Regularisation by noise with subcritical drifts or multiplicative noise

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Joint work with Konstantinos Dareiotis (University of Leeds) and Lucio Galeati (University of Bonn)

Regularisation by noise

Bad:

$$X_t = x_0 + \int_0^t b_s(X_s)\, ds$$

- Non-uniqueness of solutions (if $b \in C^\alpha$, $\alpha < 1$)
- Non-existence of solutions (if b is discontinuous)
- No meaning of the equation (if b is a distribution)

Good:

$$X_t = x_0 + \int_0^t b_s(X_s)\, ds + \text{noise}_t$$

- Weak existence/uniqueness
- Strong existence/uniqueness
- Path-by-path uniqueness
- Regular flow of solutions
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Recap: fractional Brownian motion

For $H \in (0,\infty) \setminus \mathbb{N}$,

$$B^H = \mathcal{I}^{H+1/2}\xi,$$

where ξ is white noise and $\mathcal{I}^\beta$ is 'integration' of order β.

- $H = 1/2$: Classical BM
- Gaussian
- Scale invariant $(B^H_t)_{t\geq 0} \overset{\text{law}}{=} (\lambda^{-H} B^H_{\lambda t})_{t \geq 0}$
- For any $\varepsilon > 0$, a.s. $B^H \in C^{H-\varepsilon} \setminus C^H$
- (germ) Markovian if and only if $H = k + 1/2$, $k \in \mathbb{N}$ $\rightsquigarrow$ no Itô's formula, Kolmogorov equation, Zvonkin transformation, martingale problem...

I. Drift close to criticality

Scaling

$$X_t = \int_0^t b_r(X_r)dr + B_t^H.$$

By $(B_t^H)_{t\geq 0} \stackrel{\text{law}}{=} (\lambda^{-H} B_{\lambda t}^H)_{t\geq 0}$, we can rescale and get a similar equation with new drift

$$b_t^\lambda(x) = \lambda^{1-H} b(\lambda t, \lambda^H x).$$

We call a space V of functions (or distributions) on $\mathbb{R}_+ \times \mathbb{R}^d$ critical/subcritical/supercritical if for the rescaled drift coefficient one has $\|b^\lambda\|_V = \lambda^{\gamma}\|b\|_V$ with $\gamma = 0/\gamma > 0/\gamma < 0$.

Goal: Develop solution theory in optimal regimes for *any H*

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Example $V = C_x^\alpha$

Subcriticality condition:

$$\alpha > 1 - \frac{1}{H}.$$

Weak E&U for $H \in \mathbb{N}_+ + 1/2$ [Chaudru de Raynal-Menozzi '21].

$H = 1/2$
Strong E&U for $\alpha \geq 0$ [Zvonkin '74, Veretennikov '80, Davie '07]
Weak E&U for $\alpha > -1/2$ [Flandoli-Issoglio-Russo '16]

Non-Markovian H

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Subcriticality condition (Ladyzhenskaya-Prodi-Serrin condition):

$$\frac{1}{q} + \frac{Hd}{p} < 1 - H.$$

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Critical case with $=$ [Krylov 19-20-21, Röckner-Zhao '21]

$H \in (0, 1/2)$

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Main result: full subcritical regime if $q \in (1, 2]$

$$X_t = x_0 + \int_0^t b_r(X_r)\, dr + B_t^H$$

Theorem (Galeati-G '22)

Assume

$$H \in (0, \infty) \setminus \mathbb{N}, \qquad q \in (1, 2], \qquad \alpha \in \left(1 - \frac{1}{q' H}, 1\right).$$

Then if $b \in L_t^q C_x^\alpha$, then

- *Strong existence and path-by-path uniqueness holds*
- *The solutions form a flow of diffeomorphisms*
- *The solutions are Malliavin-differentiable (in fact Fréchet differentiable in directions of $2 + \varepsilon$-variation paths)*
- *The solutions are ρ-irregular for any $\rho < 1/(2H)$*

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Brownian case

The results are also new in the classical case $H = 1/2$.
Kolmogorov PDE

$$\partial_t u - \tfrac{1}{2}\Delta u = b \cdot \nabla u.$$

If $b \in L^q_t C^\alpha_x$ with $q \in (1,2)$, naive power counting fails:

$$b \in L^q_t C^\alpha_x \rightsquigarrow u \in L^\infty_t C^{\alpha+2-2/q}_x \rightsquigarrow b \cdot \nabla u \in L^q_t C^{\alpha+1-2/q}_x.$$

Also, for $q < 2$ no Girsanov is available (even if b is spatially constant).

Overall, we do not use Girsanov. It would be applicable if and only if the critical exponent $1 - 1/(q'H) < 0$.

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ρ-irregularity

Definition (Catellier-Gubinelli '16)

A function $h \in C([0,1], \mathbb{R}^d)$ is ρ-irregular if there exists a constant $\gamma > 1/2$ such that for all $0 \le s \le t \le 1$, $\xi \in \mathbb{R}^d$

$$\left| \int_s^t e^{i\xi \cdot h_r} \, dr \right| \le |\xi|^{-\rho} |t-s|^{\gamma}$$

In terms of the local time L of h:

$$\mathrm{LHS} = \int_s^t \int_{\mathbb{R}^d} e^{i\xi \cdot x} \delta_0(x - h_r) \, dx \, dr = \int_{\mathbb{R}^d} e^{i\xi \cdot x} L_{s,t}(x) \, dx = \mathcal{F}(L_{s,t})(\xi).$$

Theorem (Galeati-G '22)

If there exists a control w such that for all $m < \infty$

$$\|\varphi_t - \mathbb{E}(\varphi_t | \mathcal{F}_s)\|_{L^m(\Omega)} \lesssim w(s,t)^{1/2} (t-s)^H,$$

then $B^H + \varphi$ is ρ-irregular for any $\rho < 1/(2H)$.

Supercritical case: counterexample to weak uniqueness

Let $q \in (1, \infty)$, $-1 < \alpha < 1 - 1/(q'H)$, $d = 1$. Define

$$b_t(x) = t^{-1/(q+\varepsilon)} \mathrm{sign}(x)|x|^\alpha \in L_t^q C_x^\alpha.$$

Lemma (Chaudru de Raynal '18, Galeati-G '22)

On some probability space there exist two continuous processes X^+, X^- and a fBM B^H, such that

$$X_t^\pm = \int_0^t b_s(X_s^\pm)\, ds + B_s^H$$

and up to a stopping time $\tau = \tau(B^H) > 0$ we have

$$X^+|_{(0,\tau]} > 0, \qquad X^-|_{(0,\tau]} < 0.$$

II. Multiplicative noise

Rough/Young/Ordinary differential equations

So far, everything with additive noise. How about multiplicative?

$$X_t = x_0 + \int_0^t b_r(X_r)\,dr + \int_0^t \sigma(X_r)\,dB_r^H.$$

Recall: B^H is not a semimartingale unless $H = 1/2$ or $H > 1$.
Noise is both friend and enemy.

Three regimes:

- $H \in (1, \infty) \setminus \mathbb{N}$: classical integration
- $H \in (1/2, 1)$: Young integration
- $H \in (1/3, 1/2]$: rough integration
  $\rightsquigarrow$ need an adapted rough path lift of B^H

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Mutliplicative noise - strong well-posedness

Theorem (Dareiotis-G '22)

If $H \in (1/3, \infty) \setminus \mathbb{N}$, $\alpha > (1 - 1/(2H)) \vee 0$, $b \in C^{\alpha}$, $\sigma \in C^{\lfloor 1/H \rfloor + 1}$, and σ is uniformly elliptic, then strong existence and path-by-path uniqueness holds.

For $H \in (1/3, 1/2)$ we get $\alpha > 0$ instead of $\alpha > 1 - 1/(2H)$.
This is not due to the distributional drift, which we can handle in a weaker form:

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Multiplicative noise - weak existence

Theorem (Dareiotis-G '22)

Assume $H \in (1/3, 1/2]$, $\alpha > 1/2 - 1/(2H)$, $b \in C^\alpha$, $\sigma \in C^2$. *Then there exists a filtered probability space* $(\bar{\Omega}, \bar{\mathbb{F}}, \bar{\mathbb{P}})$ *with a* $\bar{\mathbb{F}}$*-fBM* $\bar{B}^H$ *that has an adapted rough path lift and* $\bar{\mathbb{F}}$*-adapted stochastic processes* $(\bar{X}_t)_{t\in[s_0,1]}$ *and* $(\bar{D}_t)_{t\in[s_0,1]}$ *such that*

- $\bar{\mathbb{P}}$*-almost surely* $(\bar{X}, \sigma(\bar{X})) \in \mathcal{D}^\gamma_{B^H}$ *for some* $\gamma > 1 - H$;
- $\bar{\mathbb{P}}$*-almost surely for all* $t \in [0, 1]$ *it holds that*

$$\bar{X}_t = \bar{D}_t + \int_0^t \sigma(\bar{X}_s)\, d\bar{B}^H_s,$$

- *For any sequence* $(b^n)_{n\in\mathbb{N}}$ *of smooth functions converging to* b *in* $\mathcal{C}^\alpha$, *one has* $\bar{\mathbb{P}}$*-almost surely for all* $t \in [0, 1]$

$$\bar{D}_t = \lim_{n\to\infty} \int_0^t b^n(\bar{X}_s)\, ds.$$

Thank you for your attention!