Low degree Hurwitz stacks in the Grothendieck ring

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Motivational question

Question

How many degree d number fields are there?

Question

How many degree d covers of Spec $\mathbb{Z}$ are there?

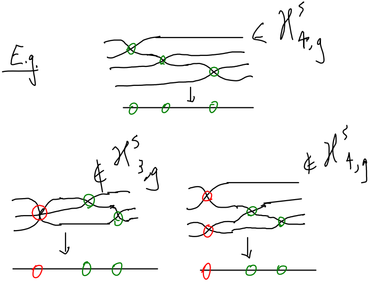
Question

How many degree d covers does $\mathbb{P}^1$ have?

Hurwitz space

Definition

Define the **Hurwitz space** $\mathscr{H}_{d,g}$ as the moduli space of (C, f) where C is a smooth proper geometrically connected genus g curve and $f : C \to \mathbb{P}^1$ is a finite degree d map with Galois closure having Galois group S_d. The **simply branched Hurwitz space** $\mathscr{H}^s_{d,g}$ parameterizes simply branched such covers, i.e., covers with squarefree discriminant.



Defining the Grothendieck Ring

Definition

The **Grothendieck ring of varieties** $K_0(Var_k)$ is the abelian group generated by isomorphism classes of finite type schemes over k with the following relations: for every such scheme X and closed subscheme $Z \subset X$

$$[X] = [Z] + [X - Z] .$$

We will work in a modified version of the Grothendieck ring where:

(1) We invert universally bijective morphisms.

(2) We invert $[\mathbb{A}^1]$ and $[\mathbb{A}^n] - 1$ for all $n \geq 1$.

(3) We complete along the dimension filtration, i.e., a sequence of classes approaches 0 if its dimension approaches $-\infty$.

The main result

Let $\mathrm{Conf}^n(X)$ denote the configuration space of n points on X. There is a covering $\mathscr{H}^s_{d,g} \to \mathrm{Conf}^{2g+2d-2}(\mathbb{P}^1)$. Naively, one might wonder if these have the nearly same class in the Grothendieck ring.

Question

For $d \geq 2$, and $\mathrm{char}(\mathbb{F}_q) \nmid d!$, does

$$\lim_{g\to\infty} \frac{[\mathscr{H}^s_{d,g}]}{[\mathbb{A}^1]^{\dim \mathscr{H}^s_{d,g}}} \stackrel{?}{=} \frac{[\mathrm{Conf}^{2g+2d-2}(\mathbb{P}^1)]}{[\mathbb{A}^1]^{\dim \mathscr{H}^s_{d,g}}} = 1 - [\mathbb{A}^1]^{-2}?$$

Theorem (L-Vakil-Wood)

The answer is "yes" for $d \leq 5$.

Comparisons

Finite field point counting	Number Rings	Cohomology	Chow Ring	Grothendieck Ring
How many covers of $\mathbb{P}^1_{\mathbb{F}_q}$ are there?	How many covers of Spec $\mathbb{Z}$ are there?	What is the cohomology of $\mathscr{H}_{d,g}$?	What is the rational Chow ring of $\mathscr{H}_{d,g}$?	What is the class of $\mathscr{H}_{d,g}$?
For $d = 3$, stabilizes in g, due to Datskovsky-Wright (over arbitrary function fields), $d \leq 5$ due to Bhargava-Shankar-Wang, $d = 3$ also investigated by Zhao and Gunther	Stabilizes when counted by discriminant: $d = 3$ Davenport and Heilbronn, $d = 4, 5$ due to Bhargava	The cohomology of $\mathscr{H}^s_{d,g}$ stabilizes (though it is not known what it stabilizes to) as $g \to \infty$ when $d = 3$, Ellenberg-Venkatesh-Westerland	Patel-Vakil stabilization of rational chow ring $d = 3$, Deopurkar-Patel stabilization of rational Picard groups $d \leq 5$, Canning-Larson stabilization of rational Chow ring $d \leq 5$.	Stabilizes in g for $d \leq 5$ L-Vakil-Wood

Table: Interpretations of counting covers of $\mathbb{P}^1$

The main result for the full Hurwitz space

We also have the following variant for the full Hurwitz space, in place of the simply branched Hurwitz space.

Question

For $d \geq 2$, and $\mathrm{char}(\mathbb{F}_q) \nmid d!$, does

$$\lim_{g \to \infty} \frac{[\mathscr{H}_{d,g}]}{[\mathbb{A}^1]^{\dim \mathscr{H}_{d,g}}} = \frac{1}{1 - [\mathbb{A}^1]^{-1}} \left(\prod_{x \in \mathbb{P}^1} (1 - [\mathbb{A}^1]^{-1}) \left(\sum_{\lambda \vdash d} [\mathbb{A}^1]^{d - |\lambda|} \right) \right)?$$

Above, $|\lambda|$ is the number of parts of the partition λ (and can also be thought of geometrically as the degree of ramification of a corresponding scheme).

Theorem (L-Vakil-Wood)

The answer is "yes" for $d \leq 5$.