

# Analysis of domain decomposition preconditioners for non-self-adjoint or indefinite problems

Marcella Bonazzoli<sup>1</sup>, Xavier Claeys<sup>2</sup>, Frédéric Nataf<sup>2</sup>,  
Pierre-Henri Tournier<sup>2</sup>

<sup>1</sup>Inria, ENSTA Paris, Institut Polytechnique de Paris, Palaiseau, France

<sup>2</sup>Sorbonne Université, Inria, LJLL, Paris, France

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# Outline

- 1 Introduction and state of art
- 2 Setting
- 3 General convergence theory
- 4 Reaction-convection-diffusion equation
- 5 Conclusion

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# Domain decomposition preconditioning

$A\mathbf{u} = \mathbf{b} \Rightarrow$  linear systems solvers:

		robustness	memory cost
direct	(LU, MUMPS...)	✓	✗
iterative	(CG, GMRES...)	✗	✓

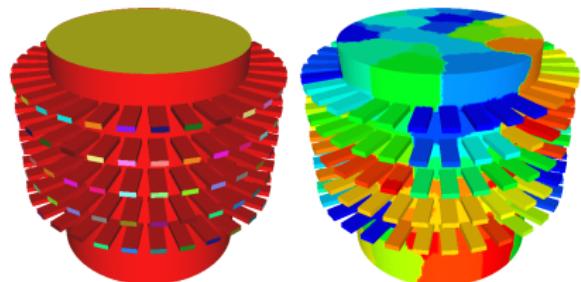
$\Rightarrow$  preconditioner necessary for iterative solvers  $M^{-1}A\mathbf{u} = M^{-1}\mathbf{b}$

$\Rightarrow$  choice: overlapping Schwarz Domain Decomposition (DD) preconditioner

Domain  $\Omega$  decomposed into  
 $N$  overlapping subdomains  $\Omega_j$

$\Rightarrow$  subproblems can be solved

- with direct solvers
- concurrently, in parallel



Transmission conditions on the interfaces

# Non-self-adjoint or indefinite problems

Helmholtz, time-harmonic Maxwell, reaction-convection-diffusion equations  
⇒  $A$  non symmetric/self-adjoint or indefinite

Example      
$$\begin{cases} c_0 u + \operatorname{div}(\mathbf{a} u) - \operatorname{div}(\nu \nabla u) = f & \text{in } \Omega \\ \nu \frac{\partial u}{\partial n} - \frac{1}{2} \mathbf{a} \cdot \mathbf{n} u + \alpha u = g & \text{on } \Gamma_R \\ u = 0 & \text{on } \Gamma_D \end{cases}$$

Find  $u \in H_{0,D}^1(\Omega) = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D \}$  such that

$$a(u, v) = F(v) \quad \text{for all } v \in H_{0,D}^1(\Omega)$$

$$a(u, v) = \int_{\Omega} \left( \tilde{c} u v + \frac{1}{2} \mathbf{a} \cdot \nabla u v - \frac{1}{2} u \mathbf{a} \cdot \nabla v + \nu \nabla u \cdot \nabla v \right) + \int_{\Gamma_R} \alpha u v$$

$$\tilde{c} = c_0 + \frac{1}{2} \operatorname{div} \mathbf{a} \quad F(v) = \int_{\Omega} fv + \int_{\Gamma_R} gv$$

$$\tilde{c}_- \leq \tilde{c}(x) \leq \tilde{c}_+, \quad \nu_- \leq \nu(x) \leq \nu_+, \quad \alpha(x) \geq 0 \quad \text{a.e. in } \Omega$$

# Convergence theory of overlapping DD preconditioners

$A$  non-self-adjoint or indefinite  $\Rightarrow$  convergence analysis is hard:

- no general analysis framework like the Fictitious Space Lemma, which is valid for Symmetric Positive Definite (SPD) problems
- no generally applicable and *descriptive* convergence bounds for GMRES (and for other Krylov methods for non-self-adjoint problems convergence theory is extremely limited)
- analysis of the spectrum not sufficient for GMRES:

# Convergence theory of overlapping DD preconditioners

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- no generally applicable and descriptive convergence bounds for GMRES (and for other Krylov methods for non-self-adjoint problems convergence theory is extremely limited)
- analysis of the spectrum not sufficient for GMRES:

[Greenbaum et al 1996]

*"Any nonincreasing convergence curve can be obtained with GMRES applied to a matrix having any desired eigenvalues"*

Given  $f(0) \geq f(1) \geq \dots \geq f(n-1) > 0$ , there exist  $A \in \mathbb{R}^{n \times n}$  and  $\mathbf{b} \in \mathbb{R}^n$  with  $\|\mathbf{r}^0\| = f(0)$  such that  $\|\mathbf{r}^m\| = f(m)$ ,  $m = 1, \dots, n-1$ .

Moreover, the matrix  $A$  can be chosen to have any eigenvalues.

# Convergence estimates for GMRES

Krylov space:  $\mathcal{K}^m(A, \mathbf{b}) = \text{span}\{\mathbf{r}^0, A\mathbf{r}^0, \dots, A^{m-1}\mathbf{r}^0\}$        $\mathbf{r}^0 = \mathbf{b} - A\mathbf{x}^0$

If  $\mathbf{x}^m \in \mathbf{x}^0 + \mathcal{K}^m(A, \mathbf{b})$ , then  $\mathbf{r}^m = p_m(A)\mathbf{r}^0$  for  $p_m \in \mathbb{P}_m$  and  $p_m(0) = 1$

GMRES finds  $\mathbf{x}^m \in \mathbf{x}^0 + \mathcal{K}^m(A, \mathbf{b})$  that minimizes the residual w.r.t.  $\|\cdot\|_2$

$$\|\mathbf{r}^m\|_2 = \min_{p_m \in \mathbb{P}_m, p_m(0)=1} \|p_m(A)\mathbf{r}^0\|_2$$

Bound:

$$\frac{\|\mathbf{r}^m\|_2}{\|\mathbf{r}^0\|_2} \leq \min_{p_m \in \mathbb{P}_m, p_m(0)=1} \|p_m(A)\|_2$$

which can be bounded in several ways ...

# Convergence estimates for GMRES

- Eigenvalue + eigenvector bound:  
if  $A$  is diagonalizable,  $A = X\Lambda X^{-1}$

$$\frac{\|\mathbf{r}^m\|_2}{\|\mathbf{r}^0\|_2} \leq \|\mathbf{X}\|_2 \|\mathbf{X}^{-1}\|_2 \min_{p_m \in \mathbb{P}_m, p_m(0)=1} \max_{\lambda_j} |p_m(\lambda_j)|$$

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- Field of value (or numerical range) bound:  
 $W(A) = \{ (\mathbf{V}, A\mathbf{V}) \mid \mathbf{V} \in \mathbb{C}^n, \|\mathbf{V}\|_2 = 1 \}$

$$\frac{\|\mathbf{r}^m\|_2}{\|\mathbf{r}^0\|_2} \leq (1 + \sqrt{2}) \min_{p_m \in \mathbb{P}_m, p_m(0)=1} \max_{z \in W(A)} |p_m(z)|$$

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- $\varepsilon$ -pseudospectrum bound:  
 $\Lambda_\varepsilon(A) = \{ z \in \mathbb{C} \mid \|(zI - A)^{-1}\| \geq \varepsilon^{-1} \}$

$$\frac{\|\mathbf{r}^m\|_2}{\|\mathbf{r}^0\|_2} \leq \frac{|\partial \Lambda_\varepsilon|}{2\pi\varepsilon} \min_{p_m \in \mathbb{P}_m, p_m(0)=1} \max_{z \in \Lambda_\varepsilon(A)} |p_m(z)|$$

# Convergence estimates for GMRES

Computable field of value bound:

## Elman convergence estimate for (preconditioned) GMRES

Consider the field of values (f.o.v.) of  $M^{-1}A$ :

$$W(M^{-1}A) = \{ \langle \mathbf{V}, M^{-1}A\mathbf{V} \rangle \mid \mathbf{V} \in \mathbb{C}^n, \|\mathbf{V}\| = 1 \}$$

and suppose  $0 \notin W(M^{-1}A)$ . Then

$$\frac{\|\mathbf{r}^m\|}{\|\mathbf{r}^0\|} \leq \sin^m(\beta) \quad \text{where} \quad \cos(\beta) := \frac{\text{dist}(0, W(M^{-1}A))}{\|M^{-1}A\|}$$

[Elman 1982], [Beckermann et al 2005] improvement, [Graham et al 2017] for weighted GMRES

# Convergence theory of overlapping DD preconditioners

Theory for (specific) non-self-adjoint or indefinite problems:

- [Chan, Zou 1996] non-symmetric parabolic problems (small perturbations of SPD operators)
- Helmholtz with absorption
  - ▶ [Graham et al 2017] Dirichlet transmission conditions, two-level (coarse grid)
  - ▶ [Graham et al 2020] Robin (impedance) transmission conditions, one-level
  - ▶ [Gong et al 2020] heterogeneous
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# Convergence theory of overlapping DD preconditioners

Here, by generalizing [Graham et al 2020],

## Analysis framework for generic problems

Identify a list of assumptions and estimates that are sufficient to obtain

- an upper bound on the norm of  $M^{-1}A$
- a lower bound on the distance of the field of values of  $M^{-1}A$  from the origin

[Bonazzoli, Claeys, Nataf, Tournier, *Journal of Scientific Computing*, 89, 19 (2021)]

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# One-level overlapping DD preconditioner

Decomposition of  $\Omega$  into  $N$  overlapping subdomains  $\Omega_j$

$\Rightarrow$  decomposition of the set of unknowns  $\mathcal{N}$  into  $N$  subsets  $\mathcal{N}_j$

## Symmetrized Optimized Restricted Additive Schwarz (SORAS)

$$M^{-1} = \sum_{j=1}^N R_j^T D_j B_j^{-1} D_j R_j$$

- $R_j$  restriction matrix from  $\mathcal{N}$  to  $\mathcal{N}_j$
- $D_j$  partition of unity matrix for  $\mathcal{N}_j$  ( $\sum_{j=1}^N R_j^T D_j R_j = I$ )
- $B_j$  local matrix of the *subproblem* on  $\Omega_j$  with **Robin-type or absorbing** transmission conditions on  $\partial\Omega_j \setminus \partial\Omega$
- $R_j^T$  extension matrix from  $\mathcal{N}_j$  to  $\mathcal{N}$

⚠  $M^{-1}$  is not self-adjoint when  $B_j$  is not self-adjoint

OBDD-H [Kimn Sarkis 2007], ORAS [St-Cyr et al 2007], SORAS [Haferssas et al 2015]

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# Weighted GMRES method

Residual minimization w.r.t **weighted norm**:

$$\|\mathbf{V}\|_{\Omega} = (\mathbf{V}, \mathbf{V})_{F_{\Omega}}^{1/2} \quad \text{where } (\mathbf{V}, \mathbf{W})_{F_{\Omega}} = (F_{\Omega} \mathbf{V}, \mathbf{W}) = \mathbf{W}^* F_{\Omega} \mathbf{V}$$

where  $F_{\Omega}$  is  $n \times n$  self-adjoint positive definite matrix,  $\mathbf{V}, \mathbf{W} \in \mathbb{C}^n$

Locally on  $\Omega_j$ :

$$\|\mathbf{V}^j\|_{\Omega_j} = (\mathbf{V}^j, \mathbf{V}^j)_{F_{\Omega_j}}^{1/2} \quad \text{where } (\mathbf{V}^j, \mathbf{W}^j)_{F_{\Omega_j}} = (F_{\Omega_j} \mathbf{V}^j, \mathbf{W}^j)$$

where  $F_{\Omega_j}$  is  $n_j \times n_j$  self-adjoint positive definite matrix,  $\mathbf{V}^j, \mathbf{W}^j \in \mathbb{C}^{n_j}$

Example

$$\|u\|_{1,c} = (u, u)_{1,c}^{1/2} \quad (u, v)_{1,c} = \int_{\Omega} (\tilde{c}uv + \nu \nabla u \cdot \nabla v)$$

$$\|u\|_{1,c,\Omega_j} = (u, u)_{1,c,\Omega_j}^{1/2} \quad (u, v)_{1,c,\Omega_j} = \int_{\Omega_j} (\tilde{c}uv + \nu \nabla u \cdot \nabla v)$$

$$\tilde{c}_- \leq \tilde{c}(x) \leq \tilde{c}_+ \text{ with } \tilde{c}_- > 0, \tilde{c}_+ > 0$$

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## General theory: Assumption 1

For  $j = 1, \dots, N$ , for all global  $\mathbf{V} \in \mathbb{C}^n$  and local  $\mathbf{W}^j \in \mathbb{C}^{n_j}$  in  $\Omega_j$

$$(\textcolor{blue}{D}_j R_j \textcolor{green}{A} \mathbf{V}, \mathbf{W}^j) = (\textcolor{blue}{D}_j \textcolor{red}{B}_j R_j \mathbf{V}, \mathbf{W}^j)$$

Ok for: local sesquilinear form like global sesquilinear form but with

- integrals on  $\Omega_j$  instead of  $\Omega$
- additional boundary integral on  $\partial\Omega_j \setminus \partial\Omega$

### Example

$$\textcolor{green}{a}(u, v) = \int_{\Omega} \left( \tilde{c}uv + \frac{1}{2}\mathbf{a} \cdot \nabla u v - \frac{1}{2}u\mathbf{a} \cdot \nabla v + v\nabla u \cdot \nabla v \right) + \int_{\Gamma_R} \alpha uv$$

$$\textcolor{red}{a}_j(u, v) = \int_{\Omega_j} \left( \tilde{c}uv + \frac{1}{2}\mathbf{a} \cdot \nabla u v - \frac{1}{2}u\mathbf{a} \cdot \nabla v + v\nabla u \cdot \nabla v \right) + \int_{\partial\Omega_j \setminus \Gamma_D} \alpha uv$$

e.g.  $\alpha(\mathbf{x}) = \sqrt{(\mathbf{a} \cdot \mathbf{n})^2 + 4c_0\nu}/2$

## General theory: Assumption 2

For all local  $\mathbf{W}^j \in \mathbb{C}^{n_j}$  in  $\Omega_j$

$$\left\| \sum_{j=1}^N R_j^T \mathbf{W}^j \right\|_{\Omega}^2 \leq \Lambda_0 \sum_{j=1}^N \|\mathbf{W}^j\|_{\Omega_j}^2$$

Continuity property of the reconstruction operator  $\{\mathbf{W}^j\}_{j=1}^N \mapsto \sum_{j=1}^N R_j^T \mathbf{W}^j$

Ok for:

- **local continuous norm** like the **global continuous norm** but with integrals on  $\Omega_j$  instead of  $\Omega$
- $\Lambda_0 = \max_j \#\Lambda(j)$ , where  $\Lambda(j) := \{ i \mid \Omega_j \cap \Omega_i \neq \emptyset \}$   
maximum number of neighboring subdomains

Example

$$\|u\|_{1,c} = (u, u)_{1,c}^{1/2} \quad (u, v)_{1,c} = \int_{\Omega} (\tilde{c}uv + \nu \nabla u \cdot \nabla v)$$

$$\|u\|_{1,c,\Omega_j} = (u, u)_{1,c,\Omega_j}^{1/2} \quad (u, v)_{1,c,\Omega_j} = \int_{\Omega_j} (\tilde{c}uv + \nu \nabla u \cdot \nabla v)$$

## General theory: Assumption 3

For all global  $\mathbf{V} \in \mathbb{C}^n$

$$\sum_{j=1}^N \|R_j \mathbf{V}\|_{\Omega_j}^2 \leq \Lambda_1 \|\mathbf{V}\|_{\Omega}^2$$

Ok for:

- local continuous norm like the global continuous norm but with integrals on  $\Omega_j$  instead of  $\Omega$
- $\Lambda_1 = \max \{ m \mid \exists j_1 \neq \dots \neq j_m \text{ such that } \text{meas}(\Omega_{j_1} \cap \dots \cap \Omega_{j_m}) \neq 0 \}$  maximal multiplicity of the subdomain intersection

## General theory: Assumptions 4, 5 and 6

For  $j = 1, \dots, N$ , for all local  $\mathbf{W}^j, \mathbf{V}^j \in \mathbb{C}^{n_j}$  in  $\Omega_j$

$$\|\mathcal{D}_j \mathbf{W}^j\|_{\Omega_j} \leq C_{D,j} \|\mathbf{W}^j\|_{\Omega_j}$$

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$$\|\textcolor{blue}{D}_j \mathbf{W}^j\|_{\Omega_j} \leq C_{D,j} \|\mathbf{W}^j\|_{\Omega_j}$$

$$|([\textcolor{blue}{D}_j, \textcolor{red}{B}_j] \mathbf{V}^j, \mathbf{W}^j)| \leq C_{DB,j} \|\mathbf{V}^j\|_{\Omega_j} \|\mathbf{W}^j\|_{\Omega_j}$$

where commutator  $[P, Q] := PQ - QP$

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Stability inf-sup condition: for all local  $\mathbf{U}^j \in \mathbb{C}^{n_j}$  in  $\Omega_j$

$$\|\mathbf{U}^j\|_{\Omega_j} \leq C_{\text{stab},j} \max_{\mathbf{W}^j \in \mathbb{C}^{n_j} \setminus \{0\}} \left( \frac{|(\textcolor{red}{B}_j \mathbf{U}^j, \mathbf{W}^j)|}{\|\mathbf{W}^j\|_{\Omega_j}} \right)$$

## General theory: Assumptions 7 and 8

For  $j = 1, \dots, N$ , for all global  $\mathbf{V} \in \mathbb{C}^n$  and local  $\mathbf{W}^j \in \mathbb{C}^{n_j}$  in  $\Omega_j$

$$(\textcolor{blue}{D}_j R_j \textcolor{green}{F}_\Omega \mathbf{V}, \mathbf{W}^j) = (\textcolor{blue}{D}_j \textcolor{red}{F}_{\Omega_j} R_j \mathbf{V}, \mathbf{W}^j)$$

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For all local  $\mathbf{V}^j, \mathbf{W}^j \in \mathbb{C}^{n_j}$  in  $\Omega_j$

$$|([\mathcal{D}_j, \mathcal{F}_{\Omega_j}] \mathbf{V}^j, \mathbf{W}^j)| \leq C_{DF,j} \|\mathbf{V}^j\|_{\Omega_j} \|\mathbf{W}^j\|_{\Omega_j}$$

where commutator  $[P, Q] := PQ - QP$

# Analysis framework for generic problems

## Theorem

Under Assumptions 1–6, then upper bound on the norm of  $M^{-1}A$ :

$$\max_{\mathbf{V} \in \mathbb{C}^n} \frac{\|M^{-1}A\mathbf{V}\|_\Omega}{\|\mathbf{V}\|_\Omega} \leq \sqrt{\Lambda_0 \Lambda_1} \max_{j=1,\dots,N} \{C_{D,j}(C_{\text{stab},j} C_{DB,j} + C_{D,j})\}$$

If in addition Assumptions 7–8, then lower bound on the distance of the field of values of  $M^{-1}A$  from the origin:

$$\begin{aligned} \min_{\mathbf{V} \in \mathbb{C}^n} \frac{|(F_\Omega \mathbf{V}, M^{-1}A\mathbf{V})|}{\|\mathbf{V}\|_\Omega^2} &\geq \frac{1}{\Lambda_0} - \Lambda_1 \max_{j=1,\dots,N} \{C_{D,j} C_{\text{stab},j} C_{DB,j}\} \\ &\quad - \Lambda_1 \max_{j=1,\dots,N} \{C_{DF,j}(C_{\text{stab},j} C_{DB,j} + C_{D,j})\} \end{aligned}$$

[Bonazzoli, Claeys, Nataf, Tournier, *Journal of Scientific Computing*, 89, 19 (2021)]

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## Application to heterogeneous reaction-convection-diffusion

- [Cai 1991; Cai & Widlund 1992] analysis of overlapping Schwarz prec, Dirichlet t.c., two-level, coarse mesh with sufficiently small elements
- [Nataf & Rogier 1995; Lube et al 2000] analysis of non-overlapping Schwarz prec, Robin or more general t.c., one-level
- [Achdou et al 2000; Alart et al 2000] analysis of Neumann–Neumann prec (substructuring DD family), one-level; two-level, coarse space not based on a coarse mesh but without convergence analysis

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Here one-level overlapping Schwarz preconditioner with Robin-type t.c.

$\Omega_j$  are uniformly star-shaped polyhedra with characteristic length scale  $H_{\text{sub}}$

Prove all the Theorem assumptions *with explicit constants*

$\Rightarrow$  obtain upper bound on the norm and lower bound on the f.o.v.

[Bonazzoli, Claeys, Nataf, Tournier, *Journal of Scientific Computing*, 89, 19 (2021)]

# Application to heterogeneous reaction-convection-diffusion

$$\begin{aligned} \min_{\mathbf{V} \in \mathbb{C}^n} \frac{|(F_\Omega \mathbf{V}, M^{-1} A \mathbf{V})|}{\|\mathbf{V}\|_\Omega^2} &\geq \frac{1}{\Lambda_0} - \Lambda_1 \max_{j=1,\dots,N} \{ C_{D,j} C_{\text{stab},j} C_{DB,j} \} \\ &\quad - \Lambda_1 \max_{j=1,\dots,N} \{ C_{DF,j} (C_{\text{stab},j} C_{DB,j} + C_{D,j}) \} \end{aligned}$$

$$\Lambda_0 = \max_{j=1,\dots,N} \# \Lambda(j), \quad \text{where } \Lambda(j) = \{ j' \mid \Omega_j \cap \Omega_{j'} \neq \emptyset \}$$

$$\Lambda_1 = \max \{ m \mid \exists j_1 \neq \dots \neq j_m \text{ such that } \text{meas}(\Omega_{j_1} \cap \dots \cap \Omega_{j_m}) \neq 0 \}$$

$$C_{\text{stab},j} = 1$$

$$C_{D,j} = \mathcal{C}(\nu_{+,j}, \tilde{c}_{-,j}, \delta) + \textcolor{blue}{C}_{\text{err},j}$$

$$C_{DB,j} = \mathcal{C}(\nu_{+,j}, \nu_{-,j}, \tilde{c}_{-,j}, \|\mathbf{a}\|_{L^\infty(\Omega_j)}, \delta) + 2 \textcolor{orange}{C}_{\text{cont},j} \textcolor{blue}{C}_{\text{err},j}$$

$$C_{DF,j} = \mathcal{C}(\nu_{+,j}, \nu_{-,j}, \tilde{c}_{-,j}, \delta) + 2 \textcolor{blue}{C}_{\text{err},j}$$

$$\textcolor{blue}{C}_{\text{err},j} = \mathcal{C}(r, d, h, \nu_{+,j}, \nu_{-,j}, \tilde{c}_{+,j}, \delta)$$

$$\textcolor{orange}{C}_{\text{cont},j} = \mathcal{C}(\nu_{+,j}, \nu_{-,j}, \tilde{c}_{+,j}, \tilde{c}_{-,j}, \|\mathbf{a}\|_{L^\infty(\Omega_j)}, \alpha, H_{\text{sub}})$$

# Application to reaction-convection-diffusion

In particular:

- The negative terms can be made arbitrarily small *if  $\delta$  is sufficiently generous*
- If the equation derives from a backward Euler scheme for the time-dependent problem,  $\tilde{c} = 1/\Delta t$ .

$C_{D,j}, C_{DB,j}, C_{DF,j}$  contain (homogeneous case):

$$\sqrt{\frac{\nu}{\tilde{c}}} \frac{1}{\delta} \quad \frac{\|a\|_{L^\infty(\Omega_j)}}{\tilde{c}} \frac{1}{\delta}$$

So  $\delta$  should be asymptotically bigger than the **square root of the diffusion area covered in a time step**, and than **the convection distance covered in a time step**

## Numerical experiments

- 2D rectangular domain,  $\Gamma_D = \Gamma$  ( $u = 1$  on the left side)
- vertical strips/METIS decomposition
- GMRES with right preconditioning (tolerance  $10^{-6}$ )
- zero/random initial guess
- local problems in each subdomain solved with a direct solver (MUMPS)
- ffddm framework of FreeFEM
- test also ORAS preconditioner

$$M_{ORAS}^{-1} = \sum_{j=1}^N R_j^T D_j B_j^{-1} \textcolor{orange}{D}_j R_j \quad M_{\textcolor{blue}{ORAS}}^{-1} = \sum_{j=1}^N R_j^T D_j B_j^{-1} R_j$$

## Numerical experiments ( $N = 5$ vertical strips)

$$\mathbf{a} = 2\pi[-(y - 0.1), (x - 0.5)]^T \text{ (rotating)}$$

	#SORAS(ORAS)			
	$\delta = 2h$	$\delta = 4h$	$\delta = 6h$	$\delta = 8h$
$c_0 = 1, \nu = 1$	21(18)	20(14)	20(12)	19(11)
$c_0 = 1, \nu = 0.001$	14(9)	13(6)	12(5)	12(5)
$c_0 = 0.001, \nu = 1$	21(20)	20(15)	20(13)	19(11)
$c_0 = 0.001, \nu = 0.001$	15(10)	14(7)	13(5)	13(5)

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$$\mathbf{a} = [-x, -y]^T \Rightarrow \tilde{c} = c_0 - 1 \text{ is zero or negative}$$

	#SORAS(ORAS)			
	$\delta = 2h$	$\delta = 4h$	$\delta = 6h$	$\delta = 8h$
$c_0 = 1, \nu = 1$	21(19)	21(14)	20(13)	20(11)
$c_0 = 1, \nu = 0.001$	16(7)	16(7)	16(6)	16(6)
$c_0 = 0.001, \nu = 1$	22(24)	22(18)	22(15)	21(13)
$c_0 = 0.001, \nu = 0.001$	17(8)	16(7)	16(7)	16(6)

# Numerical experiments: weak scaling test ( $\delta = 4h$ )

$\mathbf{a} = [1, 0]^T$  (horizontal) Streamline Upwind Petrov-Galerkin (**SUPG**) stabilization

$N$  vertical strips

$\mathbf{a} = [1, 0]^T$	#SORAS(ORAS)					
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
$c_0 = 1, \nu = 1$	18(15)	23(18)	28(19)	35(19)	36(19)	36(19)
$c_0 = 1, \nu = 0.001$	8(3)	10(5)	16(8)	23(16)	37(32)	63(62)
$c_0 = 0.001, \nu = 1$	18(15)	23(19)	29(21)	35(21)	36(21)	36(21)
$c_0 = 0.001 = \nu$	8(3)	10(5)	16(8)	24(16)	40(32)	71(64)

$N$  arbitrary-shaped subdomains (METIS)

$\mathbf{a} = [1, 0]^T$	#SORAS(ORAS)					
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
$c_0 = 1, \nu = 1$	21(17)	30(22)	40(23)	48(23)	53(23)	55(23)
$c_0 = 1, \nu = 0.001$	10(4)	12(5)	17(9)	25(17)	38(32)	63(63)
$c_0 = 0.001, \nu = 1$	21(18)	30(25)	40(28)	48(27)	54(28)	57(29)
$c_0 = 0.001 = \nu$	10(4)	12(5)	18(9)	26(17)	42(33)	73(65)

# Outline

- 1 Introduction and state of art
- 2 Setting
- 3 General convergence theory
- 4 Reaction-convection-diffusion equation
- 5 Conclusion

# Conclusion and outlook

- *Convergence analysis framework* for one-level SORAS preconditioner for non-self-adjoint or indefinite problems.
- General theory not specific to one particular boundary value problem.
- Application to one-level overlapping preconditioners with Robin-type transmission conditions for the heterogeneous reaction-convection-diffusion equation.

Outlook:

- Improve the lower bound by designing a suitable *coarse space* (two-level preconditioner) → see Niall's and Victorita's talks tomorrow!

Theory for *generic* problems?