About Landau and Landau Fermi Dirac operators

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Based on works with Ricardo Alonso (Texas A&M, Qatar), Véronique Bagland (Univ. Clermont Auvergne), Bertrand Lods (Univ. degli studi di Torino)

Landau operator, Coulomb case

Landau, 1936: For $f := f(v) \geq 0$ number density of charged particles of velocity $v \in \mathbb{R}^3$.

Collision operator:

$$Q_0(f)(v) = \nabla \cdot \int_{\mathbb{R}^3} |v - w|^{-1} \, \Pi(v - w) \left(f(w) \, \nabla f(v) - f(v) \, \nabla f(w) \right) dw,$$

where

$$\Pi_{ij}(z) := \delta_{ij} - \frac{z_i z_j}{|z|^2}$$

is the i, j-component of the orthogonal projection Π onto

$$z^{\perp} := \{ y \,/\, y \cdot z = 0 \}$$

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Landau operator, other potentials

By analogy with Boltzmann operator,

$$Q_0(f)(v) = \nabla \cdot \int_{\mathbb{R}^3} |v-w|^{\gamma+2}\, \Pi(v-w) \left(f(w)\, \nabla f(v) - f(v)\, \nabla f(w) \right) dw$$

Hard potentials: $\gamma \in]0,1[$

Maxwell molecules: $\gamma = 0$

Moderately soft potentials: $\gamma \in [-2, 0[$

Very soft potentials: $\gamma \in]-4,-2[$ (includes the Coulomb case $\gamma = -3$)

Landau-Fermi-Dirac (LFD) operator

For $f(v) \in [0, \varepsilon^{-1}]$ density of fermions,

$$Q_\varepsilon(f)(v) = \nabla \cdot \int_{\mathbb{R}^3} |v-w|^{\gamma+2}\, \Pi(v-w) \Big[f(w)\,(1-\varepsilon f(w))\, \nabla f(v)$$

$$- f(v)\,(1-\varepsilon f(v))\, \nabla f(w) \Big]\, dw$$

Quantum parameter $\varepsilon \geq 0$ modeling Pauli's exclusion principle. For electrons, one expects that ε is very small, typically $\varepsilon \sim 10^{-10}$.

The operator can be obtained by taking the grazing collision limit in the Boltzmann-Nordheim (Uehling-Uhlenbeck) operator. The taxonomy (depending on γ) is the same as that of the Landau equation.

LFD operator, weak formulation

For a smooth well suited test function φ,

$$\int_{\mathbb{R}^3} Q_\varepsilon(f)(v)\,\varphi(v)\,dv$$

$$= \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,(1-\varepsilon f(v))\,f(w)\,(1-\varepsilon f(w))\,|v-w|^{\gamma+2}$$

$$\Big(\nabla\varphi(v)-\nabla\varphi(w)\Big)^T\,\Pi(v-w)\left(\frac{\nabla f(v)}{f(v)\,(1-\varepsilon f(v))}-\frac{\nabla f(w)}{f(w)\,(1-\varepsilon f(w))}\right)dvdw.$$

Conservation of mass, momentum and energy ($\varepsilon \geq 0$)

$$\int_{\mathbb{R}^3} Q_\varepsilon(f)(v)\begin{pmatrix}1\\ v_i\\ |v|^2/2\end{pmatrix}dv = 0.$$

Entropy inequality (first part of H-theorem) for the Landau operator

Entropy production: $f := f(v) \geq 0$

$$D_0(f) := -\int_{\mathbb{R}^3} Q_0(f)(v) \ln f(v)\, dv$$

$$= \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(v)\, f(w)\, |v-w|^{\gamma+2}$$

$$\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \right)^T \Pi(v-w) \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \right) dvdw$$

$$= \frac{1}{2} \sum_{i<j} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(v)\, f(w)\, |v-w|^{\gamma} \left| (v_i - w_i) \left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(w) \right) \right.$$

$$\left. -(v_j - w_j) \left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(w) \right) \right|^2 dvdw \geq 0$$

Entropy inequality (first part of H-theorem) for the LFD operator

Entropy production: $f := f(v) \in [0, \varepsilon^{-1}]$

$$D_\varepsilon(f) := -\int_{\mathbb{R}^3} Q_\varepsilon(f)(v) \ln\left(\frac{\varepsilon f(v)}{1-\varepsilon f(v)}\right) dv$$

$$= \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Xi_\varepsilon(f)\, |v-w|^{\gamma+2}\, dvdw$$

where

$$\Xi_\varepsilon(f) := f(v)\,(1-\varepsilon f(v))\, f(w)\,(1-\varepsilon f(w)) \left(\frac{\nabla f}{f(1-\varepsilon f)}(v) - \frac{\nabla f}{f\,(1-\varepsilon f)}(w)\right)^T$$

$$\Pi(v-w) \left(\frac{\nabla f}{f\,(1-\varepsilon f)}(v) - \frac{\nabla f}{f\,(1-\varepsilon f)}(w)\right) \geq 0.$$

Case of equality in the entropy inequality (second part of H-theorem) for Landau and LFD operator

For (reasonable, non zero a.e.) $f \geq 0$, and $\varepsilon \geq 0$ small enough,

$$D_\varepsilon(f) = 0 \qquad \Rightarrow \qquad f(v) = f_{eq,\varepsilon}(v) := \frac{M(v)}{1 + \varepsilon\, M(v)},$$

where

$$M(v) = \exp(a - b|v-u|^2),$$

and $a \in \mathbb{R}$, $u \in \mathbb{R}^3$, $b > 0$.

For larger ε and/or small temperatures, degenerate steady states (characteristic functions of balls) can appear.

Entropy inequality (first part of H-theorem) for the spatially homogeneous LFD equation

Unknown:

$$f := f(t,v) \in [0, \varepsilon^{-1}],$$

Spatially homogeneous LFD equation:

$$\frac{\partial f}{\partial t}(t,v) = Q_\varepsilon(f(t,\cdot))(v).$$

Entropy inequality:

$$\frac{d}{dt} H_\varepsilon(f(t,\cdot)) = -D_\varepsilon(f(t,\cdot)) \le 0,$$

with the entropy

$$H_\varepsilon(f) := \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \Big[\varepsilon f(v) \, \ln(\varepsilon f(v)) + (1 - \varepsilon f(v)) \ln(1 - \varepsilon f(v)) \Big] \, dv.$$

Cauchy theory for the spatially homogeneous LFD equation, case of hard potentials

Theorem Alonso, Bagland, Lods 2019 There exists $\varepsilon_0 > 0$, such that for $\varepsilon \in]0, \varepsilon_0]$, $f_{in} \in [0, \varepsilon^{-1}]$ lying in $L^1_{15}(\mathbb{R}^3)$, and with finite entropy, there exists a unique weak (in a suitable sense) solution to the spatially homogeneous LFD equation $\frac{\partial f}{\partial t}(t, v) = Q_\varepsilon(f(t, \cdot))(v)$ with hard potentials ($\gamma \in]0, 1[$).

Moreover this solution is such that $f \in C^\infty([t_0, +\infty[; \mathcal{S}(\mathbb{R}^3))$ for all $t_0 > 0$. All the corresponding seminorms do not depend on $\varepsilon \leq \varepsilon_0$.

Finally one can take $t_0 = 0$ if the initial datum is sufficiently smooth and decreasing at infinity.

Cf. also previous works by Chen 2010, 2011.

Main idea of the proof: Try to estimate at the same time the L^1 and the L^2 moments of f.

Cercignani's conjecture for LFD equation with hard potential

Strong formulation of Cercignani's conjecture for an equation (Boltzmann, Landau, LFD, etc.) with entropy H and its dissipation D:

$$D(f) \geq C\,(H(f) - H(f_{eq}))$$

for some $C > 0$ depending only on conserved quantities and an upper bound of $H(f)$. Here f_{eq} is the equilibrium given by the second part of Boltzmann's H theorem..

Known to be true in the case of the Landau equation with (over)Maxwell molecules LD, Villani 2000 ; A weaker formulation holds for the Landau equation with hard potentials (C is allowed to depend on extra quantities like L^p_q norms for p, q well chosen, LD 2023).

Cercignani's conjecture for LFD equation with hard potential

Estimate for the LFD equation with hard potentials or Maxwell molecules (uniform in $\varepsilon \in [0, \varepsilon_0]$)

Proposition Alonso, Bagland, LD, Lods 2021

$$D_\varepsilon(f) \geq C \left[b_\varepsilon - 12\,\varepsilon^2 \frac{\max(||f||_\infty^2, ||f_{eq,\varepsilon}||_\infty)}{\inf(1-\varepsilon f)^4} \right] \Big(H_\varepsilon(f) - H_\varepsilon(f_{eq,\varepsilon}) \Big),$$

where

$$f_{eq,\varepsilon}(v) = \frac{a_\varepsilon \ \exp(-b_\varepsilon |v-u_\varepsilon|^2)}{1 + a_\varepsilon \ \exp(-b_\varepsilon |v-u_\varepsilon|^2)},$$

and C depends on $||f||_{L^1_2(\mathbb{R}^3)}$, $||f||_{L^2_2(\mathbb{R}^3)}$, and $\inf(1-\varepsilon f)$.

Cercignani's conjecture for LFD equation with hard potential: consequence

Estimate for the LFD equation with hard potentials or Maxwellian molecules (uniform in ε):

$$D_\varepsilon(f) \geq C \left[b_\varepsilon - 12\,\varepsilon^2 \frac{\max(||f||_\infty^2, ||f_{eq,\varepsilon}||_\infty)}{\inf(1-\varepsilon f)^4} \right] \Big(H_\varepsilon(f) - H_\varepsilon(f_{eq,\varepsilon}) \Big),$$

where

$$f_{eq,\varepsilon}(v) = \frac{a_\varepsilon \, \exp(-b_\varepsilon |v - u_\varepsilon|^2)}{1 + a_\varepsilon \, \exp(-b_\varepsilon |v - u_\varepsilon|^2)},$$

and C depends on $||f||_{L^1_2(\mathbb{R}^3)}$, $||f||_{L^2_2(\mathbb{R}^3)}$, and $\inf(1-\varepsilon f)$.

Proposition Alonso, Bagland, LD, Lods 2021 Let $f_{in} \in L^1_{15}(\mathbb{R}^3)$ be an initial datum with (values in $[0, \varepsilon_0^{-1}]$ and) finite entropy. Then there exists $\varepsilon_1 \in]0, \varepsilon_0]$ such that for any $\varepsilon \in]0, \varepsilon_1]$ and $t \geq 1$,

$$H_\varepsilon(f(t)) - H_\varepsilon(f_{eq,\varepsilon}) \leq (H_\varepsilon(f_{in}) - H_\varepsilon(f_{eq,\varepsilon}))\, e^{-\mu\,(t-1)},$$

for some $\mu > 0$ depending only on the conserved quantities and the initial entropy.

Cercignani's conjecture for LFD equation with hard potential: idea of the proof

Estimate for the LFD equation with hard potentials (uniform in ε):

$$D_\varepsilon(f) \geq C \left[b_\varepsilon - 12\,\varepsilon^2 \frac{\max(||f||^2_\infty, ||f_{eq,\varepsilon}||_\infty)}{\inf(1-\varepsilon f)^4} \right] \Big(H_\varepsilon(f) - H_\varepsilon(f_{eq,\varepsilon}) \Big),$$

where C depends on $||f||_{L^1_2(\mathbb{R}^3)}$, $||f||_{L^2_2(\mathbb{R}^3)}$, and $\inf(1-\varepsilon f)$.

First inequality:

$$D_\varepsilon(f) \geq C \int_{\mathbb{R}^3} \left| \frac{\nabla f(v)}{f(v)\,(1-\varepsilon f(v))} - K\,v \right|^2 f(v) < v >^\gamma \, dv,$$

with

$$K := \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \ln(1-\varepsilon f(v))\,dv.$$

Second inequality (cf. Carrillo, Jüngel, Markowich, Toscani, Unterreiter 2001):

$$\int_{\mathbb{R}^3} \left| \frac{\nabla f(v)}{f(v)\,(1-\varepsilon f(v))} + 2b_\varepsilon\,v \right|^2 f(v)\,dv \geq 2b_\varepsilon\,(H_\varepsilon(f) - H_\varepsilon(f_{eq,\varepsilon})).$$

Cercignani's conjecture for LFD equation with hard potential: idea of the proof

Explanation of the first step:

$$D_\varepsilon(f) = \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)\,(1-\varepsilon f(v))\, f(w)\,(1-\varepsilon f(w))\,|v-w|^{\gamma+2}$$

$$\left(\frac{\nabla f}{f(1-\varepsilon f)}(v) - \frac{\nabla f}{f\,(1-\varepsilon f)}(w)\right)^T$$

$$\Pi(v-w)\,\left(\frac{\nabla f}{f\,(1-\varepsilon f)}(v) - \frac{\nabla f}{f\,(1-\varepsilon f)}(w)\right) dvdw$$

so that

$$D_\varepsilon(f) = \frac{1}{2} \iint f\,(1-\varepsilon f)(v)\, f\,(1-\varepsilon f)(w)$$

$$\left\|(v-w)\times\left(\frac{\nabla f(v)}{f(v)(1-\varepsilon f(v))} - \frac{\nabla f(w)}{f(w)\,(1-\varepsilon f(w))}\right)\right\|^2 |v-w|^\gamma\, dvdw$$

Cercignani's conjecture for LFD equation with hard potential: idea of the proof

For showing that

$$D_\varepsilon(f) = \frac{1}{2} \iint f\,(1-\varepsilon f)(v)\, f\,(1-\varepsilon f)(w)$$

$$\left\|(v-w) \times \left(\frac{\nabla f}{f(v)(1-\varepsilon f(v))} - \frac{\nabla f(w)}{f(w)\,(1-\varepsilon f(w))}\right)\right\|^2 |v-w|^\gamma \, dvdw$$

$$\geq C \int_{\mathbb{R}^3} \left| \frac{\nabla f(v)}{f(v)\,(1-\varepsilon f(v))} - K\, v\right|^2 f(v) < v >^\gamma \; dv,$$

one uses the identity

$$\int_{\mathbb{R}^3} (v-w) \times \left(\frac{\nabla f(v)}{f(v)(1-\varepsilon f(v))} - \frac{\nabla f(w)}{f(w)\,(1-\varepsilon f(w))}\right) \Bigg|_{ij} f(w)\, w_i dw$$

$$= -\int_{\mathbb{R}^3} f(w)\, w_i^2 dw \,\frac{\partial_j f(v)}{f(v)(1-\varepsilon f(v))} + K\, v_j - \frac{1}{\varepsilon}\int_{\mathbb{R}^3} \ln(1-\varepsilon f(v))\, v_j \, dv$$

Cercignani's conjecture for LFD equation with hard potential: other approaches

Alonso, Bagland, Lods 2019: Use of a weak Cercignani conjecture (algebraic decay) coupled with spectral analysis;

LD 2023: Use of a Cercignani conjecture with simple proof, but valid only for small entropy dissipation;

Borsoni 2024: Use of a general principle transfering entropy-entropy dissipation inequalities from the classical setting to the quantum (Pauli exclusion principle) setting

Cauchy theory for the spatially homogeneous LFD equation, case of moderately soft potentials

Theorem Alonso, Bagland, LD, Lods 2022 There exists $\varepsilon_0 > 0$, such that for $\varepsilon \in]0, \varepsilon_0]$, $f_{in} \in [0, \varepsilon^{-1}]$ lying in $L^1_{2+0}(\mathbb{R}^3)$, and with finite entropy, there exists a weak (in a suitable sense) solution to the spatially homogeneous LFD equation $\frac{\partial f}{\partial t}(t,v) = Q_\varepsilon(f(t,\cdot))(v)$ with moderately soft potentials ($\gamma \in]-2, 0[$).

Moreover this solution satisfies (for all $T > t_0 > 0$) $f \in L^\infty([t_0, T]; L^\infty(\mathbb{R}^3))$, with a norm which does not depend on ε.

Cauchy theory for the spatially homogeneous LFD equation, case of moderately soft potentials

Moreover, for $q \geq 2$ if $f_{in} \in L^1_\infty(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ this solution satisfies (for all $T > 0$) $f \in L^\infty([0,T]; L^{q-0}_\infty(\mathbb{R}^3))$, with a norm which does not depend on ε.

Finally if $f_{in} \in W^{1,\infty-0}_\infty(\mathbb{R}^3)$, then this solution satisfies (for all $T > 0$) $f \in L^\infty([0,T]; W^{1,\infty-0}_\infty(\mathbb{R}^3) \cap C^{0,\alpha}([0,T] \times \mathbb{R}^3)$, with norms which do not depend on ε.

Main ideas of the proof: Once again, try to estimate at the same time the L^1 and the L^2 moments of f. The appearance of an L^∞ estimate can be obtained thanks to a De Giorgi approach.

Large time behavior of the LFD equation, case of moderately soft potentials

Proposition Alonso, Bagland, LD, Lods 2022 Let $f_{in} \in [0, \varepsilon_0]$ satisfying

$$\int_{\mathbb{R}^3} f_{in}(v) \exp(a_0 < v >^q) \, dv < +\infty$$

for $a_0 > 0$ and $0 < q < \frac{4+2\gamma}{8-\gamma}$, be an initial datum with finite entropy.

Then there exists $\varepsilon_1 \in]0, \varepsilon_0]$ such that for any $\varepsilon \in]0, \varepsilon_1]$ and $t \geq 2$,

$$H_\varepsilon(f(t)) - H_\varepsilon(f_{eq,\varepsilon}) \leq \max \left(1, H_\varepsilon(f_{in}) - H_\varepsilon(f_{eq,\varepsilon})\right) e^{-\lambda \, t^{\frac{q}{q-\gamma}}},$$

for some $\lambda > 0$ depending only on a_0, q and f_{in}.

Idea of the proof: Logarithmic interpolation

$$\frac{d}{dt}(H_\varepsilon(f(t)) - H_\varepsilon(f_{eq,\varepsilon})) = -D_\varepsilon^{(\gamma)}(f),$$

with

$$D_\varepsilon^{(\gamma)}(f) := \frac{1}{2} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Xi_\varepsilon(f)\,|v-w|^{\gamma+2}\,dvdw.$$

We first observe that (when $a, q > 0$)

$$D_\varepsilon^{(\gamma)}(f) \geq \frac{1}{2}\left[\frac{1}{a}\ln\left(\frac{\Gamma_\varepsilon^{a,q}(f)}{D_\varepsilon^{(0)}(f)}\right)\right]^{\frac{\gamma}{q}} D_\varepsilon^{(0)}(f),$$

where

$$\Gamma_\varepsilon^{a,q}(f) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Xi_\varepsilon(f)\,|v-w|^{\gamma+2}\,\exp(a|v-w|^q)\,dvdw.$$

Idea of the proof: Logarithmic interpolation

$$\Gamma_\varepsilon^{a,q}(f) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Xi_\varepsilon(f)\,|v-w|^{\gamma+2}\,\exp(a|v-w|^q)\,dvdw$$

can be estimated in the following way:

$$\Gamma_\varepsilon^{a,q}(f) \le C \int_{\mathbb{R}^3} \frac{|\nabla f(v)|^2}{f(v)} < v >^2 \exp(a < v >^q)\,dv$$

where C depends on $||f||_\infty$ and

$$\int_{\mathbb{R}^3} f(w) < w >^2 \exp(a < w >^q)$$

Idea of the proof: Estimate for the weighted Fisher information

$$\Gamma_\varepsilon^{a,q}(f) \le C \int_{\mathbb{R}^3} \frac{|\nabla f(v)|^2}{f(v)} < v >^2 \exp(a < v >^q)\, dv.$$

Provided that the initial datum has a suitable strectched exponential decay for large v, it is possible to show that (for $t \ge t_0 > 0$)

$$\int_{t_0}^{t} \int_{\mathbb{R}^3} \frac{|\nabla f(s,v)|^2}{f(s,v)} < v >^2 \exp(a < v >^q)\, dvds \le C\,(1+t)^2.$$

Idea: investigate the evolution of the "$L \ln L$-exponential moment"

$$\int_{\mathbb{R}^3} f(t,v) \ln f(t,v) \exp(b < v >^q)\, dv,$$

for suitable b.

Idea of the proof: use of a differential inequality

$$\frac{d}{dt}(H_\varepsilon(f(t)) - H_\varepsilon(f_{eq,\varepsilon})) = -D_\varepsilon^{(\gamma)}(f),$$

$$\leq -\frac{1}{2}\left[\frac{1}{a}\ln\left(\frac{\Gamma_\varepsilon^{a,q}(f)}{D_\varepsilon^{(0)}(f)}\right)\right]^{\frac{\gamma}{q}} D_\varepsilon^{(0)}(f),$$

$$\leq -C\left[C - \ln\left(D_\varepsilon^{(0)}(f)\right)\right]^{\frac{\gamma}{q}} D_\varepsilon^{(0)}(f),$$

But we know that

$$D_\varepsilon^{(0)}(f) \geq C\,(H_\varepsilon(f(t)) - H_\varepsilon(f_{eq,\varepsilon})),$$

so that $t \mapsto y(t) := H_\varepsilon(f(t)) - H_\varepsilon(f_{eq,\varepsilon})$ satisfies a differental inequality leading to the stretched exponential estimate.

In fact, the proof is a little more complex because $\Gamma_\varepsilon^{a,q}(f(t,\cdot))$ is bounded only in L^1 in time (and not in L^∞ in time).

In a work by Carrapatoso, LD, He 2017, it is shown that (under suitable assumptions on the initial datum) the weak solutions of Landau equations in the Coulomb case converge towards equilibrium with a stretched exponential rate.

The proof can be extended to the moderately soft potentials case.

It is based on a different (less conceptual ?) modification of the entropy method:

$$D(f(t)) \geq a(t) \Big(H(f(t)) - H(f_{eq}) \Big) - b(t),$$

and on a more general logarithmic Sobolev inequality.

Perspectives

- Main perspective: try to reproduce the proof of Guillen, Silvestre 2023 for LFD equation with Coulomb potential
- Use the method of logarithmic interpolation for Landau equation for all soft potentials
- Study of LBE equation
- Study of Boltzmann-Nordheim equation with some uniformity in ε, cf. Borsoni, Lods