⇐ This means that $\mathcal{E}$ is the localization of $\mathcal{D}$ by the equivalences mod $\mathcal{C}$.

of stable ∞-categories,

$$\mathsf{K}(\mathcal{C}) \to \mathsf{K}(\mathcal{D}) \to \mathsf{K}(\mathcal{E})$$

is a fibre-cofibre sequence of spectra.

Theorem (Blumberg–Gepner–Tabuada)

The functor $\mathcal{C} \mapsto \mathsf{K}(\mathcal{C})$ *is the initial localizing functor* $\mathrm{Cat}_\infty^{\mathrm{ex}} \to \mathcal{S}p$ *equipped with a natural transformation* $\mathrm{core}\,\mathcal{C} \to \Omega^\infty \mathsf{K}(\mathcal{C})$.

Unstable version ($\mathcal{C}$ any ∞-category):

$$\mathrm{uTHH}(\mathcal{C}) = \operatorname*{colim}_{[f:x\to y]\in \mathsf{TwAr}(\mathcal{C})} \mathsf{Map}_{\mathcal{C}}(y,x) \in \mathcal{S}$$

Recall: the twisted arrow category has objects $f: x \to y$ and morphisms $\begin{matrix} x & \to & y \\ \downarrow & & \uparrow \\ x' & \to & y' \end{matrix}$

Example

If $\mathcal{C}$ is an ∞-groupoid then $\mathrm{uTHH}(\mathcal{C}) \simeq \mathsf{Map}(\mathbb{T}, \mathcal{C})$.

A stable ∞-category $\mathcal{C}$ admits a canonical enrichment $\mathrm{map}_{\mathcal{C}}(x,y) \in \mathcal{S}p$ in spectra with $\mathsf{Map}_{\mathcal{C}}(x,y) = \Omega^{\infty}\mathrm{map}_{\mathcal{C}}(x,y)$.

Definition

$$\mathrm{THH}(\mathcal{C}) := \operatorname*{colim}_{[f:x\to y]\in \mathsf{TwAr}(\mathcal{C})} \mathrm{map}_{\mathcal{C}}(y,x) \in \mathcal{S}p$$

Carries a canonical natural transformation $\mathrm{uTHH}(\mathcal{C}) \to \Omega^{\infty}\mathrm{THH}(\mathcal{C})$.

Fact

The functor $\mathrm{THH}(\mathcal{C})$ is localizing

$\Rightarrow$ the composite

$$\mathrm{core}\mathcal{C} \to \mathsf{Map}(\mathbb{T}, \mathrm{core}\mathcal{C}) \simeq \mathrm{uTHH}(\mathrm{core}\mathcal{C}) \to \mathrm{uTHH}(\mathcal{C}) \to \Omega^\infty \mathrm{THH}(\mathcal{C})$$

gives us the trace map

$$\mathsf{Tr}_{\mathcal{C}} \colon \mathsf{K}(\mathcal{C}) \to \mathrm{THH}(\mathcal{C})$$

by the universal property of $\mathsf{K}(-)$.

As in the case of rings $\mathrm{THH}(\mathcal{C})$ is actually a cyclotomic spectrum

The trace map refines to a cyclotomic trace map

$$\mathsf{K}(\mathcal{C}) \to \mathrm{TC}(\mathcal{C}) := \mathrm{TC}(\mathrm{THH}(\mathcal{C})) \to \mathrm{THH}(\mathcal{C}).$$

Wish to extend the theory to the hermitian setting: instead of projective modules (or perfect complexes) over rings, consider such modules equipped with a unimodular quadratic forms (Karoubi, Knebusch, Scharlau, Schlichting...).

Observation

Unimodular forms over commutative rings come in many flavours:

- Quadratic versus symmetric (and versus even).
- Symmetric versus anti-symmetric.
- Can consider hermitian forms with respect to an involution (as often done over $\mathbb{C}$).
- Can fix an invertible R-module M with involution and consider symmetric forms with values in M.

Over non-commutative rings one has no a-priori notion, requires choosing additional structure (for example, an invertible $(R \otimes R)$-module M with involution).

A given flavour of forms over R is in essences an extra structure on $\mathcal{D}^{\mathrm{p}}(R)$ which encodes for an $X \in \mathcal{D}^{\mathrm{p}}(R)$ a notion of unimodular forms on X.

Such a structure can be *axiomatized*

Definition

A *Poincaré* ∞-category is a stable ∞-category $\mathcal{C}$ with a perfect quadratic functor

$$\Omega\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}p.$$

- Quadratic - reduced and 2-excisive in the sense of Goodwillie calculus.
- $\Rightarrow \mathrm{B}_{\Omega}(x,y) := \mathrm{fib}(\Omega(x \oplus y) \to \Omega(x) \times \Omega(y))$ is exact in each variable (bilinear part).
- $\mathrm{B}_{\Omega}(x,y)$ is also symmetric in x and y, and the cofibre

$$\Lambda_{\Omega}(x) := \mathrm{cof}[\mathrm{B}_{\Omega}(x,x)_{\mathrm{hC}_2} \to \Omega(x)]$$

  is exact in x (linear part).
- Perfect - there exists a duality $\mathrm{D}_{\Omega}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that $\mathrm{B}_{\Omega}(x,y) \simeq \mathrm{map}_{\mathcal{C}}(x, \mathrm{D}_{\Omega} y)$.

Definition

For $x \in \mathcal{C}$ the points of the space $\Omega^{\infty}\Omega(x)$ are called *hermitian forms* on x. A hermitian form β on x determines a map $\beta_{\sharp}\colon x \to \mathrm{D}x$. When $\beta_{\sharp}$ is an equivalence we say that β is unimodular, or a *Poincaré form*.

$\mathrm{He}(\mathcal{C}, \Omega)$ = the ∞-category of hermitian objects (x,β) in $\mathcal{C}$.
$\mathrm{Pn}(\mathcal{C}, \Omega)$ = the ∞-groupoid of Poincaré objects (x,β) in $\mathcal{C}$.

R - commutative (associative) ring, M - invertible R ($R \otimes R$)-module with involution.

There is a notion analogous to perfect quadratic functors on the level of additive categories. Examples $\mathrm{Proj}(R)^{\mathrm{op}} \to \mathcal{A}b$ include

- $\mathrm{Sym}_M(P) = \mathrm{Hom}_R(P \otimes_R P, M)^{\mathrm{C}_2}$ ($\mathrm{Hom}_{R\otimes R}(P \otimes P, M)^{\mathrm{C}_2}$) symmetric forms on P
- $\mathrm{Quad}_M(P) = \mathrm{Hom}_R(P \otimes_R P, M)_{\mathrm{C}_2}$ ($\mathrm{Hom}_{R\otimes R}(P \otimes P, M)_{\mathrm{C}_2}$) quadratic forms on P

We can consider *homotopy* versions of these examples:

Examples

- The functor $\Omega^{\mathrm{s}}_R \colon \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{S}p$ given by $\Omega^{\mathrm{s}}_R(X) = \mathrm{map}_R(X \otimes_R X, M)^{\mathrm{hC}_2}$ ($\mathrm{map}_{R\otimes R}(X \otimes X, M)^{\mathrm{hC}_2}$) is the Poincaré structure of homotopy symmetric forms.
- The functor $\Omega^{\mathrm{q}}_R \colon \mathcal{D}^{\mathrm{p}}(R)^{\mathrm{op}} \to \mathcal{S}p$ given by $\Omega^{\mathrm{q}}_R(X) = \mathrm{map}_R(X \otimes_R X, M)_{\mathrm{hC}_2}$ ($\mathrm{map}_{R\otimes R}(X \otimes X, M)_{\mathrm{hC}_2}$) is the Poincaré structure of homotopy quadratic forms.

We can also consider *derived* versions of these examples:

For a perfect quadratic functor $\Omega \colon \mathrm{Proj}(R)^{\mathrm{op}} \to \mathcal{A}b$ there exists an essentially unique perfect quadratic functor $\widetilde{\Omega} \colon \mathcal{D}^{\mathrm{p}}(R) \to \mathcal{S}p$ with $\widetilde{\Omega}|_{\mathrm{Proj}(R)^{\mathrm{op}}} = \mathrm{H} \circ \Omega$.

The derived functors $\Omega^{\mathrm{gs}}_M = \widetilde{\mathrm{Sym}}_M$ and $\Omega^{\mathrm{gq}}_M = \widetilde{\mathrm{Quad}}_M$ are called the genuine symmetric and genuine quadratic functors, respectively. Fit into a sequence

$$\Omega^{\mathrm{q}}_M \Rightarrow \Omega^{\mathrm{gq}}_M \Rightarrow \Omega^{\mathrm{gs}}_M \Rightarrow \Omega^{\mathrm{s}}_M.$$

To a Poincaré ∞-category $(\mathcal{C}, \Omega)$ one may associate a Grothendieck-Witt spectrum $\mathsf{GW}(\mathcal{C}, \Omega)$ and an L-theory spectrum $\mathsf{L}(\mathcal{C}, \Omega)$. Some key facts:

- There is a canonical fibre sequence

$$\mathsf{K}(\mathcal{C})_{\mathrm{hC}_2} \to \mathsf{GW}(\mathcal{C}, \Omega) \to \mathsf{L}(\mathcal{C}, \Omega)$$

- $\mathsf{GW}(-)$ and $\mathsf{L}(-)$ are both localizing: they send fibre-cofibre sequences

$$(\mathcal{C}, \Omega) \to (\mathcal{D}, \Phi) \to (\mathcal{E}, \Psi)$$

  of Poincaré ∞-categories to fibre sequences of spectra.
- There is a natural transformation $\mathrm{Pn}(-) \Rightarrow \Omega^\infty \mathsf{GW}(-)$, and GW is the initial localizing invariant with this property.
- L is similarly universally characterized by being localizing and bordism invariant.

For a ring R, an invertible module with involution M, and $r \in \{\mathrm{s}, \mathrm{q}, \mathrm{gs}, \mathrm{gq}\}$ write

$$\mathsf{GW}^r(R, M) = \mathsf{GW}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^r_M) \quad \text{and} \quad \mathsf{L}^r(R, M) = \mathsf{L}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^r_M).$$

- The spaces $\Omega^\infty \mathsf{GW}^{\mathrm{gs}}(R, M)$ and $\Omega^\infty \mathsf{GW}^{\mathrm{gq}}(R, M)$ are equivalent the symmetric and quadratic Grothendieck–Witt spaces defined by Karoubi-Villamayor [Hebestreit–Steimle]
- The spectra $\mathsf{L}^{\mathrm{s}}(R, M)$ and $\mathsf{L}^{\mathrm{q}}(R, M)$ are the 4-periodic symmetric and quadratic L-spectra of Ranicki.
- The homotopy groups of $\mathsf{L}^{\mathrm{gs}}(R, M)$ are the original non-periodic symmetric L-groups originally defined by Ranicki.

Recall

A C_2-spectrum is a spectrum X with C_2-action together with a factorization

$$X_{hC_2} \to X^{C_2} \to X^{hC_2}$$

of the transfer map $X_{hC_2} \to X^{hC_2}$. $\mathcal{S}p_{C_2}$ - the ∞-category of C_2-spectra.

The spectrum X^{C_2} is called the genuine fixed points of X, while $X^{\Phi C_2} := \mathrm{cof}[X_{hC_2} \to X^{C_2}]$ is called the geometric fixed points of X.

Example

For a Poincaré ∞-category $(\mathcal{C}, \Omega)$ and $x \in \mathcal{C}$ and object the factorization

$$\mathrm{map}(x, \mathrm{D}x)_{hC_2} \to \Omega(x) \to \mathrm{map}(x, \mathrm{D}x)^{hC_2}$$

determines a C_2-spectrum $\widetilde{\Omega}(x)$ whose underlying spectrum is $\mathrm{map}_{\mathcal{C}}(x, \mathrm{D}x)$, whose genuine fixed points is $\Omega(x)$ and whose geometric fixed point is $\Lambda_{\mathrm{Q}}(x)$.

Definition

The *real* K*-theory spectrum* of $(\mathcal{C}, \Qoppa)$ is the C_2-spectrum $\mathrm{K}\mathbb{R}(\mathcal{C}, \Qoppa)$ with underlying spectrum $\mathsf{K}(\mathcal{C})$ and factorization

$$\mathsf{K}(\mathcal{C})_{\mathrm{hC}_2} \to \mathsf{GW}(\mathcal{C}, \Qoppa) \to \mathsf{K}^{\mathrm{hC}_2}.$$

Its genuine fixed points spectrum is $\mathsf{GW}(\mathcal{C}, \Qoppa)$ and its geometric fixed points is $\mathsf{L}(\mathcal{C}, \Qoppa)$.

For a ring R, an invertible module with involution M, and $r \in \{\mathrm{s}, \mathrm{q}, \mathrm{gs}, \mathrm{gq}\}$ write

$$\mathrm{K}\mathbb{R}^r(R, M) := \mathrm{K}\mathbb{R}(\mathcal{D}^{\mathrm{p}}(R), \Qoppa^r_M)$$

.

Recall

A C_2-category is cocartesian fibration $\mathcal{E} \to \mathcal{O}_{\mathrm{C}_2}^{\mathrm{op}}$, where $\mathcal{O}_{\mathrm{C}_2}$ is the orbit category of C_2. A C_2-functor is a map over $\mathcal{O}_{\mathrm{C}_2}^{\mathrm{op}}$ preserving cocartesian arrows.

The category $\mathcal{O}_{\mathrm{C}_2}$ has two objects $\mathrm{C}_2, *$. A C_2-category $\mathcal{E}$ has

- An underlying category $\mathcal{E}_{\mathrm{C}_2}$ equipped with a C_2-action.
- A fixed point category $\mathcal{E}_*$, equipped with a C_2-equivariant functor $\mathcal{E}_* \to \mathcal{E}_{\mathrm{C}_2}$.

A C_2-category is completely determined by this data.

Examples

- The C_2-category $\widetilde{\mathcal{S}p}_{\mathrm{C}_2} \to \mathcal{O}_{\mathrm{C}_2}^{\mathrm{op}}$ whose underlying ∞-category is $\mathcal{S}p$ (with trivial action) and whose fixed points category is $\mathcal{S}p_{\mathrm{C}_2}$.
- For a Poincaré ∞-category $(\mathcal{C}, \mathfrak{Q})$, its twisted arrow category refines to a C_2-category

$$\widetilde{\mathsf{TwAr}}(\mathcal{C}, \mathfrak{Q}) \to \mathcal{O}_{\mathrm{C}_2}^{\mathrm{op}}$$

  whose underlying category is $\mathsf{TwAr}(\mathcal{C})$ with C_2-action induced by the duality and fixed points the category $\mathrm{He}(\mathcal{C}, \mathfrak{Q})$ of hermitian objects.

First introduced for ring spectra with anti-involution by Hesselholt–Madsen.

Recall

For a stable ∞-category $\mathcal{C}$ we had a canonical functor $\mathrm{map}\colon \mathsf{TwAr}(\mathcal{C}) \to \mathcal{S}p$ sending $f\colon x \to y$ to $\mathrm{map}_{\mathcal{C}}(y,x)$. The colimit of this functor is $\mathrm{THH}(\mathcal{C})$.

For a Poincaré ∞-category we have a canonical C_2-functor

$$\widetilde{\mathrm{map}}\colon \widetilde{\mathsf{TwAr}}(\mathcal{C},Ϙ) \to \widetilde{\mathcal{S}p}_{\mathrm{C}_2}.$$

- On the level of underlying categories, it is the functor $\mathrm{map}\colon \mathsf{TwAr}(\mathcal{C}) \to \mathcal{S}p$ associated to $\mathcal{C}$.
- On the level of fixed point categories, it is the functor $\mathrm{He}(\mathcal{C},Ϙ) \to \mathcal{S}p_{\mathrm{C}_2}$ sending a hermitian object (x,β) to the C_2-spectrum $\widetilde{Ϙ}(\mathrm{D}x)$.

The *real topological Hochschild homology* of $(\mathcal{C},Ϙ)$ is the C_2-colimit

$$\mathrm{THR}(\mathcal{C},Ϙ) = \operatorname*{colim}^{\mathrm{C}_2}_{\widetilde{\mathsf{TwAr}}}(\widetilde{\mathrm{map}}) \in \mathcal{S}p_{\mathrm{C}_2}$$

of the functor $\widetilde{\mathsf{TwAr}}$. It is a C_2-spectrum whose

- underlying spectrum is $\mathrm{THH}(\mathcal{C}) = \mathrm{colim}_{\mathsf{TwAr}}(\mathrm{map})$; and
- geometric fixed points $\operatorname*{colim}_{(x,\beta)\in\mathrm{He}(\mathcal{C},Ϙ)} \widetilde{Ϙ}(\mathrm{D}x)^{\Phi\mathrm{C}_2} = \operatorname*{colim}_{(x,\beta)\in\mathrm{He}(\mathcal{C},Ϙ)} \Lambda_{Ϙ}(\mathrm{D}x)$.

For a Poincaré ∞-category $(\mathcal{C}, \Omega)$, the C_2-spectrum $\mathrm{THR}(\mathcal{C}, \Omega)$ is a *real cyclotomic spectrum*.

Two definitions for this notion were proposed by J.D. Quigley and J. Shah, analogous to the two definitions of cyclotomic spectra. They then showed that they coincide when the underlying spectrum is bounded below.

In the Nikolaus-Scholze style definition, a real cyclotomic spectrum is a C_2-spectrum X with

- a $\mathbb{T}$-action, but now $\mathbb{T}$ needs to be considered as a group object in C_2-spaces; and
- Frobenius maps $\varphi_p : X \to X^{t\mu_p}$ for every prime p, but where the Tate construction is now understood in a C_2-parametrized manner.

$$\mathrm{TCR}(X) := \mathrm{Eq}\left[X^{h\mathbb{T}} \rightrightarrows \prod_p (X^{t\mu_p})^{h\mathbb{T}/\mu_p}\right] \in \mathcal{S}p_{\mathrm{C}_2}.$$

Again, the equalizer, Tate construction and homotopy fixed points are all to be understood in the C_2-parametrized sense.

For a Poincaré ∞-category $(\mathcal{C}, \Omega)$ one sets $\mathrm{TCR}(\mathcal{C}, \Omega) = \mathrm{TCR}(\mathrm{THR}(\mathcal{C}, \Omega))$. For a ring R, an invertible module with involution M, and $r \in \{\mathrm{s}, \mathrm{q}, \mathrm{gs}, \mathrm{gq}\}$ write

$$\mathrm{THR}^r(R, M) = \mathrm{THR}(\mathcal{D}^{\mathrm{p}}(R), \Omega^r_M) \quad \text{and} \quad \mathrm{TCR}^r(R, M) = \mathrm{TCR}(\mathcal{D}^{\mathrm{p}}(R), \Omega^r_M).$$

As before, one has a $\mathbb{T}$-equivariant real trace map $\mathrm{K}\mathbb{R}(\mathcal{C}, \Qoppa) \to \mathrm{TH}\mathbb{R}(\mathcal{C}, \Qoppa)$, which lifts to a real cyclotomic trace map

$$\mathrm{K}\mathbb{R}(\mathcal{C}, \Qoppa) \to \mathrm{TC}\mathbb{R}(\mathcal{C}, \Qoppa).$$

Theorem

Let $\mathcal{C}, \Qoppa$ be a Poincaré ∞-category with duality D such that $\mathcal{C}$ is generated under finite colimits and desuspensions by a set of objects $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\mathrm{map}_{\mathcal{C}}(x, y)$ is connective for $x, y \in \mathcal{C}_0$. Then on the level of geometric fixed points, the real cyclotomic trace map is the map

$$\mathsf{L}(\mathcal{C}, \Qoppa) \to \mathsf{L}^{\mathrm{nor}}(\mathcal{C}, \Qoppa) := \mathsf{cof}[\mathsf{L}(\mathcal{C}, \Qoppa^{\mathrm{q}}_{\mathrm{D}}) \to \mathsf{L}(\mathcal{C}, \Qoppa)],$$

where $\Qoppa^{\mathrm{q}}_{\mathcal{D}}(x) = \mathrm{map}_{\mathcal{C}}(x, \mathrm{D}x)_{\mathrm{hC}_2}$ is the "quadratic" Poincaré structure of D.

The condition on $\mathcal{C}$ being generated from $\mathcal{C}_0$ implies that $\mathrm{THH}(\mathcal{C})$ is bounded below (and this is the only part it plays).

Corollary

Let R be a commutative ring, $I \subseteq R$ a nilpotent ideal and a M an invertible R-module with involution. Then for every $r \in \{\mathrm{s}, \mathrm{q}, \mathrm{gs}, \mathrm{gq}\}$ the square

$$\begin{array}{ccc} \mathrm{K}\mathbb{R}^r(R, M) & \longrightarrow & \mathrm{TC}\mathbb{R}^r(R, M) \\ \downarrow & & \downarrow \\ \mathrm{K}\mathbb{R}^r(R/I, M/I) & \longrightarrow & \mathrm{TC}\mathbb{R}^r(R/I, M/I) \end{array}$$

is a cartesian square of C_2-spectra.

$$T := \mathrm{THR}(\mathcal{C}, \mathfrak{Q})^{\Phi C_2} = \operatorname*{colim}_{(x,\beta)\in \mathrm{He}(\mathcal{C},\mathfrak{Q})} \Lambda_{\mathfrak{Q}}(\mathrm{D}x) \simeq \int_{x\in\mathcal{C}} \Lambda_{\mathfrak{Q}}(x) \otimes \Lambda_{\mathfrak{Q}}(\mathrm{D}x)$$

It carries a μ_2-action and admits a Frobenius map $T \to T^{\mathrm{t}\mu_2}$. One then shows

$$\mathrm{TC}\mathbb{R}(\mathcal{C}, \mathfrak{Q})^{\Phi C_2} = \mathrm{Eq}[T^{\mathrm{h}\mu_2} \rightrightarrows T^{\mathrm{t}\mu_2}] = \mathrm{L}^{\mathrm{nor}}(\mathcal{C}, \mathfrak{Q}).$$

For the blue equivalence this is a Beck-Chevalley phenomenon of left and right functors commuting, and crucially relies on the assumption that $\mathrm{THH}(\mathcal{C})$ is bounded below.

For the red equivalence, suppose $\Lambda_{\mathrm{Q}}(-) = \mathrm{map}_{\mathcal{C}}(-, a)$ is representable. Then

$$T = \Lambda_{\mathrm{Q}}(\mathrm{D}a) = \mathrm{map}_{\mathcal{C}}(\mathrm{D}a, a) = \mathrm{B}_{\mathfrak{Q}}(\mathrm{D}a, \mathrm{D}a)$$

and one checks that

$$\mathrm{Eq}[\mathrm{B}_{\mathfrak{Q}}(\mathrm{D}a, \mathrm{D}a)^{\mathrm{h}\mu_2} \rightrightarrows \mathrm{B}_{\mathfrak{Q}}(\mathrm{D}a, \mathrm{D}a)^{\mathrm{t}\mu_2}] \simeq \mathrm{Eq}[\mathfrak{Q}(\mathrm{D}a) \rightrightarrows \mathrm{B}_{\mathfrak{Q}}(\mathrm{D}a, \mathrm{D}a)].$$

Construct a map

$$\begin{array}{ccc} \mathrm{Pn}(\mathcal{C}, \mathfrak{Q}) & \longrightarrow & \mathrm{Eq}[\Omega^\infty \mathfrak{Q}(\mathrm{D}a) \rightrightarrows \Omega^\infty \mathrm{B}_{\mathfrak{Q}}(\mathrm{D}a, \mathrm{D}a)] \\ (x, \beta) & \longmapsto & \hat{f}_\beta^* \beta \rightrightarrows \hat{f}_\beta^* f_\beta \end{array}$$

where $f_\beta : x \to a$ is the image of β in $\Lambda_{\mathrm{Q}}(x) = \mathrm{map}_{\mathcal{C}}(x, a)$, and $\hat{f}_\beta : \mathrm{D}a \to \mathrm{D}x \simeq x$ its dual $\Rightarrow$ extends to a map on normal L-theory by universal property.