On a long standing problem in the theory of Markov processes

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Contents

1. History of the problem and description of the first main result

Let $\mathbb{M} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X(t))_{t\geq 0}, (\mathbb{P}_x)_{x\in E})$ be a normal **Markov process** with state space $(E, \mathcal{B}(E))$, where E top. space with Borel σ-algebra $\mathcal{B}(E)$, having right-continuous sample paths, so we have:

$$\mathbb{P}_x[X(0) = x] = 1 \quad (\text{"normal"})$$
$$\mathbb{P}_x[X(s+t) \in A | \mathcal{F}_s] = \mathbb{P}_{X(s)}[X(t) \in A] \quad \mathbb{P}_x\text{-a.s.} \qquad (\text{"Markov property"})$$

for every $A \in \mathcal{B}(E)$, $x \in E$, $s, t \geq 0$.

Transition semigroup:

$$p_t(x, dy) := (\mathbb{P}_x \circ X(t)^{-1})(dy), \; t \geq 0, \; x \in E,$$

and set for $f : E \to \mathbb{R}$, bounded $\mathcal{B}(E)$-measurable

$$P_t f(x) := \int_E f(y) \; p_t(x, dy), \; x \in E, \; t \geq 0,$$

hence $P_t : \mathcal{B}_b(E) \to \mathcal{B}_b(E)$.

Feller property:

$$f \in C_b(E) \Rightarrow P_t f \in C_b(E), t \geq 0. \tag{F}$$

Then

$$E \ni x \mapsto p_t(x, dy) \in \mathcal{P}(E) \text{ continuous in the weak topology on } P(E)\ \forall t \geq 0. \tag{C_x}$$

$$[0, \infty) \ni t \mapsto p_t(x, dy) \in \mathcal{P}(E) \text{ right-continuous in the weak topology on } \mathcal{P}(E)\ \forall x \in E. \tag{rC_t}$$

Well-known:
In **all** interesting cases

$$[0, \infty) \in t \mapsto P_t f \in (C_b(E), \|\cdot\|_\infty) \textbf{ not} \text{ right continuous } \forall f \in C_b(E).$$

Also **not**, if $C_b(E)$ is replaced by $UC_b(E)$ and one assumes (F) with $UC_b(E)$ replacing $C_b(E)$.
So, $(P_t)_{t \geq 0}$ **not** C_0-semigroup on $(C_b(E), \|\cdot\|_\infty)$ (or $(UC_b(E), \|\cdot\|_\infty)$).

Hence the theory of C_0-semigroups on Banach spaces does **not** apply. If it did, $P_t, t \geq 0$, would be uniquely determined by its strong derivative at $t = 0$, i.e.,

$$Lf := \frac{d}{dt}_{\restriction t=0} P_t f = \| \cdot \|_\infty - \lim_{t \to 0} \frac{1}{t}(P_t f - f), \quad f \in D(L), \tag{3}$$

which defines a linear operator $L\colon D(L) \subset C_b(E) \to C_b(E)$ with $D(L)$ being the set of all $f \in C_b(E)$ for which the limit in (3) exists. In this case $P_t, t \geq 0$, can be recalculated from the operator $(L, D(L))$, called *infinitesimal generator* of $(P_t)_{t \geq 0}$, through Euler's formula. But as said, this is in general not possible on $(C_b(E), \| \cdot \|_\infty)$.

Ways out:
a) Assume E **locally compact** and

$$P_t(C_\infty(E)) \subset C_\infty(E),\ t \geq 0. \tag{F_∞}$$

Then $(P_t)_{t\geq 0}$ C_0-semigroup on $(C_\infty(E), \|\cdot\|_\infty)$! And e.g., if $E = \mathbb{R}^d$, (F_∞) can be verified in quite a large number of cases. But **need** locally compact, so all infinite dimensional E are out, hence e.g. **all** SPDEs are out.
b) Drop "C_0-" and define "generator" L in a different way, namly by $D(L) := R_1 C_b(E)$ and

$$L := 1 - R_1^{-1},\ \text{where } R_1 f(x) := \int_0^\infty e^{-t} P_t f(x)\, dt, f \in C_b(E).$$

But its definition uses the whole semigroup $P_t, t \in [0, \infty)$, so definitely can **not** be called an **infinitesimal** generator.

c) Consider $P_t, t \geq 0$, not as operators on $C_b(E)$, but on $L^p(E, \mu), p \in [1, \infty)$, for some suitable **reference measure** μ on $(E, \mathcal{B}(E))$, as e.g. an **invariant** or **symmetrizing** measure for $P_t, t \geq 0$. Clearly, the latter do not always exist. But it was proven in [R/Trutnau 2007] that for any semigroup $(P_t)_{t \geq 0}$ as above one can construct a probability measure μ so that each P_t uniquely extends to a continuous operator on $L^p(E, \mu)$ and these extensions form a C_0-semigroup on $L^p(E, \mu)$ which then have an infinitesimal generator $(L^\mu, D(L^\mu))$ on $L^p(E, \mu)$.
But μ not unique and generator depends on μ (so no "pointwise" theory, but only "μ-a.e. theory").
Worse: The construction μ depends on the whole semigroup $P_t, t \in [0, \infty)$. So, **again** $(L^\mu, D(L^\mu))$ is **not** really an **infinitesimal** generator of the transition semigroup of our Markov process $\mathbb{M}$.

Remark

For references to all approaches (a)-(c) above we refer to our paper [G/N/R 2022: arXiv: 2204.07484].

So, concluding: It has been an open problem whether the transition semigroup of a general Markov process $\mathbb{M}$ as above, which has the Feller property (F), is infinitesimally generated by its strong derivative at zero in a "suitable" topology on $C_b(E)$.
The first main contribution of this paper is to prove that such a "suitable" topology is the well-known mixed topology $\tau_1^{\mathscr{M}}$ on $C_b(E)$, i.e., the strongest locally convex topology on $C_b(E)$ which on $\|\cdot\|_\infty$-bounded subsets of $C_b(E)$ coincides with the topology of uniform convergence on compact subsets of E, provided $(P_t)_{t\geq 0}$, satisfies the following very general condition:

$$[0,\infty)\times K \ni (t,x) \to \; p_t(x,dy) \in \mathcal{P}(E) \text{ is continuous in the weak topology on } \mathcal{P}(E) \text{ for all compact } K \subset E. \qquad (C_{t,x})$$

In fact, this is true for very general state spaces E, more precisely, those which satisfy:

Hypothesis (E)

E is a completely regular Hausdorff topological space, such that

1. *compact subsets of E are metrizable,*
2. $\mathcal{B}(E) = \sigma(C_b(E))$.
3. *a function $\varphi : E \to \mathbb{R}$ is continuous if and only if φ is continuous on every compact subset of E.*

So, in such a very general case the transition semigroup of a Markov process with right continuous sample paths is uniquely determined by its strong derivative at zero with respect to the mixed topology $\tau_1^{\mathscr{M}}$ on $C_b(E)$ and can be reconstructed from it through an Euler formula.

2. Basic definitions

Fix a continuous weight function $\kappa : E \to (0, \infty)$ and define

$$C_\kappa(E) := \frac{1}{\kappa} C_b(E),$$
$$\|\varphi\|_\kappa := \|\kappa\, \varphi\|_\infty, \quad \varphi \in C_\kappa(E).$$

For compact $C \subset E$, define the seminorm on $C_\kappa(E)$

$$p_{\kappa,C}(\varphi) := \sup_{x \in C} |\kappa(x)\, \varphi(x)|, \quad \varphi \in C_\kappa(E)$$

and

$$\tau_\kappa^C := \text{ topology generated by the seminorms } p_{\kappa,C}, C \subset E, C \text{ compact.}$$

For every zero sequence $a_n \in [0, \infty)$, $n \in \mathbb{N}$, and every sequence $(C_n)_{n \in \mathbb{N}}$ of compacts sets in E define the seminorm on $C_\kappa(E)$

$$p_{\kappa,(C_n),(a_n)}(\varphi) := \sup_n \big(a_n\, p_{\kappa,C_n}(\varphi)\big), \quad \varphi \in C_\kappa(E).$$

and

$$\tau_\kappa^{\mathcal{M}} := \text{ topology generated by all the seminorms } p_{\kappa,(C_n),(a_n)}.$$

$\tau_\kappa^{\mathcal{M}}$ is called **mixed topology** on $C_\kappa(E)$. Clearly, $\tau_\kappa^C \subset \tau_\kappa^{\mathcal{M}} \subset \tau_{\|\cdot\|_\kappa}$ and $(C_\kappa(E), \tau_\kappa^{\mathcal{M}})$ is a locally convex topological vector space.

Remark

For $\kappa \equiv 1, \tau_1^{\mathscr{M}}$ well-known, well studied. See e.g. [Wiweger 1961], [Fremlin/Garling/Haydon 1972], [LeCam 1957], [Sentilles 1972] and the Appendix in our paper. Here we only list the following important properties:

(i) *$\tau_\kappa^{\mathscr{M}}$ is the strongest locally convex topology on $C_\kappa(E)$, which coincides on $\|\cdot\|_\kappa$-bounded sets with τ_κ^C.*

(ii) *$\tau_\kappa^{\mathscr{M}}$ is (in general) not metrizable.*

(iii) *If E satisfies Hypothesis (E), then $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ is complete.*

(iv) *A sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_\kappa(E)$ is $\tau_\kappa^{\mathscr{M}}$-convergent to $\varphi \in C_\kappa(E)$ if and only if $\{\varphi_n | n \in \mathbb{N}\}$ is $\|\cdot\|_\kappa$-bounded and τ_κ^C-convergent to φ.*

(v) *For the topological dual of $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})'$ of $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ we have*

$$(C_\kappa(E), \tau_\kappa^{\mathscr{M}})' = M_\kappa(E) = \kappa \cdot M_b(E),$$

where $M_b(E) :=$ all Radon measures on $(E, \mathcal{B}(E))$ of bounded total variation.

Below $M_\kappa(E)$ will be equipped with the weak* topology.

The **proof for (v)** in the literature is quite involved. So, let us give a short and easy proof here based on (iv) (which in turn is very easy to prove) and on the Daniell-Stone theorem, assuming that $(E, \mathcal{B}(E))$ is a Radon space, i.e. every finite positive measure on $(E, \mathcal{B}(E))$ is Radon, (which is a very weak condition):
For simplicity we assume $\kappa \equiv 1$.
Obviously, each $\mu \in M_b(E)$ is in the topological dual $(C_b(E), \tau_1^{\mathscr{M}})'$ of $(C_b(E), \tau_1^{\mathscr{M}})$. To prove the converse we first note that it is well-known that every element ℓ of the latter can be written as a difference $\ell = \ell^+ - \ell^-$, with $\ell^+, \ell^- \in (C_b(E), \tau_1^{\mathscr{M}})'$ and both are nonnegative on nonnegative elements in $C_b(E)$. Hence we may assume that ℓ itself has this property. Since $C_b(E)$ is a Stone vector lattice, which by assumption (2) in Hypothesis (E) generates $\mathscr{B}(E)$, we only have to show Daniell continuity, because then ℓ is represented by a unique finite nonnegative measure μ, which, since E is a Radon space, is in $M_b(E)$. But if $\varphi_n \in C_b(E)$, $\varphi_n \geq 0$, $n \in \mathbb{N}$, such that $\varphi_n \downarrow 0$ pointwise on E, then by (iv) and Dini's Theorem, we conclude that

$$\tau_1^{\mathscr{M}} - \lim_{n\to\infty} f_n = 0,$$

hence $\lim\limits_{n\to\infty} \ell(f_n) = 0$, and Daniell continuity holds.

3. Strongly continuous semigroups on $C_\kappa(E)$ with mixed topology $\tau_\kappa^{\mathscr{M}}$

Definition I

A family of (possibly nonlinear) operators $P = (P_t)_{t\geq 0}$ on $C_\kappa(E)$ is called a semigroup on $C_\kappa(E)$ if

(i) $P_0\varphi = \varphi$ *for all* $\varphi \in C_\kappa(E)$.

(ii) $P_sP_t\varphi = P_{s+t}\varphi$ *for all* $s, t \geq 0$ *and* $\varphi \in C_\kappa(E)$.

The family P is called a C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ if it additionally satisfies:

(iii) *The semigroup P is locally $\tau_\kappa^{\mathscr{M}}$-uniformly equicontinuous, i.e., for every $T \geq 0$, $\varepsilon > 0$, and every seminorm $p_{\kappa,(K_n),(a_n)}$, there exists a seminorm $p_{\kappa,(C_n),(b_n)}$ and $\delta > 0$ such that, for every $0 \leq t \leq T$ and $\varphi_1, \varphi_2 \in C_\kappa(E)$,*

$$p_{\kappa,(K_n),(a_n)}(P_t\varphi_1 - P_t\varphi_2) < \varepsilon \quad \text{if} \quad p_{\kappa,(C_n),(b_n)}(\varphi_1 - \varphi_2) < \delta.$$

(iv) *The semigroup P is strongly $\tau_\kappa^{\mathscr{M}}$-right continuous, i.e., for all $\varphi \in C_\kappa(E)$ and every seminorm $p_{\kappa,(K_n),(a_n)}$,*

$$\lim_{t\to 0} p_{\kappa,(K_n),(a_n)}(P_t\varphi - \varphi) = 0.$$

Theorem I

Let $P = (P_t)_{t\geq 0}$ be a semigroup of <u>linear</u> operators on $C_\kappa(E)$. Then, the following conditions are equivalent.

(a) *The semigroup P is a C_0-semigroup on $\left(C_\kappa(E), \tau_\kappa^{\mathscr{M}}\right)$.*

(b) *There exists a family of Borel measures $\{\mu_t(x,\cdot)\colon x \in E,\ t \geq 0\} \subset M_\kappa(E)$ such that:*

(1) *The map $E \ni x \mapsto \mu_t(x, B)$ is measurable for every $B \in \mathcal{B}(E)$ and $t \geq 0$.*

(2) *For every $t \geq 0, \mu_t(\cdot, dy)$ represents P_t, i.e.,*

$$P_t\varphi(x) = \int_E \varphi(y)\mu_t(x, dy)\ \forall\varphi \in C_\kappa(E), x \in E. \text{ ("measure representation")}$$

(3) *For every $T \geq 0$,*

$$\sup_{t\leq T}\sup_{x\in E}\Big(\kappa(x)\int_E \frac{|\mu_t|(x, dy)}{\kappa(y)}\Big) < \infty\,.$$

(4) *For every $T \geq 0$ and every compact $C \subset E$, the family of measures*

$$\left\{\frac{\kappa(x)\,|\mu_t|\,(x, dy)}{\kappa(y)} : x \in C,\ t \in [0, T]\right\}$$

is tight.

(5) *For every $x \in E$ and any sequence $(x_n) \subset E$ with $\lim_{n\to\infty} x_n = x$ (in E), we have*

$$\lim_{(t,x_n)\to(0,x)} \mu_t(x_n,\cdot) = \delta_x \ (= \text{Dirac measure in } x) \text{ in } M_\kappa(E).$$

Proposition I

(4) and (5) above holds, if for all compact $C \subset E$ and all $\varphi \in C_\kappa(E)$

$$[0,\infty) \times C \ni (t,x) \longmapsto \int_E \varphi(y)\, \mu_t(x,dy) \text{ is continuous,} \qquad (C_{t,x})$$

provided E is a Prokhorov space, i.e. every compact subset of M_b^+ is tight (which is the case for all examples below).

Example 1 (Variational solutions to locally monotone SDEs on Hilbert spaces with norm topology) Let:
$E := H$ be a separable Hilbert space with dual H^*.
$V :=$ reflexive Banach space such that $V \subset H$ continuously and densely.
Hence

$$V \subset H \,(\equiv H^*) \subset V^* \text{ ("Gelfand triple")}$$

continuously and densely. Let $W(t),\ t \in [0, \infty)$, be a cylindrical Wiener process in a separable Hilbert space U on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $\mathcal{F}_t,\ t \in [0, \infty)$. Consider the following SDE on H:

$$dX(t) = A(X(t))dt + B(X(t))dW(t),\ X(0) = x \in H, \qquad (SDE)_v$$

where

$$A : V \to V^*;\ B : V \to L_2(U, H)$$

are measurable, where $L_2(U, H)$ denotes the set of all Hilbert-Schmidt operators from U to H.

Assume that the usual local monotonicity conditions from [Liu/R. 2015] are fulfilled. Hence by [Liu/R. 2015] $(SDE)_v$ has a unique solution $X(t,x)$, $t \geq 0$, $x \in H$. In particular, the laws

$$\mathbb{P}_x := \mathbb{P} \circ X(\cdot, x)^{-1}, x \in H,$$

form a Markov process with state space H.
Consider the corresponding transition semigroup, i.e. for $\varphi \in C_b(H)$, $x \in H$, $t \geq 0$,

$$\begin{aligned} P_t\varphi(x) :&= \mathbb{E}[\varphi(X(t,x))] = \int_\Omega \varphi(X(t,x)(\omega))\mathbb{P}(d\omega) \\ &= \int \varphi(y)\ \mu_t(x,dy), \end{aligned}$$

where

$$\mu_t(x,dy) := (\mathbb{P} \circ X(t,x)^{-1})(dy).$$

Let $m \in [1, \infty)$ and

$$\kappa(x) := (1 + \|x\|_H^m)^{-1}, \quad x \in H.$$

Then:

Theorem I + Proposition I $\Rightarrow (P_t)_{t\geq 0}$ is C_0-semigroup on $(C_\kappa(H), \tau_\kappa^{\mathscr{M}})$

Remark

The above framework from [Liu/R. 2015] covers a large class of SPDEs including the stochastic heat equation, the stochastic p-Laplace equation, the stochastic slow diffusion-porous media equation, the stochastic fast diffusion-porous media equation, both with general diffusivity, the perturbed stochastic Burgers equation and the stochastic 2D Navier-Stokes equation.

Example 2 (Mild solutions to SDEs on Hilbert spaces with bounded weak topology)
Let

$H :=$ separable Hilbert space equipped with bounded weak topology τ_{bw}.

Recall:

$B \subset H$ is τ_{bw}-closed if its intersection with every weakly compact set in H is weakly closed. Then $C_b((H, \tau_{bw})) =$ bounded sequentially weakly continuous functions on H.

Consider the following SDE on H:

$$dX(t) = (AX(t) + F(X(t)))\,dt + G(X(t))\,dW(t), \quad X(0) = x \in H, \qquad (SDE)_m$$

where W is a cylindrical Wiener process on a separable Hilbert space U, as in Example 1.
Assume:

- A generates a C_0-semigroup $T_t,\ t \geq 0$, on H,
- $F : H \to H$ Lipschitz continuous.
- $G : E \to L(U, E)$ ($:=$ all continuous linear operators from U to H) is strongly measurable such that

$$\|T_t G(x)\|^2_{L_2(U,H)} \leq k(t)(1 + \|x\|^2_H)\,, \quad x \in H,$$

and

$$\|T_t(G(x) - G(y))\|^2_{L_2(U,H)} \leq k(t)\|x - y\|^2_H\,, \quad x, y \in H,$$

where $k \in L^1_{loc}(0, \infty)$, $k \geq 0$.

Then $(SDE)_m$ has a unique mild solution $X(t,x)$, $t \geq 0$, $x \in H$, in H.
As in Example 1 let for $\varphi \in C_b(H)$, $t \geq 0$, $x \in H$,

$$P_t\varphi(x) := \mathbb{E}[\varphi(X(t,x))],$$

and for $m \geq 1$,

$$\kappa(x) : (1 + \|x\|_H^m)^{-1}.$$

Then, if each $T_t = e^{tA}$, $t > 0$, is compact on $(H, \|\cdot\|_H)$:

Theorem I + Proposition I $\Rightarrow$ $(P_t)_{t\geq 0}$ is C_0-semigroup on $(C_\kappa(H, \tau_{bw}), \tau_\kappa^{\mathscr{M}})$.

Example 3 (Generalized Mehler semigroups on Banach spaces)
Let E be a separable Banach space and let $\kappa = 1$. Let $(T_t)_{t\geq 0}$ be C_0-semigroup of linear operators on E. Furthermore, let μ_t, $t \in [0,\infty)$, be probability measures on $(E, \mathcal{B}(E))$ such that:

$$[0,\infty) \ni t \longmapsto \mu_t \in P(E) \text{ is weakly continuous.} \tag{GMS1}$$

$$\mu_{t+s} = \left(\mu_t \circ T_s^{-1}\right) * \mu_s \quad t,s \in [0,\infty). \tag{GMS2}$$

Define for $t \in [0,\infty)$, $x \in E$

$$P_t\varphi(x) := \int_E \varphi(T_t x + y)\, \mu_t(dy), \quad \varphi \in C_b(E).$$

Then $(P_t)_{t\geq 0}$ is by (GMS2) a semigroup of linear operators on $C_b(E)$, called "generalized Mehler semigroup" = transition semigroup of Ornstein-Uhlenbeck process with Levy noise, i.e. solutions to the following SDEs on E

$$dX(t) = AX(t)dt + dY(t),$$

where A is the generator of (T_t) on E and $Y(t)$, $t \geq 0$, is the underlying Levy process. (See [Bogachev/R./Schmuland: PTRF 1995], [Fuhrman/R.: POTA 2000]).

Then:

Theorem I + Proposition I $\Rightarrow (P_t)_{t\geq 0}$ is C_0-semigroup on $(C_b(E), \tau_1^{\mathscr{M}})$ and, provided E is reflexive, also on $(C_b(E, \tau_{bw}), \tau_1^{\mathscr{M}})$.

4. Infinitesimal generators

Definition II

Let $(P_t)_{t\geq 0}$ be a C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$. Then, we define its infinitesimal generator $(L, D(L))$ by

$$L\varphi := \tau_\kappa^{\mathscr{M}} - \lim_{t\to 0} \frac{P_t\varphi - \varphi}{t} \quad \text{for } \varphi \in D(L) := \left\{ \psi \in C_\kappa(E) \colon \exists\, \tau_\kappa^{\mathscr{M}} - \lim_{t\to 0} \frac{P_t\psi - \psi}{t} \right\}.$$

Proposition II

Let $P = (P_t)_{t\geq 0}$ be a C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ consisting of linear operators with generator $(L, D(L))$. Then, the following holds:

(a) *$\tau_\kappa^{\mathscr{M}}$-closure of $D(L) = C_\kappa(E)$.*

(b) *$(L, D(L))$ is $\tau_\kappa^{\mathscr{M}}$-closed.*

(c) *For every $\varphi \in D(L)$, $P_t\varphi \in D(L)$ and $LP_t\varphi = P_tL\varphi$. In particular, $P_t \colon D(L) \to D(L)$ is continuous in the $\tau_\kappa^{\mathscr{M}}$-graph topology of L on $D(L)$.*

Proposition II (continued)

(d) *For every* $\varphi \in D(L)$

$$P_t\varphi - \varphi = \int_0^t P_s L\varphi \, ds \, .$$

Moreover, for every $\varphi \in C_\kappa(E)$

$$\int_0^t P_s\varphi \, ds \in D(L), \quad \textit{and} \quad P_t\varphi - \varphi = L\int_0^t P_s\varphi \, ds \, .$$

(e) *The Euler formula holds, i.e., for all* $\varphi \in C_\kappa(E)$

$$P_t\varphi = \tau_\kappa^{\mathscr{M}} - \lim_{n\to\infty} \left(\frac{n}{t}\left(\frac{n}{t} - L\right)^{-1} \right)^n \varphi.$$

Remark

It is very easy to check that in the linear case our C_0*-semigroups on* $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ *are special cases of the bi-continuous semigroups introduced in [Kühnemund: Semigroup Forum 2003].*

Definition III

Let $P = (P_t)_{t\geq 0}$ *be a* C_0*-semigroup on* $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$. *We say that* $\varphi \in D(L_w) \subset C_\kappa(E)$ *if and only if there exists some* $f \in C_\kappa(E)$ *such that* $\frac{1}{t}(P_t\varphi - \varphi) \xrightarrow{t\to 0} f$ *weakly in* $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$, *i.e.,*

$$\lim_{t\to 0} \int_E \frac{P_t\varphi(x) - \varphi(x)}{t} \, \nu(dx) = \int_E f(x) \, \nu(dx) \quad \text{for each } \nu \in M_\kappa(E)$$

In this case, we define the operator L_w *by the formula*

$$L_w\varphi = f.$$

We say that L_w *is the weak generator of the* C_0*-semigroup* P *on* $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ *with domain* $D(L_w)$.

Theorem III

Let $(P_t)_{t\geq 0}$ be a C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ consisting of linear operators. Then,

$$L = L_w.$$

Moreoever, $\varphi \in D(L)$ if and only if

$$\sup_{t\leq 1}\left(\frac{1}{t}\left\|P_t\varphi - \varphi\right\|_\kappa\right) < \infty,$$

$$f(x) := \lim_{t\to 0}\frac{P_t\varphi(x) - \varphi(x)}{t} \quad \text{exists for all } x \in E,$$

and $f \in C_\kappa(E)$. In this case, $f = L\varphi$.

Definition IV

Let P_t, $t \geq 0$, be a C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathcal{M}})$ with infinitesimal generator $(L, D(L))$ and let $(L_0, D(L_0))$ be a densely defined (i.e., $D(L_0)$ is dense in $(C_\kappa(E), \tau_\kappa^{\mathcal{M}})$) linear operator on $C_\kappa(E)$ such that $L_0 \subset L$ (i.e., $D(L_0) \subset D(L)$ and $L_0\varphi = L\varphi$ for all $\varphi \in D(L_0)$).

(i) *The operator $(L_0, D(L_0))$ is called a core operator for $(L, D(L))$ if the closure of its graph $\Gamma(L_0) = \{(\varphi, L_0\varphi) \in C_\kappa(E) \times C_\kappa(E) \mid \varphi \in D(L_0)\}$ in $(C_\kappa(E), \tau_\kappa^{\mathcal{M}}) \times (C_\kappa(E), \tau_\kappa^{\mathcal{M}})$ coincides with the graph $\Gamma(L)$.*

(ii) *Suppose that κ is bounded and that $(P_t)_{t\geq 0}$ is Markov, i.e. $C_\kappa(E) \ni \varphi \geq 0 \Rightarrow P_t\varphi \geq 0$, $t \geq 0$; and $P_t 1 = 1$, $t \geq 0$. The operator $(L_0, D(L_0))$ is called a Markov core operator for $(L, D(L))$ if $(L, D(L))$ is the only operator with $L_0 \subset L$, which is the infinitesimal generator of a Markov C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathcal{M}})$.*

Remark

core operator $\Rightarrow$ Markov core operator

Theorem IV (Sufficient condition for Markov uniqueness)

Let κ be bounded and $(P_t)_{t\geq 0}$ be a Markov C_0-semigroup on $(C_\kappa(E),\, \tau_\kappa^{\mathcal{M}})$ with infinitesimal generator $(L,\, D(L))$ and let $(L_0,\, D(L_0))$ be a densely defined linear operator on $C_\kappa(E)$ such that $L_0 \subset L$. Suppose that, for every $x \in E$, the Fokker-Planck-Kolmogorov equation

$$\int \varphi(y)\, \nu_t(dy) = \int \varphi(y)\, \delta_x(dy) + \int_0^t \int L_0\varphi(y)\, \nu_s(dy)\, ds, \quad t \geq 0,\, \varphi \in D(L_0),$$

(see [BKRS 2015]) has a unique solution $(\nu_t)_{t\geq 0} \in C([0,\infty),\, M_\kappa^+(E))$, such that $\nu_t(E) = 1$ for all $t \in [0,\infty)$ and such that

$$\int_0^T \int_E \frac{1}{\kappa}\, d\nu_t\, dt < \infty, \quad T > 0.$$

Then $(L_0,\, D(L_0))$ is a Markov core operator for $(L,\, D(L))$ on $(C_\kappa(E), \tau_\kappa^{\mathcal{M}})$.

Application I (SDEs on $\mathbb{R}^d$)

Let $E := \mathbb{R}^d$ and $(\Omega, F, \mathbb{P})$ be a complete probability space with normal filtration $\mathscr{F}_t$, $t \geq 0$, and $(W_t)_{t \geq 0}$ be a (standard) $(\mathscr{F}_t)$-Wiener process on $\mathbb{R}^{d_1}$. Let $M(d \times d_1, \mathbb{R})$ denote the set of real $d \times d_1$-matrices equipped with the Hilbert-Schmidt norm $\|\cdot\|$ and let

$$\sigma\colon \mathbb{R}^d \to M(d \times d_1, \mathbb{R}),$$
$$b\colon \mathbb{R}^d \to \mathbb{R}^d,$$

be continuous maps satisfying the following standard assumptions. There exist $K \in L^1_{loc}([0,\infty))$ and $C \in [0,\infty)$ such that for all $R \geq 0$,

$$2\langle x-y,\, b(x)-b(y)\rangle + \|\sigma(x)-\sigma(y)\|^2 \leq K(R)|x-y|^2, \;\; x, y \in \mathbb{R}^d, |x|, |y| \leq R, \tag{$b\sigma1$}$$

and

$$2\langle x, b(x)\rangle + \|\sigma(x)\|^2 \leq C(1+|x|^2), \quad \text{for all } x \in \mathbb{R}^d. \tag{$b\sigma2$}$$

Here $\langle\, ,\, \rangle$ denotes the Euclideam inner product on $\mathbb{R}^d$ and $|\cdot|$ the corresponding norm.

Then it is well-known that

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \quad X(0) = x \in \mathbb{R}^d, \qquad (SDE)_d$$

has a unique strong solution $X(t,x)$, $t \geq 0$. Let for $m \geq 1$

$$\kappa(x) := (1 + |x|^m)^{-1}$$

and for $\varphi \in C_\kappa(\mathbb{R}^d)$, $t \geq 0$, $x \in \mathbb{R}^d$,

$$P_t\varphi(x) := \mathbb{E}_{\mathbb{P}}[\varphi(X(t,x))] = \int \varphi(y)\mu_t(x,dy),$$

where

$$\mu_t(x,dy) := (\mathbb{P} \circ X(t,x)^{-1})(dy) \in M_\kappa(\mathbb{R}^d).$$

Then:

$$\boxed{\text{Theorem I} + \text{Proposition I} \Rightarrow (P_t)_{t\geq 0}\ C_0\text{-semigroup on } (C_\kappa(\mathbb{R}^d), \tau_\kappa^{\mathscr{M}}).}$$

Let $(L, D(L))$ be its infinitesimal generator and let us consider the Kolmogorov operator $(L_0, D(L_0))$ corresponding to $(SDE)_d$, defined as

$$L_0\varphi(x) := \frac{1}{2}\sum_{i,j=1}^{d}(\sigma(x)\sigma(x)^T)_{i,j}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\varphi(x) + \langle b(x), \nabla\varphi(x)\rangle, \quad x \in \mathbb{R}^d,$$
$$\varphi \in D(L_0) := C_b^2(\mathbb{R}^d).$$

Assume

$$\sup_{x\in\mathbb{R}^d}\frac{|b(x)| + \|\sigma(x)\|}{1+|x|^m} < \infty \qquad (b\sigma 3)$$

and:

$$\text{For every compact } K \subset \mathbb{R}^d \text{ there exists } c_K \in (0,\infty) \text{ such that for all } \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \qquad (\sigma 1)$$
$$\sum_{i,j=1}^{d}(\sigma(x)\sigma(x)^T)_{i,j}\xi_i\xi_j \geq c_K|\xi|^2, \quad x \in \mathbb{R}^d.$$

Each $(\sigma\sigma^T)_{i,j}$ is locally in $VMO(\mathbb{R}^d)$. $(\sigma 2)$

We recall that a $\mathscr{B}(\mathbb{R}^d)$-measurable function $g : \mathbb{R}^d \to \mathbb{R}$ belongs to the class $VMO(\mathbb{R}^d)$, if it is bounded and for

$$O(g,R) := \sup_{x\in\mathbb{R}^d}\sup_{r\le R}|B_r(x)|^{-2}\iint\limits_{y,z\in B_r(x)}|g(y)-g(z)|\,dydz, \quad R\in(0,\infty),$$

we have

$$\lim_{R\to 0} O(g,R) = 0,$$

where $B_r(x)$ denotes the ball in $\mathbb{R}^d$ of radius r, centered at $x \in \mathbb{R}^d$, and $|B_r(x)|$ its Lebesgue measure. g belongs locally to the class $VMO(\mathbb{R}^d)$ if $\zeta g \in VMO(\mathbb{R}^d)$ for every $\zeta \in C_0^\infty(\mathbb{R}^d)$.
Then:

Theorem IV + Theorem 9.3.6 in [BKRS 2015] $\Rightarrow (L_0, D(L_0))$ is Markov core operator for $(L, D(L))$.

Remark

(i) *We have corresponding applications for SDE's on Hilbert spaces, hence for SPDEs.*

(ii) *Consider the* C_0*-semigroup on* $(C_b(H), \tau_1^{\mathscr{M}})$ *given by a generalized Mehler semigroup on a Hilbert space H (see Example 3 above), i.e. the transition semigroup corresponding to*

$$dX(t) = AX(t)dt + dY(t),\ X(0) = x, \quad \text{"OU-process with Levy noise"}$$

and let $(L, D(L))$ *be its infinitesimal generator.*
Then we have proved (under fairly weak conditions) that the corresponding Kolmogorov operator $(L_0, D(L_0))$ *is even a core operator for* $(L, D(L))$*. The Kolmogorov operator in this case is a pseudo differential operator, more precisely,*

$$L_0\varphi(x) = \int_H \big(i\langle A^*\xi, x\rangle - \lambda(\xi)\big)e^{i\langle \xi, x\rangle}\nu(d\xi), \quad x \in H,$$

where $\varphi(x) := \int e^{i\langle x,\xi\rangle}\nu(d\xi)$, $x \in E$, *and* ν *runs in a suitable large enough class of finite measures on* $(H, \mathcal{B}(H))$, A^* *is the adjoint of A and* λ *is the symbol of the Kolmogorov operator associated to the Levy process* $Y(t)$, $t \geq 0$.

5. Convex C_0-semigroups on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$, viscosity solutions and nonlinear Markov processes

Now consider a C_0-semigroup on $(C_\kappa(E), \tau_\kappa^{\mathscr{M}})$ consisting of convex increasing operators on $C_\kappa(E)$ with infinitesimal generator $(L, D(L))$ and the corresponding abstract differential equation of the form

$$u'(t) = Lu(t), \quad \text{for all } t > 0. \tag{ade}$$

In the following, an operator $T\colon C_\kappa(E) \to C_\kappa(E)$ is called *increasing* if

$$T\varphi_1 \leq T\varphi_2 \quad \text{for all } \varphi_1, \varphi_2 \in C_\kappa(E) \text{ with } \varphi_1 \leq \varphi_2,$$

and *convex* if

$$T\big(\lambda\varphi_1 + (1-\lambda)\varphi_2\big) \leq \lambda T\varphi_1 + (1-\lambda)T\varphi_2$$

for all $\lambda \in [0,1]$ and $\varphi_1, \varphi_2 \in C_\kappa(E)$.

Definition V

Let $L\colon D \to C_\kappa(E)$ be a nonlinear operator, defined on a nonempty set $D \subset C_\kappa(E)$. We say that $u\colon [0,\infty) \to C_\kappa(E)$ is a D-viscosity subsolution to the abstract differential equation (ade) *if u is continuous w.r.t. the mixed topology $\tau_\kappa^{\mathscr{M}}$ and, for every $t > 0$, $x \in E$, and every differentiable function $\psi\colon (0,\infty) \to C_\kappa(E)$ with $\psi(t) \in D$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$, and $\psi(s) \geq u(s)$ for all $s > 0$,*

$$\big(\psi'(t)\big)(x) \leq \big(L\psi(t)\big)(x).$$

Analogously, u is called a D-viscosity supersolution to (ade) *if $u\colon [0,\infty) \to C_\kappa(E)$ is continuous and, for every $t > 0$, $x \in M$, and every differentiable function $\psi\colon (0,\infty) \to C_\kappa(E)$ with $\psi(t) \in D$, $\big(\psi(t)\big)(x) = \big(u(t)\big)(x)$, and $\psi(s) \leq u(s)$ for all $s > 0$,*

$$\big(\psi'(t)\big)(x) \geq \big(L\psi(t)\big)(x).$$

We say that u is a D-viscosity solution to (ade) *if u is a viscosity subsolution and a viscosity supersolution.*

Theorem V

Let $(P_t)_{t\geq 0}$ be a C_0-semigroup on $(C_\kappa, \tau_\kappa^{\mathscr{M}})$ consisting of convex increasing operators with infinitesimal generator $(L, D(L))$. Then, for every $\varphi \in C_\kappa(E)$, the function $u\colon [0,\infty) \to C_\kappa(E),\ t \mapsto P_t\varphi$ is a $D(L)$-viscosity solution to the abstract initial value problem

$$\begin{aligned} u'(t) =& Lu(t), \quad \text{for all } t > 0, \\ u(0) =& \varphi. \end{aligned}$$

Finally, we have a stochastic representation for $(P_t)_{t\geq 0}$ using convex expectations. We recall:

Definition VI

Let $(\Omega, \mathcal{F})$ be a measurable space. A functional $\mathcal{E}\colon B_b(\Omega, \mathcal{F}) \to \mathbb{R}$ is called a convex expectation if, for all $X, Y \in B_b(\Omega, \mathcal{F})$ and $\lambda \in [0, 1]$,

(i) $\mathcal{E}(X) \leq \mathcal{E}(Y)$ *if* $X \leq Y$,

(ii) $\mathcal{E}(m) = m$ *for all constants* $m \in \mathbb{R}$,

(iii) $\mathcal{E}\big(\lambda X + (1-\lambda)Y\big) \leq \lambda\mathcal{E}(X) + (1-\lambda)\mathcal{E}(Y)$.

We say that $(\Omega, \mathcal{F}, \mathcal{E})$ is a convex expectation space if there exists a set of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{F})$ and a function $\alpha\colon \mathcal{P} \to [0, \infty)$ such that

$$\mathcal{E}(X) = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}(X) - \alpha(\mathbb{P}) \quad \text{for all } X \in B_b(\Omega, \mathcal{F}),$$

where $\mathbb{E}_{\mathbb{P}}(\cdot)$ denotes the expectation w.r.t. to the probability measure $\mathbb{P}$.

The following theorem is a consequence of [Denk/ Kupper/Nendel 2018] and the fact that the $\tau_\kappa^{\mathscr{M}}$-continuity of P_t implies Daniell continuity of P_t, for $t \geq 0$.

Theorem VI

Assume that E is a Polish space, $\kappa \equiv 1$, and P is a C_0-semigroup of increasing convex operators with $P_t m = m$ for all $t \geq 0$ and $m \in \mathbb{R}$. Then, there exists a quadruple $(\Omega, \mathcal{F}, (\mathcal{E}^x)_{x\in E}, (X(t))_{t\geq 0})$ such that

(i) *$X(t)\colon \Omega \to E$ is $\mathcal{F}$-$\mathcal{B}$-measurable for all $t \geq 0$,*

(ii) *$(\Omega, \mathcal{F}, \mathcal{E}^x)$ is a convex expectation space with $\mathcal{E}^x(\varphi(X(0))) = \varphi(x)$ for all $x \in E$ and $\varphi \in C_b(E)$,*

(iii) *For all $0 \leq s < t$, $n \in \mathbb{N}$, $0 \leq t_1 < \ldots < t_n \leq s$ and $\psi \in C_b(E^{n+1})$,*

$$\mathcal{E}^x\big(\psi(X(t_1), \ldots, X(t_n), X(t))\big) = \mathcal{E}^x\left(\big(P_{t-s}\psi(X(t_1), \ldots, X(t_n), \,\cdot\,)\big)(X(s))\right).$$

In particular,

$$\big(P_t\varphi\big)(x) = \mathcal{E}^x(\varphi(X(t))).$$

for all $t \geq 0$, $x \in E$, and $\varphi \in C_b(E)$.

The quadruple $(\Omega, \mathcal{F}, (\mathcal{E}^x)_{x\in E}, (X(t))_{t\geq 0})$ can be seen as a nonlinear version of a Markov process.

THANK YOU VERY MUCH, NICK!

AND ALL THE BEST FOR MANY MORE HAPPY AND SCIENTIFICALLY PRODUCTIVE YEARS!