Supersymmetry and trace formulas

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(**Based on the joint work with Changha Choi, arXiv:2112.07942, arXiv:2306.13636 & arXiv:2502.10210**)

I. Introduction

- Trace type formula is a statement

  "Spectral Trace = Matrix Trace"

  where spectral trace $\mathrm{Tr}\, A$ of an operator A is the sum of eigenvalues, and the matrix trace is the sum (integral) of diagonal elements of its matrix (kernel).

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- If $A = e^{-\beta\hat{H}}$, where $\beta > 0$ and $\hat{H}$ is a Schrödinger operator (Laplace operator on some manifold M) with purely discrete spectrum, the matrix trace is a Feynman (Wiener) path integral of e^{-S_E} ("Euclidean action") over the space of maps $\mathbb{R}/\beta\mathbb{Z} \to M$ (Feynman-Kac formula).

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- Can compute the matrix trace only when $\beta \to 0$.
- For a Dirac operator $\not{\partial}$ of a Levi-Civita connection on a spin Riemannian manifold M the difference $\operatorname{Tr}\left(e^{-\beta \not{\partial}^* \not{\partial}} - e^{-\beta \not{\partial} \not{\partial}^*}\right)$ — an integer — is the analytic index of $\not{\partial}$.

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- Supersymmetry, a global symmetry between bosons and fermions, provides invaluable insights to the non-perturbative aspects of general strongly coupled quantum field theories, and is deeply related to various areas of mathematics.

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- The Hilbert space of a supersymmetric quantum theory

$$\mathscr{H} = \mathscr{H}^+ \oplus \mathscr{H}^-$$

is $\mathbb{Z}_2$-graded by the fermion number operator F.

- The quantity

$$I = \mathrm{Str}\, e^{-\beta \hat{H}} = \mathrm{Tr}(-1)^F e^{-\beta \hat{H}}$$

called Witten index by physicists, carries precise non-perturbative information about the ground states of a supersymmetric quantum Hamiltonian $\hat{H}$.

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- Example: Dirac operator $\not{\partial} = \gamma^\mu(\boldsymbol{x})\nabla_\mu$ on a spin manifold,

$$\hat{H} = \not{\partial}^2 = \begin{pmatrix} \not{\partial}_+ \not{\partial}_+^* & 0 \\ 0 & \not{\partial}_+^* \not{\partial}_+ \end{pmatrix},$$

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- Quantum supersymmetric system with the real supercharge $\hat{Q}$,

$$\hat{Q}^2 = \hat{H},$$

where quantum Hamiltonian $\hat{H}$ acts in the Hilbert space $\mathscr{H}$.

- The Witten index is given by the path integral

$$I = \mathrm{Tr}(-1)^F e^{-\beta \hat{H}} = \int e^{-S_E[x,\psi]} \mathscr{D}x \mathscr{D}\psi,$$

  where

$$S_E[x,\psi] = \int_0^\beta \mathcal{L}_E(x, \dot{x}; \psi, \dot{\psi}) dt$$

  is the Euclidean action, and $\mathscr{D}x\mathscr{D}\psi$ is a path integration 'measure' for the bosonic and fermionic degrees of freedom.

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- The integration goes over periodic boundary conditions and

$$\delta S_E = 0 \quad \text{and} \quad \delta(\mathscr{D}x\mathscr{D}\psi) = 0.$$

  Here δ is (Wick rotated) classical supersymmetry transformation generated by a supercharge Q,

$$\delta x^\mu = \{Q, x^\mu\} = \psi^\mu, \quad \delta\psi^\mu = \{Q, \psi^\mu\} = -\dot{x}^\mu.$$

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- Let $V[x,\psi]$ be an invariant deformation, a functional of classical fields satisfying

$$\delta^2 V = 0.$$

- **The key fact:** for all real λ we have

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- The Witten index ($\mathcal{L}(M)$ is a free loop space of M)

$$I = \mathrm{Str}\, e^{-\beta\hat{H}} = \int_{\Pi T\mathcal{L}(M)} e^{-S_E}\mathscr{D}x\mathscr{D}\psi$$

localizes on constant loops (Witten 1982, Atiyah 1985); explicit computation (L. Alvarez-Gaumé, 1983) gives AS formula for the index of Dirac operator.

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  2. In the Hilbert space $\mathscr{H} = \mathscr{H}_B \otimes \mathscr{H}_F$ the Majorana fermions $\hat{\chi}_1, \dots, \hat{\chi}_n$ satisfy

$$c_n \hat{\chi}_1 \cdots \hat{\chi}_n = 2^{-\frac{n}{2}} (-1)^F,$$

where $c_n = i^{n(n-1)/2}$, so since $\boxed{(-1)^F \cdot (-1)^F = 1}$ we have

$$\operatorname{Str} \hat{\chi}_1 \cdots \hat{\chi}_n e^{-\beta \hat{H}} = 2^{-\frac{n}{2}} \operatorname{Tr}_{\mathscr{H}} e^{-\beta \hat{H}} = \int \chi_1 \cdots \chi_n e^{-S_E} \mathscr{D}x \mathscr{D}\psi.$$

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- Note that condition **(A)** is rather natural, condition **(B)** is standard, while condition **(C)**, the absence of fermion zero modes in V and δV, is a completely new requirement. It is rather constraining and forces V to explicitly depend on the first time derivatives of fermion degrees of freedom.

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where V is a deformation satisfying conditions (B)–(C).
- If bosonic and fermionic degrees of freedom decouple

$$\mathscr{H} = \mathscr{H}_B \otimes \mathscr{H}_F \quad \text{and} \quad \hat{H} = \hat{H}_B \otimes I_F + I_B \otimes \hat{H}_F,$$

then

$$\mathrm{Str}\, \hat{\chi}_1 \cdots \hat{\chi}_n e^{-\beta \hat{H}} = 2^{-n/2}\, \mathrm{Tr}_{\mathscr{H}_F}\, e^{-\beta \hat{H}_F} \cdot \mathrm{Tr}_{\mathscr{H}_B}\, e^{-\beta \hat{H}}.$$

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- If $\hat{H}_F = 0$, we have

$$\boxed{\mathrm{Str}\, \hat{\chi}_1 \cdots \hat{\chi}_n e^{-\beta\hat{H}} = \mathrm{Tr}_{\mathscr{H}_B}\, e^{-\beta\hat{H}_B}}$$

Thus we obtain a pure bosonic trace formula by localizing the supersymmetric path integral in the limit $\lambda \to \infty$ to the zero locus of V.

III. Examples

- Poisson summation formula (Jacobi): free supersymmetric particle on $\mathrm{U}(1)$

$$\boxed{\sum_{n=-\infty}^{\infty} e^{-n^2\beta/2} = \sqrt{\frac{2\pi}{\beta}} \sum_{n=-\infty}^{\infty} e^{-2\pi^2 n^2/\beta}}$$

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- Eskin trace formula: supersymmetric sigma-model with a flat left-invariant connection on a compact semi-simple Lie group G (Л.Д. Эскин "Уравнение теплопроводности на группах Ли", Сб. памяти Н.Г. Чеботарева, Изд. КГУ, Казань, 1964)

$$K_\beta(e^h) = \mathrm{Tr}\left[L_g e^{-\frac{\beta}{2}\Delta_G}\right] = \sum_{\pi\in \mathrm{Irrep}\, G} d_\pi\, \chi_\pi(h) e^{-\frac{1}{2}\beta C_2(\pi)}$$

$$= \frac{\mathsf{Vol}(G) e^{\frac{1}{2}\beta\langle\rho,\rho\rangle}}{(2\pi\beta)^{n/2}} \sum_{\gamma\in\Gamma}\prod_{\alpha\in R_+} \frac{\frac{1}{2}\langle\alpha, h+\gamma\rangle}{\sinh\frac{1}{2}\langle\alpha, h+\gamma\rangle} e^{-\frac{1}{2\beta}\langle h+\gamma, h+\gamma\rangle}$$

($g = e^h$, $h\in\mathfrak{t}$ is regular, ρ is Weyl vector, Γ is co-character lattice, $T = \mathfrak{t}/\Gamma$).

- Frenkel trace formula: supersymmetric gauged sigma-model on $G \times G$ for compact $G \simeq G \times G/G$ (where $(g_1, g_2) \mapsto g_1 g_2^{-1}$)

$$Tr\left[L_{g_l} R_{g_r}^{-1} e^{-\frac{1}{2}\beta\Delta_G}\right] = \sum_{\pi \in \mathsf{Irrep}\, G} \chi_\pi(g_l)\chi_\pi(g_r^{-1}) e^{-\frac{1}{2}\beta C_2(\pi)}$$

$$= \frac{\mathsf{Vol}(\mathbb{T}) e^{\frac{1}{2}\beta\langle\rho,\rho\rangle}}{(2\pi\beta)^{r/2}\mathfrak{s}(h_l)\mathfrak{s}(-h_r)} \sum_{(w,\gamma)\in W\times 2\pi i Q^\vee} \epsilon(w) e^{-\frac{1}{2}\beta\|h_l - wh_r + \gamma\|^2}$$

Here $r = \dim \mathbb{T}$ where $\mathbb{T}$ is the maximal torus, $g_{l,r} = e^{h_{l,r}}$ with regular $h_{l,r} \in \mathfrak{t}$, $Q^\vee$ is the corout lattice, W is the Weyl group and $\mathfrak{s}(h)$ is a denominator of the Weyl character formula

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- Selberg trace formula: K-gauged supersymmetric sigma-model on $\Gamma\backslash G$; G — real, semisimple, non-compact, K — maximal compact subgroup. Classic case: $G = \mathrm{SL}(2,\mathbb{R})$, $K = \mathrm{SO}(2)$, $\mathbb{H} = G/K$ is Lobachevsky plane, and Γ is co-compact Fuchsian group.

IV. Details

(a) Poisson summation formula

- Free supersymmetric particle of mass $m = 1$ on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ with the following Lagrangian and real supercharge

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + i\psi\dot{\psi}), \qquad Q = i\dot{x}\psi,$$

and the Hamiltonian

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- Quantum supercharge and the Hamiltonian operator are

$$\hat{Q} = \psi P \quad \text{and} \quad \hat{H} = \frac{1}{2}\hat{Q}^2 = \frac{1}{2}P^2.$$

- The partition function is

$$Z(\beta) = \operatorname{Tr} e^{-\beta \hat{H}} = \sum_{n \in \mathbb{Z}} e^{-\beta n^2/2}, \quad \beta > 0.$$

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- Using path integral,

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- In the limit $\lambda \to \infty$ the path integral localizes on the classical trajectories $\ddot{x} = 0$, and one can compute $Z(\beta)$ exactly.

(b) Eskin summation formula on compact G

- $0+1$ supersymmetric sigma model — supersymmetric particle on compact simple Lie group G with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\langle \dot{x}, \dot{x} \rangle + \frac{i}{2}\langle \boldsymbol{\psi}, \nabla^{-}_{\dot{x}} \boldsymbol{\psi} \rangle, \quad \boldsymbol{\psi} \in \Pi T_{x(t)} G,$$

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- In Cartan moving frame formalism $J = g^{-1}\dot{g} \in \mathfrak{g}$ and $\psi = L_{g^{-1}} \boldsymbol{\psi} \in \Pi\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G and

$$\mathcal{L} = \frac{1}{2}\langle J, J \rangle + \frac{i}{2}\langle \psi, \dot{\psi} \rangle.$$

- Real supercharge

$$Q = \langle \psi, J \rangle + \frac{i}{6} \langle \psi, [\psi, \psi] \rangle$$

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- Hamiltonian operator $\hat{H} = \hat{Q}^2$ is given by

$$\hat{H} = \frac{1}{2}\Delta + \frac{R}{12}\hat{I}$$

where Δ is the Laplace operator on $L^2(G)$ and the second term ($R = n/4$ is scalar curvature) is the 'notorious' DeWitt term.

- Fermion zero modes

$$\chi^a = \frac{1}{\beta}\int_0^\beta \psi^a \, dt,$$

so (factor $c_n = i^{n(n-1)/2}$ is included)

$$\mathsf{Str}\,\hat{\chi}^1 \dots \hat{\chi}^n e^{-\beta\hat{H}} = e^{-\frac{1}{12}\beta R}\,\mathrm{Tr}\,e^{-\frac{1}{2}\beta\Delta}.$$

and

$$\mathsf{Str}\,\hat{\chi}^1 \dots \hat{\chi}^n e^{-\beta\hat{H}+i\langle h,\hat{r}\rangle} = V_G e^{-\frac{1}{12}\beta R} K_\beta(e^h),$$

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- Path integral representation

$$\mathsf{Str}\,\hat{\chi}^1 \ldots \hat{\chi}^n e^{-\beta \hat{H} + i\langle h, \hat{r}\rangle} = \int\limits_{\Pi TLG} \chi^1 \ldots \chi^n e^{-S_E^h}\, \mathscr{D}g \mathscr{D}\psi,$$

where

$$S_E^h = \frac{1}{2}\int_0^\beta (\langle J^h, J^h\rangle + \langle \psi, \dot{\psi}\rangle) dt, \quad J^h = J + \frac{1}{\beta}\mathsf{Ad}_{g^{-1}} h.$$

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$$\delta g = g\psi, \quad \delta\psi = -J^h - \psi\psi, \quad \delta J^h = (\partial_t + \mathsf{ad}_{J^h})\psi.$$

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- When $h \in \mathfrak{t}$ is regular, on ΩG solutions are isolated geodesics and we obtain Eskin summation formula.

(c) Frenkel trace formula

- Gauged supersymmetric sigma-model on $G \times G$

$$\begin{aligned}&\mathcal{L}_{(G\times G)/G}\\ &= \langle J_{1,A}, J_{1,A}\rangle + \langle J_{2,A}, J_{2,A}\rangle + i\langle \psi_1, D_A\psi_1\rangle + i\langle \psi_2, D_A\psi_2\rangle,\end{aligned}$$

where A is a connection in the principal G-bundle over $\mathbb{R}/\beta\mathbb{Z}$, $J_{k,A} = g_k^{-1}\dot{g}_k - A$ for $k = 1, 2$ and $D_A = \partial_t + \mathrm{ad}_A$.

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$$g_k \mapsto g_k g(t), \quad A \mapsto g(t)^{-1}Ag(t) + g(t)^{-1}\dot{g}(t).$$

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- Non-chiral twist $L_{g_l}R_{g_r}^{-1}$ becomes a left twist $L_{g_l}^{(1)}L_{g_r}^{(2)}$, and as in Eskin formula, we replace in the Lagrangian

$$J_{k,A} \to \tilde{J}_{k,A} = J_{k,A} + \frac{1}{\beta}g_k^{-1}h_k g_k, \quad k = 1, 2,$$

where $h_1 = h_l, h_2 = h_r$.

- The corresponding Euclidean action S_E for this Lagrangian is supersymmetric, where

$$\delta g_k = g_k \psi_k, \quad \delta \psi_k = -\tilde{J}_{k,A} - \psi_k \psi_k \quad \text{and} \quad \delta A = 0.$$

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- There are $2r = r + r$ fermion zero modes $\chi_k^1, \ldots, \chi_k^r$ coming from the kernel of $\nabla_A = d + \mathrm{ad}_A$ for each ψ_k, and we put

$$\chi_k(A) = c_r 2^{r/2} \chi_k^1 \ldots \chi_k^r, \quad k = 1, 2.$$

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- The Gauss law $J_{1,A} + J_{2,A} = 0$ reduces $L^2(G \times G)$ and $L^2(G)$ and

$$\mathrm{Tr}_{L^2(G)} \left[L_{g_l} R_{g_r}^{-1} e^{-\frac{1}{2}\beta \Delta_G} \right] = \frac{e^{\frac{1}{2}\beta \langle \rho, \rho \rangle}}{\mathrm{vol}(\mathcal{G})}$$

$$\times \int \frac{\chi_1(A)\chi_2(A) e^{-S_E}}{\left[\mathrm{Pf}' \left(i(\mathrm{Hol}_{S_\beta^1}^{-1/2}(\nabla_A) - \mathrm{Hol}_{S_\beta^1}^{1/2}(\nabla_A)) \right) \right]^2} \mathscr{D}A \prod_{k=1,2} \mathscr{D}g_k \mathscr{D}\psi_k.$$

- By gauge fixing A to the constant gauge $A = h/\beta$, the integration over A reduces to the integration over the holonomy $\boldsymbol{t} = e^h \in \mathbb{T}$:

$$\mathrm{Tr}_{L^2(G)}\left[L_{g_l} R_{g_r}^{-1} e^{-\frac{1}{2}\beta\Delta_G}\right] = \frac{e^{\frac{1}{2}\beta\langle\rho,\rho\rangle}}{|W|\mathsf{vol}(\mathbb{T})}$$
$$\times \int_{\mathbb{T}} |\delta(\boldsymbol{t})|^2 \left(\int \frac{\chi_1(h/\beta)\chi_2(h/\beta)e^{-S'_E}}{\mathfrak{s}(h)^2} \prod_{k=1,2} \mathscr{D}g_k \mathscr{D}\psi_k\right) d\boldsymbol{t},$$

where

$$\delta(\boldsymbol{t}) = \det(1 - \mathsf{Ad}_{\boldsymbol{t}})_{\mathfrak{g}/\mathfrak{t}} = \prod_{\alpha\in R_+}\left(e^{\frac{\langle\alpha,h\rangle}{2}} - e^{-\frac{\langle\alpha,h\rangle}{2}}\right).$$

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- Invariant deformation $S'_E \to S'_E + \lambda\delta(V_1 + V_2)$ with

$$V_k = -\int_0^\beta \langle \dot{\tilde{J}}_k, \dot{\psi}_k \rangle dt \quad \text{where} \quad \tilde{J}_k = \tilde{J}_{k,0}, \quad k = 1, 2.$$

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$$\int_{G/\mathbb{T}} e^{\langle X, g^{-1}\lambda g\rangle} dg = \frac{(2\pi)^{\dim(G/\mathbb{T})/2}}{\boldsymbol{\pi}(X)\boldsymbol{\pi}(\lambda)} \sum_{w\in W} \epsilon(w) e^{\langle w\lambda, X\rangle},$$

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- As a result, we obtain Frenkel trace formula.
- The path integral derivation of Selberg trace formula is similar but more involved.

Verbania, Provincia Verbano-Cusio-Ossola, Piemonte 2015

Саша, с Днем Рождения!

Skoltech, 2018

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