SPDE in critical spaces

Mark Veraar (TU Delft, The Netherlands)

Harmonic Analysis, Stochastics and PDEs
in Honour of the 80th Birthday of Nicolai Krylov

Joint work Antonio Agresti (IST, Austria)

Introduction

Consider a parabolic SPDEs of the form:

$$\begin{cases} du + Audt = F(u)dt + (Bu + G(u))dW, & t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{1}$$

- $X_1 \hookrightarrow X_0$ special Banach spaces (e.g. L^q, $H^{s,q}$
- $q \in [2,\infty)$ spatial integrability, $p \in [2,\infty)$ temporal integrability.
- $X_\theta = [X_0, X_1]_\theta$ (complex interpolation)
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1. Stochastic maximal regularity (SMR)
2. Local well-posedness
3. Global well-posedness
4. Variational setting
5. Stochastic Navier-Stokes equations for turbulent flows

What is stochastic maximal regularity (SMR)?

SMR is an L^p-estimate for the linear part of the SPDE (1) with $u(0) = 0$.

$$du + Audt = fdt + (Bu + g)dW, \qquad t \in (0, T). \tag{2}$$

Definition

Let $p \in [2, \infty)$.

1. $(A, B) \in \mathrm{SMR}_p$ if $\forall f \in L^p((0, T) \times \Omega; X_0)$ and $\forall g \in L^p((0, T) \times \Omega; X_{1/2}(\ell^2))$ there exists a unique solution $u \in L^p((0, T) \times \Omega; X_1)$ to (2) and

$$\|u\|_{L^p((0,T)\times\Omega;X_1)} \leq C\|f\|_{L^p((0,T)\times\Omega;X_0)} + C\|g\|_{L^p((0,T)\times\Omega;X_{1/2}(\ell^2))}.$$

2. $(A, B) \in \mathrm{SMR}_p^+$ if also $\partial_t^\theta u \in L^p((0, T) \times \Omega; X_{1-\theta})$ for all $\theta \in [0, 1/2)$ and

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In (2) for $p = 2$: use $u \in L^2(\Omega; C([0, T]; X_{1/2}))$ instead.

Weighted variant $\mathrm{SMR}_{p,\kappa}$: replace $L^p(0, T)$ by $L^p((0, T), t^\kappa dt)$.

(2) is important for SPDEs. It implies: $u \in L^p(\Omega; C([0, T]; X^{\mathrm{Tr}}_{\kappa,p}))$ (Meyries-V. '14)

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SMR: Examples

Variational setting

$p = 2$, $V = X_1$, $H = X_{1/2}$ (Hilbert space), $V^* = X_0$: Coercivity $\Rightarrow (A, B) \in \mathrm{SMR}_2^+$
Bensoussan and Temam '72, Pardoux '75, Krylov-Rozovskii '79, Liu-Röckner '15

Real interpolation scale

A sectorial, $X_0 = D_A(0, p)$, $X_1 = D_A(1, p) \Rightarrow (A, 0) \in \mathrm{SMR}_p$.
Da Prato '83, Brzezniak '95, Da Prato-Lunardi '98, Brzezniak-Hausenblas '09, Lorist-V. '21

Krylov's '99 L^p-theory $p \in [2, \infty)$

$X_0 = H^{s,p}$. A is 2^{nd}-order, B is 1^{st}-order. Coercivity + regularity $\Rightarrow (A, B) \in \mathrm{SMR}_p$
Krylov since '94, Lotoskii, Mikulevicius, Rozovskii, Kim K.H., Kim I., ...

Functional calculus approach

A bounded H^∞-calculus, $X_0 = H^{s,q} \Rightarrow (A, 0) \in \mathrm{SMR}^+_{p,\kappa}$.
van Neerven-V.-Weis '12, Agresti-V. '20

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Da Prato '83, Brzezniak '95, Da Prato-Lunardi '98, Brzezniak-Hausenblas '09, Lorist-V. '21

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van Neerven-V.-Weis '12, Agresti-V. '20

SMR: Examples

Variational setting

$p = 2$, $V = X_1$, $H = X_{1/2}$ (Hilbert space), $V^* = X_0$: Coercivity $\Rightarrow (A, B) \in \mathrm{SMR}_2^+$
Bensoussan and Temam '72, Pardoux '75, Krylov-Rozovskii '79, Liu-Röckner '15

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Overview

Local well-posedness

- Either $p \in (2, \infty)$ and $\kappa \in [0, \frac{p}{2} - 1)$, or
- $p = 2$ and $\kappa = 0$ and X_0, X_1 are Hilbert spaces.

$$\begin{cases} du + Audt = F(u)dt + (Bu + G(u))dW, & t \in (0, T), \\ u(0) = u_0. \end{cases} \tag{3}$$

Theorem (Local well-posedness, AV20a)

Let F and G be as below. Let $u_0 \in X^{\mathrm{Tr}}_{\kappa,p}$ a.s., and suppose $(A, B) \in \mathrm{SMR}_{p,\kappa}$, then there exists a (unique) maximal local solution (u, σ) to (3) with $\sigma > 0$, and a.s.

$$u \in H^{\theta,p}_{\mathrm{loc}}(0, \sigma; X_{1-\theta}) \;\; \forall \theta \in [0, 1/2) \;\; \text{and} \;\; u \in C((0, \sigma); X^{\mathrm{Tr}}_{0,p}) \text{ (instant regularity).}$$

Here $F(t, \omega, \cdot) : X_\beta \to X_0$: $\forall n \geq 1 \, \exists C_n \geq 0 \, \forall x, y \in X_\beta$ with $\|x\|_{X^{\mathrm{Tr}}_{\kappa,p}}, \|y\|_{\|x\|_{X^{\mathrm{Tr}}_{\kappa,p}}} \leq n$

$$\|F(t, \omega, x)\|_{X_0} \leq C_n \|x\|^{\rho+1}_{X_\beta}$$

$$\|F(t, \omega, x) - F(t, \omega, y)\|_{X_0} \leq C_n (\|x\|^{\rho}_{X_\beta} + \|y\|^{\rho}_{X_\beta}) \|x - y\|_{X_\beta}$$

where $\rho \geq 0$ and β satisfy the (sub)criticallity condition:

$$1 - \frac{1 + \kappa}{p} < \beta \leq 1 - \frac{1 + \kappa}{p} \frac{\rho}{\rho + 1}.$$

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Example of a quadratic nonlinearity

Burgers/Navier-Stokes nonlinearity

Let $X_\theta = H^{-1+2\theta,q}$ and $F(u) = \nabla(u^2)$
When do we have $\|F(t,\omega,u)\|_{X_0} \leq C\|u\|_{X_\beta}^{\rho+1}$ and $1 - \frac{1+\kappa}{p} < \beta \leq 1 - \frac{1+\kappa}{p}\frac{\rho}{\rho+1}$?

$$\|F(u)\|_{X_0} \lesssim \|u^2\|_{L^q} = \|u\|_{L^{2q}}^2 \lesssim \|u\|_{X_\beta}^2 \quad \to \quad \rho = 1$$

if $-1 + 2\beta - \frac{d}{q} \geq -\frac{d}{2q}$. Or equivalently: $\beta \geq \frac{d}{4q} + \frac{1}{2}$.
Thus we can find admissible β if and only if

$$\frac{d}{4q} + \frac{1}{2} \leq 1 - \frac{1+\kappa}{2p} \quad \text{or equivalently} \quad \frac{d}{2q} + \frac{1+\kappa}{p} \overset{(*)}{\leq} 1.$$

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Overview

Global well-posedness via blow-up criteria

Theorem (AV21b)

Suppose that the conditions of the local well-posedness theorem hold. Then

$$P(\sigma < \infty, \sup_{t\in[0,\sigma)} \|u(t)\|_{X^{\mathrm{Tr}}_{\kappa,p}} < \infty) = 0 \quad \textit{if the nonlinearity is subcritical}$$

$$P(\sigma < \infty, \|u\|_{L^p(0,\sigma;X_{1-\frac{\kappa}{p}})} < \infty) = 0 \quad \textit{if the nonlinearity is quadratic.}$$

Strategy: check $\|u\|_{L^p(0,\sigma;X_{1-\frac{\kappa}{p}})} < \infty$ on $\{\sigma < \infty\}$. This leads to

$$P(\sigma < \infty) = P(\sigma < \infty, \|u\|_{L^p(0,\sigma;X_{1-\frac{\kappa}{p}})} < \infty) = 0.$$

This gives $\sigma = \infty$ a.s., and thus would lead to global well-posedness.

Weights in time make it possible to

- obtain higher order regularity by new weighted bootstrap method
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Strategy: check $\|u\|_{L^p(0,\sigma;X_{1-\frac{\kappa}{p}})} < \infty$ on $\{\sigma < \infty\}$. This leads to

$$P(\sigma < \infty) = P(\sigma < \infty, \|u\|_{L^p(0,\sigma;X_{1-\frac{\kappa}{p}})} < \infty) = 0.$$

This gives $\sigma = \infty$ a.s., and thus would lead to global well-posedness.

Weights in time make it possible to

- obtain higher order regularity by new weighted bootstrap method
- transfer global well-posedness results between different spaces

Overview

Variational setting

Let $V := X_1 \hookrightarrow X_0 =: V^*$ Hilbert spaces, $p = 2$, $\kappa = 0$.
Then $X^{\mathrm{Tr}}_{0,2} = X_{1/2} =: H$. Locally Lipschitz F and G as before.
(sub)criticallity condition: $1/2 < \beta \leq \frac{1}{2} + \frac{1}{2(\rho+1)}$.

Theorem (AV22a)

Suppose that $\forall T > 0, \exists \theta, C, M > 0, \phi \in L^2((0,T)\times\Omega)$ *s.t.* $\forall v \in V\, \forall t \in [0,T]$,

$$\langle v, Av + F(\cdot, v)\rangle - \tfrac{1}{2}\|Bv + G(\cdot, v)\|^2_{\mathscr{L}_2(\ell^2,H)} \geq \theta\|v\|^2_V - M\|u\|^2_H - |\phi(t)|^2 \quad \textit{(coercivity)}$$

$$\|G(\cdot, v)\|_{\mathcal{L}(U,H)} \leq C(1 + \|v\|_H)(\phi(t) + \|v\|_V) \quad \textit{(quadratic growth)} \qquad (4)$$

Then $\forall u_0 \in L^0(\Omega; H)$, *there exists a unique global solution* u *of* (3) *such that a.s.*

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Moreover, in this space u *depends continuously on* u_0 *(in probability).*

Replacing $\frac{1}{2}\|Bv + G(\cdot, v)\|^2_{\mathscr{L}(\ell^2,H)}$ by $(\frac{1}{2} + \varepsilon)\|Bv + G(\cdot, v)\|^2_{\mathscr{L}_2(\ell^2,H)}$ one can omit (4).
Many applications! No monotonicity condition.

Independent results due to Röckner-Shang-Zhang '22

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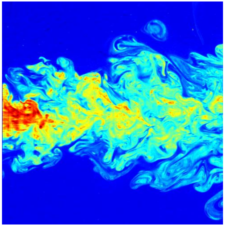
Overview

Stochastic Navier-Stokes equations for turbulent flows

Stochastic Lagrangian approach

Reynolds 1880:

$$\text{Velocity field} = \underbrace{u}_{\substack{\text{Slow oscillating part} \\ \text{(deterministic)}}} + \underbrace{b\dot{W}}_{\substack{\text{Fast oscillating part} \\ \text{(random)}}}$$



Kraichnan's turbulence theory (1968)

1. Statistic modeling: $b \in H^{\eta,\xi}$ for some $\eta > d/\xi$ and $\xi \in [2, \infty)$;
2. Newton's law yields

$$du - \Delta u\, dt = \big(- \nabla p - (u \cdot \nabla) u \big) dt + \textstyle\sum_{n \geq 1} \underbrace{(b_n \cdot \nabla) u}_{\substack{\text{Stochastic} \\ \text{transport}}} \, dW_t^n. \tag{5}$$

Eq. (5) is called the Navier-Stokes equations for turbulent flows.
Contributors include: Brzezniak, Capinski, Ferrario, Flandoli, Gatarek, Hofmanova, Leahy, Mikulevicius, Motyl, Nilssen, Peszat, Rozovskii

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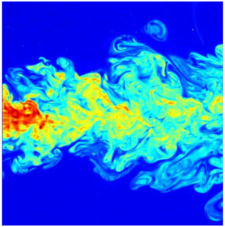

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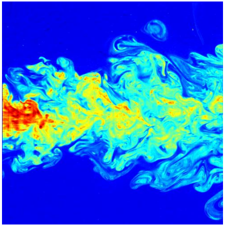

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SNS: transport noise as a model for small-scales

Derivation in Mikulevicius-Rozovskii 2004:

Separation of scales

Suppose that $u = u_L + u_S$ where L stands for "Large" and S for "Small" scale and

$$\partial_t u_L - \Delta u_L = -\nabla P_L - ((u_L + u_S) \cdot \nabla) u_L,$$
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Then the sum $u = u_L + u_S$ solves the Navier-Stokes equations.

Turbulent regime

In a turbulent regime one can model u_S as an approximation of white noise, so that

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Thus the large scale component u_L solves

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$\forall u_0 \in L^0(\Omega; L^d(\mathbb{T}^d))$ with $\operatorname{div} u_0 = 0$*, there exists a maximal local solution* (u, σ) *to* (6)*, and* $u \in L^q_{\rm loc}(0,\sigma; W^{1,q}) \cap C^{\theta_1,\theta_2}_{\rm loc}((0,\sigma)\times\mathbb{T}^d)$ *a.s.* $\forall q \in (2,\infty), \theta_1 \in (0,\frac{1}{2}), \theta_2 \in (0,1)$.

1. u_0 can even have negative (Besov) regularity down to $-\frac{1}{2}$.
2. Even $u \in C^{\theta_1,\theta_2+\eta-\frac{d}{\xi}}_{\rm loc}((0,\sigma)\times\mathbb{T}^d)$ a.s. $\forall q \in (2,\infty), \theta_1 \in [0,1/2), \theta_2 \in (0,1)$
3. Previous results: high smoothness of u_0 and b, not much regularity for u.
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5. Results are also new if non-gradient noise is used

SNS: local well-posedness and regularity

Consider

$$\begin{cases} du - \Delta u\,dt = (-\nabla p - (u\cdot\nabla)u)dt + \sum_{n\geq 1}(b_n\cdot\nabla)u dW_t^n & \text{on } \mathbb{T}^d, \\ \mathrm{div} u = 0, \quad u(0) = u_0, & \text{on } \mathbb{T}^d. \end{cases} \tag{6}$$

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There exists a unique solution $u \in Z := L^2(0,T;H^1) \cap C([0,T];L^2)$ a.s. to SNS (6) and

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The above is connected to the critically of F.

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Stochastic version of the recent Prüss-Wilke-Simonett 2018 theory.

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In several cases the linear theory is missing: $p \neq q$, $B \neq 0$

Many follow-up questions need to be answered:

- Large deviations
- Existence of invariant measures
- New a priori estimates for nonlinear SPDEs

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Selection of some of our recent papers which appeared in the talk

AV20a Agresti, & Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces Part I. Stochastic maximal regularity and local existence. To appear in *Nonlinearity*

AV20b Agresti, & Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces Part II. Blow-up criteria and instantaneous regularization. Online first in *Journal of evolution equations*

AV21a Agresti, & Veraar. Stochastic maximal $L^p(L^q)$-regularity for second order systems with periodic boundary conditions, arxiv

AV21b Agresti, & Veraar. Stochastic Navier-Stokes equations for turbulent flows in critical spaces, arxiv

AV22a Agresti, & Veraar. The critical variational setting for stochastic evolution equations. arxiv

AV22b Agresti, & Veraar. Reaction-diffusion equations with transport noise and critical superlinear diffusion, in preparation

LV21 Lorist, & Veraar. Singular stochastic integral operators, Analysis and PDE Vol. 14 (2021), No. 5, 1443-1507

PV21 Portal, & Veraar. Stochastic maximal regularity for rough time-dependent problems, SPDE: Analysis and Computations 7(4), 2019, 541-597.

to professor Krylov